Calculation of the pion decay constant in the framework of the Bethe-Salpeter equation

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We elaborate on the calculation of the pion decay constant f_π in the context of QCD and in the limit of purely spontaneous chiral-symmetry breaking. We use the expressions given by relativistic bound-state theory which require the simultaneous solution of the Schwinger-Dyson equation and of the Bethe-Salpeter equation developed to first order in the pion momentum. The equations are solved by numerical iteration in the Landau gauge and in the ladder approximation, but without any further approximations. We use a model for the gluon propagator with ultraviolet behavior determined by renormalization-group considerations and discuss its influence on the determination of f_π. We analyze approximations that have been used in the literature for the calculation of f_π and discuss their limitations.

I. INTRODUCTION

The calculation of the π meson decay constant f_π involves a variety of fundamental theoretical issues such as spontaneous and explicit chiral-symmetry breaking, the infrared and ultraviolet behavior of the effective quark-gluon coupling, and of the solutions of the Schwinger-Dyson (SD) and Bethe-Salpeter (BS) equations. This applies not only to quantum chromodynamics (QCD) but also to higher symmetry theories such as technicolor.

The relativistically covariant calculations of f_π and analogous decay constants found in the literature are based on, or related to, the Jackiw-Johnson sum rule [1] for dynamical symmetry breaking, which expressed f_π^2 in terms of properties of the fermion propagator and of the axial-vector current. In the case in which chiral symmetry is spontaneously broken, Pagels and Stokar (PS) obtained [2] from the sum rule a widely used [3,4] approximate formula for f_π^2 which requires just the knowledge of the quark propagator $S^{-1}(q)=\not{q}A(q^2)-B(q^2)$. The PS approximation scheme sets $A(q^2)=1$ and therefore their formula depends exclusively on the function $B(q^2)$, which can be obtained from the SD equation for the quark propagator.

The assumption $A=1$ turns out to be correct only when the gluon-fermion-fermion vertex appearing in the Landau-gauge SD equation is taken to be the bare vertex, that is, in the ladder approximation, and only when the gluon propagator is the free one. If instead the free gluon propagator is realistically replaced by the effective one, that is, if the QCD coupling α_s is allowed to run as a function of momentum according to renormalization-group analysis, then the assumption $A=1$ is incorrect. One further approximation leading to the PS expression for f_π is the use of the amputated, or vertex, BS pion wave function evaluated just to zeroth order in the pion momentum.

The purpose of the present work is to improve on previous calculations by treating the Landau-gauge SD equation without making the assumption $A=1$ and by solving the BS equation to the order needed for an accurate calculation of f_π. At the same time we probe the running coupling $\alpha_s(q^2)$ by testing the effect of its low-momentum behavior on the numerical evaluation of f_π.

In Sec. II we give an exact expression for f_π which can be derived directly from the relativistic bound-state formalism developed by Nishijima and Mandelstam [5]. This is an alternative to the sum-rule derivation [1] and it expresses f_π in terms of the BS pion wave function evaluated to first order in the pion momentum. The Nambu-Goldstone-boson wave-function existence is directly related to the existence of a broken-chiral-symmetry solution [6] to the SD equation which we discuss in Sec. III. We do not attempt here to go beyond the ladder approximation for two reasons. First, we want to be able to compare our results with those of the PS approximation. Second, there is not, to our knowledge, a quantitatively reliable treatment of the nonperturbative quark-quark-gluon vertex applicable simultaneously to the SD and BS equations [7]. In Sec. IV we give the necessary details related to the Bethe-Salpeter equation for a massless pseudoscalar.

Our numerical results are presented in Sec. V, where we also elaborate on our modeling of the low- and intermediate-momentum behavior of the gluon propagator. The calculations involve the solution of coupled integral SD equations and subsequently of coupled BS equations. In both cases the kernel involves the running coupling $\alpha_s((q-k)^2)$, which in most previous treatments of the SD equation has been expressed in the approximation where $\alpha_s((q-k)^2)$ is replaced by $\theta(k^2-q^2)\alpha_s(k^2)+\theta(q^2-k^2)\alpha_s(q^2)$. This approximation has been criticized as quantitatively unreliable [8]. We have instead used the procedure of solving the integral SD equation by numerical iteration after integrating over polar angles [9], a procedure which we extend here to the solution of the BS equation. We performed the calculations for a variety of choices of parameters and for several approximations, including the PS approximation to the BS equation. Our calculations indicate that the major contribution to f_π comes from values of $\alpha_s(q^2)$ at energies below a few GeV and that the PS approximation behaves poorly for any of the models discussed here. We summarize our results in Sec. VI.

