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Fermion mass in three dimensions and the renormalization group

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The spontaneous generation of a parity-even mass for fermions in three dimensions is analyzed by using effective Lagrangians and the renormalization group. To leading order in ϵ, an expansion about $4-\epsilon$ dimensions predicts the spontaneous generation of mass for two or more flavors of four-component fermions.

I. INTRODUCTION

The spontaneous generation of fermion mass is of fundamental significance for QCD and theories of technicolor. One of the oldest concepts in the generation of fermion mass is that of a "critical coupling." In their model of chiral-symmetry breaking via four-fermion couplings, Nambu and Jona-Lasino [1] showed that chiral-symmetry breaking only occurs when the four-fermion coupling exceeds a certain threshold. A critical coupling is also seen in theories of fermions coupled to gauge fields, when the Schwinger-Dyson equations are solved approximately by the summation of rainbow-type diagrams [2].

In QCD the concept of a critical coupling for chiral-symmetry breaking is a bit murky, for by dimensional transmuation the coupling constant turns into a mass scale anyway. I suggest that while a critical coupling is seen in the models of Refs. [1] and [2], it is not universal, and may not be characteristic of theories of fermions coupled to gauge fields. Indeed, if one hopes to produce the bare quark and lepton masses by means of some hidden gauge symmetry, then it cannot be so: for the same mechanism to produce scales that range in magnitude from the up to the top-quark masses, it must apply over a wide range of coupling.

The problem can be studied in three [3] as well as in four dimensions. Consider a gauge theory in three dimensions coupled to massless fermions. The only dimensionful parameter is the fine-structure constant α, so any mass which is generated dynamically must be proportional to α. In these theories the effective, dimensionless coupling constant is $1/N$: at small N the theory is in strong coupling, but at large N the theory is in a regime of weak coupling.

Because there is no γ_5 matrix in $2+1$ dimensions, there is no true chiral symmetry, and the global symmetries are entirely those of flavor. For N flavors of massless, complex, four-component fermions, the global flavor symmetry is $U(2N)$ [4]. One difference from four dimensions is the role of parity. In three dimensions both gauge fields and fermions have mass terms which are odd under parity [3]; for fermions the parity-odd mass is symmetric under $U(2N)$. Fermions can also have a parity-even mass, but it reduces the $U(2N)$ symmetry to $U(N)\times U(N)$. Thus in three dimensions I can consider a theory which is always parity symmetric, with no parity-odd masses for either the fermion or the gauge field. The fermion is then given a bare, parity-even mass. As this bare mass is turned to zero, the dynamics then determines whether or not mass is generated spontaneously, with the concomitant breaking of the $U(2N)$-flavor symmetry.

In Ref. [5] I argued that a (parity-even) mass is spontaneously generated at large N in Abelian gauge theories, with the dynamical mass exponentially small in $1/N$. Further studies appeared to confirm this [6–8]. The pattern of flavor-symmetry breaking, from $U(2N)$ to $U(N)\times U(N)$, accords with general arguments by Vafa and Witten [9] and by Polychronakos [10]. These general arguments, however, only state that if flavor-symmetry breaking occurs, then it must be in this manner, which is the maximal breaking of $U(2N)$ possible; they do not guarantee that the $U(2N)$ symmetry must break.

That the dynamics does not always favor the spontaneous generation of mass was argued by Appelquist, Nash, and Wijewardhana [11,12]. They approximated the full Schwinger-Dyson equations by summing a class of rainbow-type diagrams. Under this approximation they find that flavor-symmetry breaking only occurs when $N \leq 3$. In essence, Appelquist, Nash, and Wijewardhana find that in terms of the coupling $1/N$, there is a critical coupling for spontaneous mass generation.

The Schwinger-Dyson equations have also been studied by Pennington *et al.* [13]. In particular, Pennington and Walsh attempt to solve the Schwinger-Dyson equations by going beyond the summation of rainbow diagrams, incorporating nontrival wave-function and vertex renormalization. This alters the form of the Schwinger-Dyson equations into a form in which mass is spontaneously generated for all $N \geq 1$.

