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Local dynamics on a gauge-invariant basis of non-Abelian gauge theories

Ramesh Anishetty*
The Institute of Mathematical Sciences, Madras—600 113, India
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A gauge-invariant basis in terms of electric field strength is given. Explicitly, for SU(2) Yang-Mills theory in 3+1 dimensions, it is shown that the gauge-invariant basis states are given by "dreibeins." The Hamiltonian quantum dynamics on this basis is shown to be manifestly local and rotational invariant.

Understanding nonperturbative aspects of non-Abelian gauge theories is a long pursued subject. Confinement due to topological degrees of freedom such as monopoles, dynamical Higgs mechanism, chiral-symmetry breaking, etc. [1], are widely believed to be nonperturbative properties of non-Abelian gauge theories. In the standard Lagrangian or the Hamiltonian formulation we define the theory along with so-called unphysical degrees of freedom, i.e., gauge degrees of freedom. This renders further analysis tedious and sometimes raises questions of validity. In the Hamiltonian formulation, Gribov [2] tried to remove the redundant degrees of freedom in the Coulomb gauge and found that it is not possible to do so for all coupling constants. Others have tried it in the Lagrangian formalism and encountered similar difficulties. In radial gauges [3] it is found that a complete gauge fixing can be achieved and thus true physical degrees of freedom can be elucidated; however, the dynamics is nonlocal and needs regularizations which are not aesthetically pleasing. The situation is quite different in Weinberg-Salam theory where we have an explicit Higgs field and a complete gauge fixing is possible and useful, namely the 't Hooft unitary gauge. Indeed the low-energy physics becomes transparent in this gauge. 't Hooft [4] attempted to achieve similar ends for pure non-Abelian gauge theories.

Mandelstam [5] suggested that the natural gauge-invariant objects of interest are Wilson loops and in principle we can rewrite the dynamics in terms of these objects, which therefore will yield a nonlocal theory. Recently [6] on the lattice, non-Abelian gauge theory has been reformulated as a local theory of local gauge-invariant objects. Here we will show that in the continuum also the Hamiltonian quantum dynamics on explicit gauge-invariant states is manifestly local and rotational invariant. To begin with, we first work with SU(2) Yang-Mills theory in 3+1 dimensions given by the Hamiltonian

$$H=\int d^3x\left[\frac{g^2}{2}[E_i^a(x)]^2+\frac{1}{4g^2}[F_{ij}^a(x)]^2\right], \tag{1}$$

where

$$F_{ij}^a(x)\equiv\partial_i A_j^a(x)-\partial_j A_i^a(x)+\epsilon^{abc}A_i^b A_j^c , \tag{2}$$

g is the coupling constant, and the canonical conjugates E_i^a and A_i^a are the electric field strength and the gauge vector potential satisfying equal-time local commutation algebra. The physical states of the theory satisfy the first-class, Gauss-law constraint

$$\partial_i E_i^a(x)+\epsilon^{abc}A_i^b(x)E_i^c(x)=0 . \tag{3}$$

The left-hand side is the infinitesimal gauge transformation generator.

For the unconstrained system a complete basis is that of electric field strengths $E_i^a(x)$. Consider the metric

$$g_{ij}(x)\equiv E_i^a(x)E_j^a(x) . \tag{4}$$

It is a gauge-invariant, positive, symmetric metric with six degrees of freedom per space point, which are exactly the physical degrees of freedom we expect. $g_{ij}(x)$ cannot form a complete physical basis since there is another gauge-invariant object

$$\det E=\frac{1}{3!}\epsilon^{abc}E_i^a(x)E_j^b(x)E_k^c(x)\epsilon_{ijk} , \tag{5}$$

with $(\det E)^2=\det g$. The sign of $\det E$ cannot be recovered from g_{ij}. Indeed from (4), E_i^a is almost the square root of g_{ij}. By choosing a gauge choice that E_i^a is symmetric matrix with space and color indices or equivalently the gauge choice

$$\mathcal{E}^a\equiv\tfrac{1}{2}\epsilon_{iab}E_i^b=0 , \tag{6}$$

we can define the "unique" square root of g_{ij} which forms a complete basis for physical states. To begin with, the electric field strength has nine degrees of freedom and (6) removes three gauge degrees of freedom per space point.