II. THE PION DECAY CONSTANT

In the absence of isospin breaking, the decay constant f_π for a π meson of momentum p_μ is given by

$$if_\pi p_\mu \delta^{ab} = \langle 0 | A_\mu^a(0) | \pi^b(p) \rangle , \tag{2.1}$$

where $A_\mu^a(x) = \bar{q}(x)\gamma_\mu\gamma_5 \frac{1}{2}\tau^a q(x)$ is the axial-vector current. A summation over color quark indices is understood.

The matrix element in Eq. (2.1) can be more explicitly expressed in terms of a $q\bar{q}$ bound-state wave function $\eta_{ij}^b(p,x-y) = \delta_{ij}(1/\sqrt{N_c})(1/\sqrt{2})\tau^b \eta(p,x-y)$, where i,j are color indices, N_c is the number of colors, and $\eta(p,x-y)$ is the properly normalized solution of the Bethe-Salpeter equation for a pseudoscalar singlet of momentum p. It follows that [5]

$$\begin{aligned} if_\pi p_\mu &= \frac{\sqrt{N_c}}{\sqrt{2}} \mathrm{Tr}[\gamma_\mu\gamma_5 \eta(p,0)] \\ &= \frac{\sqrt{N_c}}{\sqrt{2}} \mathrm{Tr} \left[\gamma_\mu\gamma_5 \int \psi(p,q) \frac{d^4q}{(2\pi)^4} \right] , \end{aligned} \tag{2.2}$$

where $\psi(p,q)$ is the Fourier transform of $\eta(p,x-y)$.

The pion bound-state function can be expressed in terms of scalar functions ψ_a as

$$\psi(p,q) = \gamma_5\psi_0 + \gamma_5 \not{p}\psi_1 + \gamma_5 \not{q} p\cdot q \psi_2 + \gamma_5 [\not{q},\not{p}]\psi_3 , \tag{2.3}$$

where charge-conjugation invariance requires that the ψ_a's be even functions of $p\cdot q$. In terms of these functions we can write

$$ip_\mu f_\pi = \left[\frac{N_c}{2} \right]^{1/2} 4 \int [p_\mu \psi_1(p,q) + q_\mu p\cdot q \psi_2(p,q)] \frac{d^4q}{(2\pi)^4} . \tag{2.4}$$

If the pseudoscalar meson is massless and if we assume that ψ_1 and ψ_2 can be expanded in a power series in $p\cdot q$ under the integral then we have the exact, and deceptively simple, expression

$$f_\pi = -i \left[\frac{N_c}{2} \right]^{1/2} \int [4\psi_1(0,q) + q^2\psi_2(0,q)] \frac{d^4q}{(2\pi)^4} . \tag{2.5}$$

We see from Eq. (2.5) that to calculate f_π we need the pion wave function only to first order in p. In the following sections we undertake its calculation. For completeness we give the equation for f_π obtained after rotation to Euclidean variables ($q_0 \to iq_4$) and angular integration:

$$f_\pi = \left[\frac{N_c}{2} \right]^{1/2} \frac{1}{16\pi^2} \int_0^\infty [4\psi_1(0,q) - q^2\psi_2(0,q)] q^2 dq^2 . \tag{2.6}$$

III. SCHWINGER-DYSON EQUATION IN QCD

In order to find the solutions to the Bethe-Salpeter equation one needs to have a compatible expression for the quark propagator, as explained in the Introduction. Therefore, we discuss first the Schwinger-Dyson equation for the quark propagator in the ladder approximation. Since we do not consider in this article the consequences of flavor breaking we do not need any internal symmetry indices on the propagator, which we write as

$$S^{-1}(q) = \not{q} A(q^2) - B(q^2) . \tag{3.1}$$

In the Landau gauge the gluon propagator can be expressed as

$$G_{\mu\nu}(k) = -\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] G(k^2) . \tag{3.2}$$