The flavor symmetries of three-dimensional fermions can be transcribed onto the lattice [14]. Results from strong-coupling expansions [15], Monte Carlo simulations [16–18], and variational calculations [19] all seem to indicate that for a single flavor, $N=1$, the flavor symmetry is spontaneously broken. Monte Carlo simulations with larger values of N have been carried out by Dagotto, Kocic, and Kogut [17]: in agreement with Appelquist, Nash, and Wijewardhana, they only find flavor-symmetry breaking for $N \leq 3$.

In this paper I study the spontaneous generation of fermion mass in three dimensions through the renormalization group. This method is complementary to studies of the Schwinger-Dyson equations [5–8,11–13] and on the lattice [14–19]. In Sec. II, I construct an effective Lagrangian for the U($2N$)-flavor symmetry. While my interest is principally in a parity-even mass, the analysis encompasses a parity-odd mass term as well. For a theory which is parity even, I show that the dynamics near a second-order phase transition is controlled by an effective theory constructed from an adjoint field in SU($2N$).

In Sec. III, I use the techniques of the renormalization group. Typically, one expects that as the bare mass is tuned to zero there is a point of second-order phase transition. This assumes, though, that there exists an infrared stable fixed point in the appropriate universality class. If there is no such infrared stable fixed point, then even as the bare mass is tuned to zero, fluctuations generate a nonzero mass [20]. The renormalization-group flows for an adjoint SU($2N$) field are analyzed by an expansion about $4-\epsilon$ dimensions [21]. To leading order in ϵ, calculation of the β functions shows that there is no infrared stable fixed point for $N > \sqrt{5}/2$. Assuming that this result holds down to $\epsilon=1$ implies that a mass gap is dynamically generated for two or more flavors. In other words, mass is generated even at large N, where the effective coupling constant $1/N$ is small.

The renormalization-group analysis makes no prediction as to how large this mass is. In Sec. IV, I discuss [5] why in QED at large N it must be *very* small, exponential in $1/N$.

II. EFFECTIVE LAGRANGIAN FOR FLAVOR SYMMETRIES

The simplest way of constructing three-dimensional fermions is by the obvious reduction from four dimensions. The fermion Lagrangian is

$$\mathcal{L}_{\text{ferm}} = \sum_{a=1}^{N} (\bar{\Psi}_a \gamma^\mu D_\mu \Psi_a + m \bar{\Psi}_a \Psi_a) \ . \tag{1}$$

I take N flavors of four-component spinors Ψ_a each with mass m. The fermions are coupled to the gauge field A_μ through the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$. I assume that the fermions lie in a complex representation of the local gauge group, such as for a U(1) gauge group. (The case of real representations is discussed at the end of Sec. III.) Implicitly I assume Euclidean space-time. In three dimensions only two component spinors are required by the Euclidean group, so (1) can be simplified. In two-component form the γ_μ's and the Ψ_a's can be chosen as

$$\gamma^\mu = \begin{bmatrix} \sigma^\mu & 0 \\ 0 & -\sigma^\mu \end{bmatrix} , \quad \Psi_a = \begin{bmatrix} \psi_a \\ \psi_{N+a} \end{bmatrix} , \tag{2}$$

where the σ^μ's are the Pauli matrices, $\mu = 1,2,3$. Then (1) becomes

$$\mathcal{L}_{\text{ferm}} = \sum_{a=1}^{N} [\bar{\psi}_a (\not{D} + m)\psi_a - \psi_{N+a}(\not{D} - m)\psi_{N+a}] \ , \tag{3}$$

$\not{D} = D_\mu \sigma^\mu$. Recognizing that the overall sign of the fermion Lagrangian is irrelevant, flipping the overall sign of the Lagrangian for the ψ_{N+a} fields gives

$$\mathcal{L}_{\text{ferm}} = \bar{\psi}(\not{D}\mathbf{1}_{2N} + m\mathbf{Q}_{2N})\psi \ , \quad \mathbf{Q}_{2N} = \begin{bmatrix} +\mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{bmatrix} . \tag{4}$$

In (4) ψ is promoted to a $2N$ component vector in flavor space; the mass m enters through the flavor matrix $\mathbf{Q}_{2N}$. In this form it is evident that for massless fermions, $m=0$, the global flavor symmetry is U($2N$); with a mass m, the symmetry reduces to U(N)×U(N). It is also possible to add another mass to (4), $m_{\text{odd}}\bar{\psi}\mathbf{1}_{2N}\psi$; as this mass term is proportional to the unit matrix in flavor space, it does not spoil the U($2N$) symmetry.