Making the following general decomposition

$$\begin{aligned}E_i^a(x)&\equiv e_i^a(x)+\epsilon_{iab}\mathcal{E}^b(x) ,\\ A_i^a(x)&\equiv \pi_i^a(x)+\epsilon_{iab}\mathcal{A}^b(x) ,\end{aligned} \tag{7}$$

where $e_i^a(x)$ and $\pi_i^a(x)$ are symmetric matrices. In the gauge (6), $g_{ij}=e_i^a e_j^a$. In this gauge the color index is identified with the space index. If we make a global or local (if gravity is also coupled to the gauge fields) spatial rotation then we simultaneously have to make a rotation in the color space to be in the same gauge as (6) and thus we note that $e_i^a(x)$ transforms as a proper covariant rotation symmetric tensor. A complete gauge-invariant basis is given by "dreibeins" $e_i^a(x)$.

In the gauge (6) we can rewrite the quantum dynamics using Dirac brackets corresponding to the constraints (3) and (6). Equivalently we can solve for $\mathcal{A}^b(x)$ using (3) and (7) and then impose (6). We find

$$\mathcal{A}^a(x)=\left[\frac{1}{1(\mathrm{tr}e)-e}\right]^{ab}(\hat{D}_i^{bc}e_i^c)\ , \tag{8}$$

where

$$(\hat{D}_i)^{bc}\equiv\partial_i\delta^{bc}+\epsilon^{bdc}\pi_i^d\ .$$

Thus we find the gauge-invariant quantum dynamics is defined by

$$H=\int d^3x\left[\frac{g^2}{2}e_i^2(x)+\frac{1}{4g^2}[F_{ij}^a(x)]^2\right]\ , \tag{9}$$

where $F_{ij}^a(x)$ given by (2) with

$$A_i^a(x)=\pi_i^a(x)+\epsilon_{iab}\left[\frac{1}{\mathrm{tr}e-e}\right]^{bc}(\hat{D}\cdot e)^c \tag{10}$$

and the local equal-time commutation algebra

$$[e_i^a(x),\pi_j^b(y)]_{\mathrm{ET}}=i(\delta_{ij}\delta^{ab}+\delta_j^a\delta_i^b)\delta^3(x-y)\ . \tag{11}$$

The above quantum dynamics is manifestly rotational invariant and local. By eliminating the Gauss-law constraint we have now obtained a nonpolynomial Hamiltonian. In this formulation $e_i^a(x)$ are equivalent to gauge-invariant variables; however, $\pi_i^a(x)$ are not necessarily so, just as in the canonical formulation $E_i^a(x)$ is covariant but $A_i^a(x)$ is not covariant under gauge transformation.

Addition of matter fields to the dynamics can be envisaged easily; (10) will be modified to

$$A_i^a(x)=\pi_i^a(x)+\epsilon_{iab}\left[\frac{1}{\mathrm{tr}e-e}\right]^{bc}[(\hat{D}_ie_i)-\rho]^c\ , \tag{12}$$

where ρ^c is the matter charge density.

Exactly the same procedure can be followed for SU(2) in $2+1$ dimensions and the results are identical to the case as before except that all indices take values $i,a=1,2$.

Our analysis for SU(2) is simple since the local gauge algebra of SU(2) and the space O(3) are isomorphic. For SU(3) this simplicity is lost. The metric $g_{ij}(x)$ given by (4) for SU(3) group does not exhaust all the gauge-invariant degrees of freedom. To proceed further, we have to classify all independent ways of embedding locally O(3) algebra in SU(3). This discussion is postponed to a later publication.

It is evident from the above analysis that we can reformulate the dynamics on another choice of basis corresponding to gauges such as $\epsilon_{iab}A_i^a(x)=0$ or $\partial_i A_i^a(x)=0$. In either case we will have nonlocal interactions in the Hamiltonian and perhaps its associated Gribov ambiguities. These ambiguities are evidently consequences of wrong choice of gauges as opposed to inherent difficulties of the theory.

A similar analysis can also be done even for U(1) local gauge theories. Owing to the fact that Gauss's law $\partial_iE_i-\rho=0$ does not involve the vector potential, we are naturally led to use the Coulomb gauge with its nonlocal charge density interactions. It is surprising that in this respect the non-Abelian Gauss's law yields a local quantum dynamics.

Finally we remark that much further work is necessary to understand the dynamical consequences of this theory. In this reformulation even the weak coupling expansion is nontrivial. It shall be discussed in a later communication. A similar analysis as here can be envisaged for quantum gravity formulated as complexified SU(2) gauge theory [7].

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*Electronic address: ramesha%imsc@shiva.ernet.in.bitnet.

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