With no explicit quark mass term in the QCD Lagrangian the SD equation is

$$\begin{aligned} S^{-1}(q) = \not{q} - i \int \gamma_\mu S(k) \gamma_\nu \left[g_{\mu\nu} - \frac{(k-q)_\mu (k-q)_\nu}{(k-q)^2} \right] \\ \times G((k-q)^2) \frac{d^4k}{(2\pi)^4} . \end{aligned} \tag{3.3}$$

We can also write two separate but coupled integral equations for the functions A and B defined by Eq. (3.1). After a rotation to Euclidean variables, introducing a mass scale parameter Λ and dimensionless variables

$$\begin{aligned} &y = \frac{k^2}{\Lambda^2}, \quad x = \frac{q^2}{\Lambda^2} , \\ &\frac{(q-k)^2}{\Lambda^2} = x + y - 2\sqrt{xy}\cos\theta, \quad B(x) = \frac{B(q^2)}{\Lambda} , \end{aligned} \tag{3.4}$$

the equations read

$$B(x) = \frac{2}{\pi} \int_0^\infty \frac{B(y)}{yA^2(y) + B^2(y)} K_1(x,y) y\, dy , \tag{3.5a}$$

$$A(x) = 1 + \frac{2}{3\pi x} \int_0^\infty \frac{A(y)}{yA^2(y) + B^2(y)} K_2(x,y) y\, dy . \tag{3.5b}$$

The kernels $K_1(x,y)$ and $K_2(x,y)$ involve integrations over a four-dimensional polar angle θ and their expressions are given in the Appendix. Equation (3.5a) admits a solution $B(x)=0$ which corresponds to a situation in which there is no spontaneous breaking of global chiral symmetry. The physically interesting case is, instead, when $B(x) \neq 0$. In this case Goldstone's theorem predicts the existence of massless bosons. We will discuss the details of the solutions to Eqs. (3.5) in Sec. V.

IV. BOUND-STATE BETHE-SALPETER EQUATIONS

It has been known for a long time [6] that the BS equation predicts a massless pseudoscalar particle when solved in conjunction with the SD equation, both in the ladder approximation. In this approximation we have, for the BS function $\psi(p,q)$ of Sec. II,

$$S^{-1}(q+\tfrac{1}{2}p)\psi(p,q)S^{-1}(q-\tfrac{1}{2}p) = -i\int \gamma_\mu \psi(p,k)\gamma_\nu G_{\mu\nu}(k-q)\frac{d^4k}{(2\pi)^4} . \quad (4.1)$$

Since the kernel $\gamma_\mu \otimes \gamma_\nu G_{\mu\nu}(k-q)$ does not depend on the eigenvalue p, the normalization of the wave function is given by

$$\frac{i}{(2\pi)^4}\mathrm{Tr}\int \bar{\psi}(p,q)\left[\frac{\partial}{\partial p_\mu}S^{-1}(q+\tfrac{1}{2}p)\psi(p,q)S^{-1}(q-\tfrac{1}{2}p) + S^{-1}(q+\tfrac{1}{2}p)\psi(p,q)\frac{\partial}{\partial p_\mu}S^{-1}(q-\tfrac{1}{2}p)\right] \times d^4q = 2p_\mu , \quad (4.2)$$

where $\bar{\psi}(p,q)\equiv\gamma_0\psi^\dagger(p,q)\gamma_0$.

We can verify easily that if the SD equation (3.5a) has a nonzero solution $B(q^2)$ then the BS equation (4.1) has, for vanishing four-momentum p, a solution of the form

$$\psi(0,q)=\gamma_5\psi_0(0,q)=\frac{\gamma_5 NB(q^2)}{[q^2A^2(q^2)-B^2(q^2)]} , \quad (4.3)$$

where N is a normalization factor. Obviously, this solution is not adequate for the evaluation of f_π as given by Eq. (2.6), since one needs there the terms of $\psi(p,q)$ which are linear in p.

Alternatively, one can consider the vertex BS function

$$\zeta(p,q)=S^{-1}(q+\tfrac{1}{2}p)\psi(p,q)S^{-1}(q-\tfrac{1}{2}p) , \quad (4.4)$$

which satisfies the equation

$$\zeta(p,q)=-i\int \gamma_\mu S(k+\tfrac{1}{2}p)\zeta(p,k) \times S(k-\tfrac{1}{2}p)\gamma_\gamma G_{\mu\nu}(k-q)\frac{d^4k}{(2\pi)^4} . \quad (4.5)$$

This equation has a zero-momentum solution

$$\zeta(0,q)=\gamma_5 NB(q^2) . \quad (4.6)$$

Using Eqs. (4.6) and (4.4) one can then obtain approximate expressions for $\psi_1(0,q)$ and $\psi_2(0,q)$ and use Eq. (2.6) to calculate f_π. The further approximation of setting $A(q^2)=1$ in the SD equation yields, as discussed in the next section, the extensively used PS expression for f_π.