Under parity inversion [3] the kinetic term for a fermion is invariant, but the mass for a single flavor changes sign, $\bar{\psi}_a\psi_a \rightarrow -\bar{\psi}_a\psi_a$. Thus the U($2N$)-symmetric mass term $m_{\text{odd}}\bar{\psi}\mathbf{1}_{2N}\psi$ is odd under parity. The U(N)×U(N)-symmetric mass term $m\bar{\psi}\mathbf{Q}_{2N}\psi$ can be defined to be parity even. Since it is a discrete symmetry, combine parity inversion with the interchange of $\psi_a \leftrightarrow \psi_{N+a}$. This changes $\mathbf{Q}_{2N} \rightarrow -\mathbf{Q}_{2N}$, and compensates for the change of sign from parity inversion. In other words, for an even number of flavors a parity-even mass is constructed by pairing up fermions with equal masses of opposite sign.

To construct an effective Lagrangian for flavor symmetry I form the scalar field Φ from fermion bilinears:

$$\Phi_{ij} = \bar{\psi}_i \psi_j \ , \tag{5}$$

$i,j = 1, \ldots, 2N$. By its definition the Φ field is Hermitian, $\Phi^\dagger = \Phi$. Φ can be decomposed into a traceless part and its trace:

$$\phi = \Phi - \frac{\mathbf{1}_{2N}}{2N}\text{tr}(\Phi) \ , \quad \chi = \text{tr}(\Phi) \ . \tag{6}$$

Consider a global U($2N$) rotation of the fermion field,

$$\psi \rightarrow \psi \mathbf{U} e^{i\theta} \ , \tag{7}$$

where U is an element of SU($2N$), and $e^{i\theta}$ the phase for the U(1) of fermion number. Under this transformation,

$$\phi \rightarrow \mathbf{U}^\dagger \phi \mathbf{U} \ , \quad \chi \rightarrow \chi \ . \tag{8}$$

The overall U(1) of fermion number does not affect either field: only SU($2N$) matters. Under SU($2N$), ϕ transforms as an adjoint field, while χ is a scalar. Both the ϕ and χ fields are odd under parity inversion.

It is worth contrasting the transformation properties of these fields with their counterpart in four dimensions. Because of the γ_5 matrix in four dimensions, the Euclidean group distinguishes between left- and right-handed fermions. Classically, the global symmetry group for N flavors of fermions is $U_L(N)\times U_R(N)$. The field which replaces Φ is $\Phi_4=\bar{\psi}_L\psi_R$; because left- and right-handed fields are distinct, the Φ_4 field is not Hermitian. Under a $U_L(N)\times U_R(N)$ rotation, the Φ_4 fields transform as $\Phi_4\rightarrow e^{i(\theta_R-\theta_L)}U_L^\dagger\Phi_4 U_R$. The U(1) phase is the rotation for axial fermion number, which is broken quantum mechanically by the axial anomaly; as in three dimensions, the rotation for total fermion number drops out from the transformation of Φ_4.

For any number of dimensions the effective Lagrangian is constructed by writing down all terms consonant with the global symmetries. In four dimensions there is only one invariant field Φ_4; there is no separate χ field, since $\mathrm{tr}(\Phi_4)$ is not an invariant. The number of invariant terms in the effective Lagrangian is very limited by the $U_L(N)\times U_R(N)$ symmetry. There is one mass term, $\mathrm{tr}(\Phi_4^\dagger\Phi_4)$, no cubic couplings, and only two quartic couplings, $[\mathrm{tr}(\Phi_4^\dagger\Phi_4)]^2$ and $\mathrm{tr}\,(\Phi_4^\dagger\Phi_4)^2$.

In three dimensions, up to quartic order the most general effective Lagrangian invariant under SU(2N) is

$$\begin{aligned}\mathcal{L}\mathrm{eff}=&\tfrac{1}{2}\mathrm{tr}(\partial_\mu\phi)^2+\tfrac{1}{2}(\partial_\mu\chi)^2+\tfrac{1}{2}m_\phi^2\mathrm{tr}(\phi^2)+\tfrac{1}{2}m_\chi^2\chi^2\\&+H_1\mathrm{tr}(\phi^3)+H_2\chi\,\mathrm{tr}(\phi^2)+H_3\chi^3+G_1[\mathrm{tr}(\phi^2)]^2\\&+G_2\mathrm{tr}(\phi^4)+G_3\chi\,\mathrm{tr}(\phi^3)+G_4\chi^2\mathrm{tr}(\phi^2)+G_5\chi^4\ .\end{aligned}\tag{9}$$

The effective Lagrangian has many more terms in three dimensions than in four. First of all, because $\mathrm{tr}(\Phi)=\chi$ is invariant under SU(2N), there are two independent fields, ϕ and χ. Second, because the ϕ is a Hermitian field, more couplings of ϕ with itself are allowed, as well as its couplings with χ. These include cubic couplings, and new quartic couplings.

For small N there are even more couplings than those of (9). A determinantal coupling

$$\mathcal{L}_{\mathrm{det}}=K\,\det(\phi)\tag{10}$$

is also invariant under SU(2N) [but not U(2N)]. For $N=2$ this is not an independent term, as it reduces to the vectorlike coupling of (15) below. For $N\geq 3$ it is an independent invariant of order N, and so must be included for $N=3$ and $N=4$; additionally, when $N=3$ the coupling $\chi\det(\phi)$ is allowed.

It is elementary to predict the phase transitions of (9). There are two fundamental fields, and so there are transitions in either the ϕ or the χ field. At a generic point of phase transition, both $\langle\phi\rangle$ and $\langle\chi\rangle$ change. Without further ado, since the effective Lagrangian contains terms of cubic order the phase transition is of first order. The only exception is the multicritical point at the border of first-order transitions, where all cubic couplings vanish. This case is treated in the next section.

Let me now assume that the effective Lagrangian describes an underlying theory which is parity symmetric, such as (1). I assume that parity is not spontaneously broken, so the vacuum is parity symmetric; this assumption agrees with dynamical studies [7] and general arguments [9,10]. Since the χ field is parity odd, $\langle\chi\rangle=0$. The ϕ field can acquire a vacuum expectation value, but only if $\langle\phi\rangle\sim Q_{2N}$ (up to global rotations). Because each and every term in the fundamental Lagrangian is symmetric under parity, we can require the same for the effective Lagrangian. As ϕ and χ are odd under parity, this implies that all cubic terms in the effective Lagrangian vanish: $H_1=H_2=H_3=0$. For $N=3$ and $N=4$, the determinantal terms also vanish, $K=0$. There is no restriction on the quartic terms.

We are still left with a theory with two fields and five quartic couplings [six for $N=3$, including the coupling between χ and $\det(\phi)$]. Since χ cannot acquire a vacuum expectation value, the χ mass term must be positive, $m_\chi^2>0$. For the ϕ field the critical point is determined by the limit in which $m_\phi^2\rightarrow 0$. Near such a critical point χ is simply a massive field which decouples over long distances. Thus about the critical point we can drop the χ field and consider simply the interactions of a massless ϕ field with itself; the two quartic couplings are those proportional to G_1 and G_2 in (9).