It is clear that the approach outlined above misses some terms linear in p in the description of $\psi(p,q)$, and consequently the expression for f_π will be flawed. Therefore we undertake here the development of accurate, order-p expressions for $\psi_1(p,q)$ and $\psi_2(p,q)$.

To first order in p we have from Eq. (2.3)

$$\psi(p,q)=\gamma_5\psi_0(0,q)+\gamma_5\Delta\psi(q)+O((p\cdot q)^2) , \quad (4.7)$$

where $\psi_0(0,q)$ is given by Eq. (4.3) and

$$\Delta\psi(p,q)\equiv\not{p}\psi_1(0,q)+\not{q}p\cdot q\psi_2(0,q)+[\not{q},\not{p}]\psi_3(0,q) . \quad (4.8)$$

Keeping just terms of order p in the BS equation (4.1) we obtain the inhomogeneous integral equation which determines $\Delta\psi(p,q)$:

$$\Delta\psi(p,q)=\{\not{p}BA+2\not{q}p\cdot q(BA'-AB')+\tfrac{1}{2}[\not{p},\not{q}]A^2\}\left[\frac{\psi_0(0,q)}{q^2A^2-B^2}\right] + [\gamma_5 S(q)\gamma_5]\left[i\int\gamma_\mu\Delta\psi(k)\gamma_\nu G_{\mu\nu}(k-q)\frac{d^4k}{(2\pi)^4}\right]S(q) . \quad (4.9)$$

In this expression $A=A(q^2)$ and $B=B(q^2)$ and the prime means the derivative with respect to q^2.

Once the propagator functions A and B are calculated by solving the SD equations (3.5a) and (3.5b), the component $\psi_0(0,q)$ of the pion wave function is determined by Eq. (4.3). Subsequently, Eq. (4.9) can be projected into three coupled linear integral equations for the functions $\psi_i(0,q)$. More conveniently, going to Euclidean variables and introducing dimensionless quantities defined by

$$x\equiv\frac{q^2}{\Lambda^2},\quad B(x)\equiv\frac{B(q^2)}{\Lambda},\quad \chi_0(x)\equiv N^{-1}\Lambda\psi_0(0,q)=-\frac{B(x)}{[xA^2(x)+B^2(x)]} ,$$
$$\chi_1(x)\equiv N^{-1}\Lambda^2\psi_1(0,q),\quad \chi_2(x)\equiv N^{-1}\Lambda^4\psi_2(0,q),\quad \chi_3(x)\equiv N^{-1}\Lambda^3\psi_3(0,q) , \quad (4.10)$$

and with the notation $D=xA^2+B^2$, we have for the $\chi_i(x)$ the set of equations

$$\chi_1=\left[-\frac{\chi_0 AB}{D}\right]+\left[\frac{B^2-xA^2}{D^2}I_1(x)+\frac{4ABx}{D^2}I_3(x)\right] , \quad (4.11a)$$

$$\chi_2=\left[2\chi_0\frac{A'B-AB'}{D}\right]+\left[\frac{-2A^2}{D^2}I_1(x)+\frac{1}{D}I_2(x)+\frac{4AB}{D^2}I_3(x)\right] , \quad (4.11b)$$

$$\chi_3=\left[\frac{\chi_0 A^2}{2D}\right]-\left[\frac{AB}{D^2}I_1(x)+\frac{(xA^2-B^2)}{D^2}I_3(x)\right] . \tag{4.11c}$$

In these equations $I_1(x)$, $I_2(x)$, and $I_3(x)$ are integrals involving combinations of the functions χ_1, χ_2, and χ_3, and their expressions are given in the Appendix. Keeping just the inhomogeneous terms on the right-hand side of the equations corresponds to the approximation discussed above involving the function $\zeta(0,q)$ of Eq. (4.6). We shall see in the next section that the terms containing the integrals produce large changes in the values of χ_1,χ_2,χ_3 and consequently of f_π.

To complete the calculation of the BS wave function we need the normalization factor N of Eq. (4.3). With the use of Eqs. (4.2), (4.7), (4.8), and (4.10) it is determined by the Euclidean integral relationship

$$\frac{1}{N^2}=-\frac{\Lambda^2}{16\pi^2}\int_0^\infty x\,dx\{\chi_0^2[A^2+x^2(A')^2+x(B')^2-2xAA'-2BB'-x^2AA''-xBB''] \\ -2\chi_0\chi_1(AB'-BA')x+4\chi_0\chi_1AB+2x^2\chi_0\chi_2(AB'-BA')-x\chi_0\chi_2AB-6x\chi_0\chi_3A^2\} . \tag{4.12}$$

A relationship between the BS wave-function normalization factor N and f_π can be obtained [1] from the axial-vector Ward identity

$$p_\mu\Gamma_\mu^{5a}(q+p,q)=\tfrac{1}{2}\tau^a[S^{-1}(q+p)\gamma_5+\gamma_5S^{-1}(q)] . \tag{4.13}$$

The limit $p\to 0$ is nonzero if $B(q)\neq 0$, that is, if there is spontaneous chiral-symmetry breaking. In such a case the presence of the massless Nambu-Goldstone boson induces a pole at $p^2=0$ in the axial-vector vertex, and from Eqs. (2.1), (4.3), and (4.13) it follows that

$$N=\sqrt{2N_c}f_\pi^{-1} . \tag{4.14}$$

V. NUMERICAL RESULTS

A. Gluon propagator model

The calculation of f_π as given by Eq. (2.6) and its comparison with the PS or other approximations requires the specification of the gluon propagator function $G(q^2)$. The ultraviolet asymptotic behavior of $G(k^2)$ is known from renormalization group analysis to be

$$G(k^2)\equiv\frac{16\pi}{3}\frac{\alpha_s(k^2)}{k^2}\approx\frac{16\pi^2}{3k^2}\frac{d}{\ln(k^2/\Lambda_{\rm QCD}^2)} , \tag{5.1}$$

where $d=12/(33-2n)$, n is the number of quark flavors, which we set to six for our calculations, $\Lambda_{\rm QCD}$ is the QCD scale parameter, and we have just kept the one-loop asymptotic expression for $\alpha_s(k^2)$. The asymptotic form (5.1) appears to be compatible with experiment for $\Lambda_{\rm QCD}$ about ≈ 200 MeV and for q larger than a few GeV [10], an assumption which we will make here. Very little is known, theoretically or experimentally, about the low-momentum behavior of $G(k^2)$. An infrared behavior $G(k^2)\approx 1/k^4$, presumably leading to confinement, has been assumed [11] or obtained from approximate calculations in covariant [12] and axial gauges [13]. Such behavior, however, makes the SD and BS equations highly singular. On the other hand, the $1/k^4$ infrared behavior has been contested by lattice calculations in the Landau gauge [14] and by theoretical and lattice calculations in the axial gauge [15]. These objections do not necessarily apply to the form $G(k^2)\approx\delta^4(k)$, a regularized alternative to the behavior $1/k^4$, which may still be indicative of confinement [16].

In view of the above we will consider for our numerical calculations two models for $G(k^2)$ which display its asymptotic ultraviolet behavior and which exemplify two different types of infrared behavior:

$$G_1(k^2)=(2\pi)^4\frac{4}{3}\eta^2\delta^4(k)+\frac{16\pi^2}{3k^2}\frac{d}{\ln(x_0+k^2/\Lambda_{\rm QCD}^2)} , \tag{5.2a}$$

$$G_2(k^2)=(2\pi)^4\frac{4}{3}\frac{1}{\mu^2}e^{-k^2/k_0^2}+\frac{16\pi^2}{3k^2}\frac{d}{\ln(x_0+k^2/\Lambda_{\rm QCD}^2)} . \tag{5.2b}$$