III. THE ϵ EXPANSION FOR AN ADJOINT SU(2N) FIELD

I am led to analyze the critical behavior of the theory with the bare Lagrangian

$$\mathcal{L}_{\mathrm{eff}}=\tfrac{1}{2}\mathrm{tr}(\partial_\mu\phi)^2+\frac{8\pi^2\mu^\epsilon}{4!}\{g_1[\mathrm{tr}(\phi^2)]^2+g_2\mathrm{tr}(\phi^4)\}\ .\tag{11}$$

I compute in $4-\epsilon$ dimensions at small ϵ. Anticipating the results at one-loop order, I redefine the overall normalization of the coupling constants, exchanging G_1 and G_2 for g_1 and g_2; μ is a renormalization-group mass. For simplicity, I change from the variable N to

$$n=2N\ .\tag{12}$$

In (11) ϕ is traceless, Hermitian matrix:

$$\mathrm{tr}\phi=0\ ,\quad \phi=\phi^\dagger\ .\tag{13}$$

Consequently, I can trade the n-by-n matrix ϕ for its independent components ϕ^a:

$$\phi_{ij}=\sqrt{2}\phi^a(t^a)_{ij}\ .\tag{14}$$

The indices $i,j,\ldots$ run from $1,\ldots,n$, while the indices $a,b,\ldots$ run from $1,\ldots,n^2-1$; the t^a's are matrices in the fundamental representation of SU(n), normalized so that $\mathrm{tr}(t^at^b)=\delta^{ab}/2$. Then

$$\mathcal{L}_{\text{eff}}=\tfrac{1}{2}(\partial_\mu\phi^a)^2+\frac{8\pi^2\mu^\epsilon}{4!}\left[\left[g_1+\frac{g_2}{n}\right](\phi^a\phi^a)^2+2g_2 d^{abe}d^{cde}\phi^a\phi^b\phi^c\phi^d\right]. \quad (15)$$

The d^{abc}'s are the structure constants for the group SU(n).

From (15) there are two simplifying limits which provide a check on the β function. First, for arbitrary n when $g_2=0$ the theory reduces to an O(n^2-1) theory in the vector representation, with coupling g_1. There is a further simplification when $n=2$. For the group SU(2) the d^{abc}'s vanish, so that even for $g_2\neq 0$, (15) reduces to a vector O(3) model with coupling $g_1+g_2/2$.

The clever way of computing the β function for SU(n) would be to use the representation of (15). This requires knowing how the product of four d^{abc}'s reduces to a sum over products of two d^{abc}'s. As this reduction is unknown to me, I take a more prosaic course and use the representation of (11). While inelegant, it is easily automated.

Since ϕ is traceless I take $\phi=\Phi-1_n\text{tr}(\Phi)/n$, and rewrite (15) as

$$\mathcal{L}_{\text{eff}}=\tfrac{1}{2}\{\text{tr}(\partial_\mu\Phi)^2-(1/n)[\text{tr}(\partial_\mu\Phi)]^2\}+\frac{8\pi^2\mu^\epsilon}{4!}\{g_1[\text{tr}(\Phi^2)]^2+g_2\text{tr}(\Phi^4)+g_3\text{tr}(\Phi)\text{tr}(\Phi^3)+g_4[\text{tr}(\Phi)]^2\text{tr}(\Phi^2)+g_5[\text{tr}(\Phi)]^4\}. \quad (16)$$

The couplings g_3, g_4, and g_5 are not independent, but are related to g_1 and g_2 as

$$g_3=-\frac{4}{n}g_2\,,\quad g_4=-\frac{2}{n}g_1+\frac{6}{n^2}g_2\,,\quad g_5=\frac{1}{n^2}g_1-\frac{3}{n^3}g_2\,. \quad (17)$$

As a Hermitian field the ϕ propagator is represented graphically by two directed lines, with the arrows running in opposite directions. At tree level the propagator for a field with momentum p is

$$\langle\Phi^{a_1a_2}(p)\Phi^{a_3a_4}(-p)\rangle=\left[\delta^{a_1a_4}\delta^{a_2a_3}-\frac{1}{n}\delta^{a_1a_2}\delta^{a_3a_4}\right]\frac{1}{p^2}\,. \quad (18)$$