B. The quark propagator

For any given choice of $G(k^2)$ the coupled integral Eqs. (3.5a) and (3.5b) for the propagator functions A and B were solved numerically by simultaneous iterations which converged typically after about 100 iterations. In order to obtain solutions $A(x)$ and $B(x)$ for sufficiently large values of x, the integration variable y in Eqs. (5.2a) and (5.2b) was expressed as $y=e^z$, with z taking values in the interval -10 to 10 with a step $\Delta z=0.025$. The iterations were started with trial functions $A(y)=A_0$, a constant, and $B(y)=m[(y+x_0)\ln(x_0+y)]^{-1}$. We verified that the iterative solution of the equations was unique independently of the choices of the values A_0 and $B(0)$ in the zeroth-order trial functions. The large-x behavior of this unique solution was found to be $A(x)\to 1$ and $B(x)\to\lesssim 1/x$, in agreement with the asymptotic behavior of the regular solution found in the operator-product-expansion analysis and corresponding to the occurrence of spontaneous symmetry breaking. We found that for the cases discussed below the low-x behavior of $A(x)$ and $B(x)$ is dominated by the infrared term in the gluon propagator, Eqs. (5.2a) and (5.2b). In Figs. 1 and 2 we display $A(x)$ and $B(x)$ for several cases.

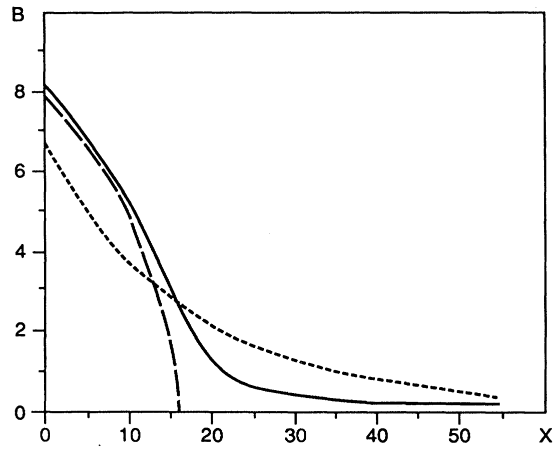

FIG. 1. Plot of $B(x)$ for three different cases: (i) $G(k^2)=G_1(k^2)$, Eq. (5.2a) (solid curve); (ii) neglecting the second term in $G_1(k^2)$ (long dashed curve); (iii) $G(k^2)=G_2(k^2)$, Eq. (5.2b) (short dashed curve).

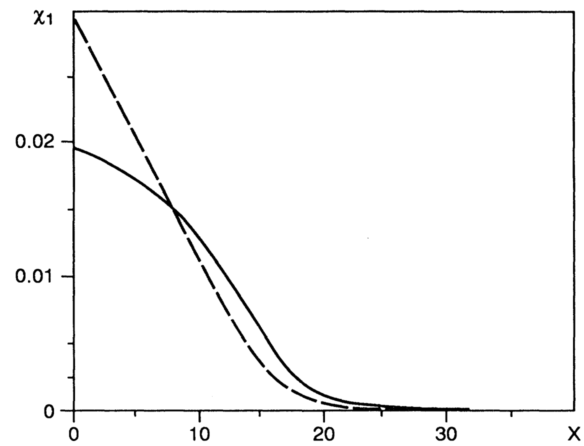

FIG. 3. The dimensionless Bethe-Salpeter wave function $\chi_1(x)$, defined in Eq. (4.10), for the exact solution (solid curve) and for the first approximation discussed in Sec. V C (dashed curve). The second approximation was found to be the same as the exact result within the accuracy of the figure. The curves correspond to the model $G(k^2)=G_1(k^2)$, Eq. (5.2a).

C. Solutions to the Bethe-Salpeter equations and calculation of f_π

The complete set of BS equations (4.11) was solved numerically by iteration starting with the inhomogeneous terms as input. As in the case of the SD equation the accuracy required at any point was $\frac{1}{1000}$. The process converged more rapidly, typically after about 30 iterations. Figures 3–5 show graphs for $\chi_1(x)$, $\chi_2(x)$, and $\chi_3(x)$. Once the χ_i's were found, f_π was calculated by using Eqs. (2.6) and (4.10). The normalization factor N was computed according to Eq. (4.12), and we used agreement with Eq. (4.14) as a further test of our numerical results.

We chose for our calculations values $\Lambda_{\text{QCD}}=230$ MeV and $x_0=2$, and subsequently determined the rest of the parameters in $G(q^2)$ by requiring that $f_\pi=93$ MeV. We obtained $\eta=920$ MeV when using G_1 of Eq. (5.2a). For G_2 of Eq. (5.2b) we chose a value $k_0=380$ MeV and obtained $\mu=600$ MeV. The values of these parameters varied just a few percent when x_0 varied between 1.4 and 100 and when Λ_{QCD} varied between 150 and 400 MeV. In all cases the expression (4.14), derived from the axial Ward identity, was satisfied within 2 to 3%.

In order to test the role of the ultraviolet part of the gluon propagator in the evaluation of f_π, we performed the calculation by keeping only the second term in the ex-

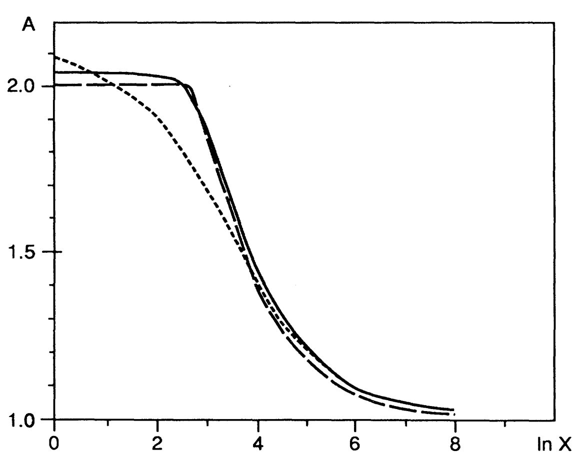

FIG. 2. Plot of $A(x)$ for three different cases: (i) $G(k^2)=G_1(k^2)$, Eq. (5.2a) (solid curve); (ii) neglecting the second term in $G_1(k^2)$ (long dashed curve); (iii) $G(k^2)=G_2(k^2)$, Eq. (5.2b) (short-dashed curve).

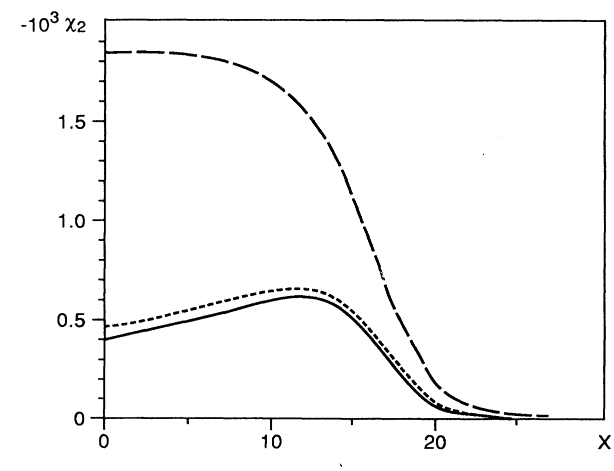

FIG. 4. The dimensionless Bethe-Salpeter wave function $-10^3\chi_2(x)$, where $\chi_2(x)$ is defined in Eq. (4.10), for the exact solution (solid curve), for the first approximation (long dashed curve), and for the second approximation (short dashed curve). $G(k^2)=G_1(k^2)$.

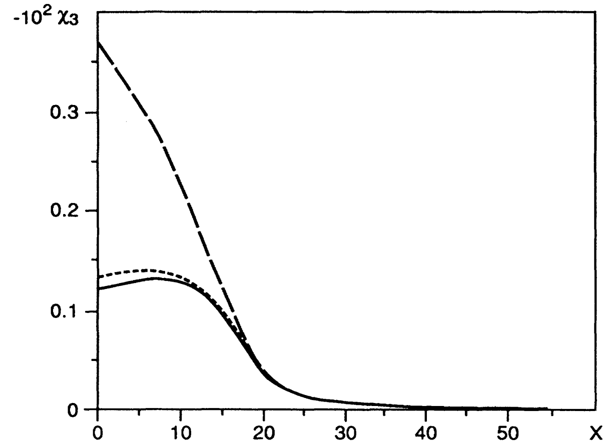

FIG. 5. The dimensionless Bethe-Salpeter wave function $-10^2\chi_3(x)$, where $\chi_3(x)$ is defined in Eq. (4.10), for the exact solution (solid curve), for the first approximation (long dashed curve), and for the second approximation (short dashed curve). $G(k^2)=G_1(k^2)$.

pressions (5.2a) and (5.2b) for $G(k^2)$, and we obtained $f_\pi=11$ MeV, indicating that the main contribution to f_π comes from $G(k^2)$ at energies below a few GeV if the present phenomenological analysis is correct [10].

To obtain the PS approximation we set $A=1$ in the SD equation (3.5a) and kept just the first term on the right-hand side of the BS equations (4.11). The value of f_π calculated this way was about three times larger than the value obtained with the full solution for any of the models for $G(k^2)$.

We also studied the effect of two other approximations to the BS equations. In the first approximation we dropped all integrals in Eqs. (4.11). The values of f_π calculated this way were 30% too small. A second approximation was to drop the logarithmic ultraviolet part from the integrals in Eqs. (4.11). In this case the value of f_π was only 5 to 10% smaller than the value calculated with the full solution and the relationship (4.14) was satisfied within 5 to 10%. This result is consistent with our comment above on the weak influence of the ultraviolet in the calculation of f_π.

VI. CONCLUSIONS

We have given here the basic expressions for the calculation of the decay constant f_π in the limit of purely spontaneously broken-chiral symmetry, together with an accurate procedure for its numerical evaluation. The formalism is only limited by the ladder approximation and it requires the knowledge of the gluon propagator function $G(k^2)$, whose ultraviolet asymptotic behavior we take to be given by renormalization group considerations. Our calculations put on a stronger quantitative basis some previous estimations [2,9,17] that the value of f_π is dominated by the low-energy behavior of $G(k^2)$, presumably [10] below a few GeV.

We have also discussed the effects on the calculation of f_π of some approximations to the solutions of the SD and/or BS equations. In particular, we have found that the widely used PS approximation seriously overestimated the value of f_π in all the models we considered.

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APPENDIX

The kernels appearing in the SD equations (5.2a) and (5.2b) are given by the expressions

$$K_1(x,y)=\frac{3}{16\pi^2}\int_0^\pi G(x,y,\cos\theta)\sin^2\theta\, d\theta\ , \tag{A1a}$$

$$K_2(x,y)=\frac{3}{16\pi^2}\int_0^\pi\left[\sqrt{xy}\cos\theta+\frac{2(\sqrt{xy}\cos\theta-y)(x-\sqrt{xy}\cos\theta)}{(x+y-2\sqrt{xy}\cos\theta)}\right]G(x,y,\cos\theta)\sin^2\theta\, d\theta\ , \tag{A1b}$$

where

$$G(x,y,\cos\theta)\equiv\Lambda^2G(k-q)^2)\ , \tag{A1c}$$

q,k Euclidean. The integrals appearing in the BS equations (5.5a)–(5.5c) are of the forms

$$I_1(x)=-\frac{2}{3\pi}\int_0^\infty\chi_1(y)K_1(x,y)y\,dy+\frac{2}{9\pi}\int_0^\infty[-2\chi_1(y)K_3(x,y)+(3y+x)\chi_2(y)K_3(x,y)-4\sqrt{xy}\,\chi_2(y)K_4(x,y)]y^2dy\ , \tag{A2a}$$

$$I_2(x)=-\frac{2}{9\pi x}\int_0^\infty\{[8yK_3(x,y)-6K_1(x,y)]\chi_1(y)+[9K_1(x,y)-(12y+10x)K_3(x,y)+16\sqrt{xy}K_4(x,y)]y\chi_2(y)\}y\,dy\ , \tag{A2b}$$

$$I_3(x) = -\frac{2}{3\pi}\int_0^\infty \chi_3(y) K_5(x,y) y\, dy \ . \tag{A2c}$$

The kernels $K_1(x,y)$ and $K_2(x,y)$ are the same ones appearing in Eqs. (A1a) and (A1b). The remaining kernels are given by the expressions

$$K_3(x,y) = \frac{3}{16\pi^2}\int_0^\pi \frac{G(x,y,\cos\theta)}{x+y-2\sqrt{xy}\cos\theta}\sin^4\theta\, d\theta \ , \tag{A3a}$$

$$K_4(x,y) = \frac{3}{16\pi^2}\int_0^\pi \frac{G(x,y,\cos\theta)}{x+y-2\sqrt{xy}\cos\theta}\cos\theta\sin^4\theta\, d\theta \ , \tag{A3b}$$

$$K_5(x,y) = \frac{3}{16\pi^2}\int_0^\pi \left[-\left[\frac{y}{x}\right]^{1/2}(x+y)\cos\theta + 2y - \frac{2}{3}y\sin^2\theta \right] \frac{G(x,y,\cos\theta)}{x+y-2\sqrt{xy}\cos\theta}\sin^2\theta\, d\theta \ . \tag{A3c}$$

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