The four-point function is given by

$$\langle\Phi^{a_1a_2}\Phi^{a_3a_4}\Phi^{a_5a_6}\Phi^{a_7a_8}\rangle=\mathcal{V}(a_1,a_2;a_3,a_4;a_5,a_6;a_7,a_8)=\sum_{s=1}^{5}g_s\mathcal{V}_s(a_1,a_2;a_3,a_4;a_5,a_6;a_7,a_8)\,, \quad (19)$$

where

$$\mathcal{V}_1(a_1,a_2;a_3,a_4;a_5,a_6;a_7,a_8)=\tfrac{1}{3}(\delta^{a_1a_4}\delta^{a_2a_3}\delta^{a_5a_8}\delta^{a_6a_7}+2\text{ permutations})\,,$$
$$\mathcal{V}_2(a_1;a_2;a_3,a_4;a_5,a_6;a_7,a_8)=\tfrac{1}{6}(\delta^{a_1a_8}\delta^{a_2a_3}\delta^{a_4a_5}\delta^{a_6a_7}+5\text{ permutations})\,,$$
$$\mathcal{V}_3(a_1,a_2;a_3,a_4;a_5,a_6;a_7,a_8)=\tfrac{1}{8}(\delta^{a_1a_6}\delta^{a_2a_3}\delta^{a_4a_5}\delta^{a_7a_8}+7\text{ permutations})\,, \quad (20)$$
$$\mathcal{V}_4(a_1,a_2;a_3,a_4;a_5,a_6;a_7,a_8)=\tfrac{1}{6}(\delta^{a_1a_4}\delta^{a_2a_3}\delta^{a_5a_6}\delta^{a_7a_8}+5\text{ permutations})\,,$$
$$\mathcal{V}_5(a_1,a_2;a_3,a_4;a_5,a_6;a_7,a_8)=\delta^{a_1a_2}\delta^{a_3a_4}\delta^{a_5a_6}\delta^{a_7a_8}\,.$$

For $\mathcal{V}_1-\mathcal{V}_4$, each term is a sum over permutations of the different ways in which the isospin indices can be tied together. The different permutations are given explicitly by Berg and Weisz [22]. While they studied a rather different problem (computing the exact S matrix for elastic two-body scattering in 1+1 dimensions), the isospin decomposition of the four-point function is the same. In terms of the notation of Berg and Weisz, the three terms which contribute to $\mathcal{V}_1$ are B, G_1, and G_2; the six terms for $\mathcal{V}_2$ are F_1 through F_4, H_1 and H_2; the eight terms for $\mathcal{V}_3$ are E_1 through E_8; the six terms for $\mathcal{V}_4$, are C_1, C_2, and D_1 through D_4; and the one term for $\mathcal{V}_5$ is A.

To compute the β function at one-loop order I follow the classic analysis of Brezin, Le Guillou, and Zinn-Justin [21]. The momentum integrals are the same as for a vector theory, with the only complication arising for the isospin indices. At one-loop order just the coupling-constant renormalization affects the β function. The wave-function renormalization was also computed, and used to determine the anomalous dimension of the ϕ field. I do not present this result, since it does not affect any of the conclusions.

At one-loop order the coupling-constant renormalization is given by typing two four-point vertices together. The combination which enters is

$$\sum_{b_1\cdots b_4=1}^{n}\mathcal{V}(a_1,a_2;a_3,a_4;b_1,b_2;b_3,b_4)\mathcal{V}(a_5,a_6;a_7,a_8;b_2,b_1;b_4,b_3)+2\text{ permutations}=\sum_{s=1}^{5}\delta g_s\mathcal{V}_s(a_1,a_2;a_3,a_4;a_5,a_6;a_7,a_8)\,. \quad (21)$$

The propagator of (18) contains two terms; the second $\delta^{a_1a_2}\delta^{a_3a_4}/n$, does not contribute to (21) because each of the vertices $\mathcal{V}$ is traceless in any pair of indices, and so the trace term in the propagator can be dropped. The three permutations on the left-hand side of (21) correspond to the usual channels for tying two four-point functions together. On the right-hand side, the δg_s are five functions, each quadratic in g_1 and g_2. There are only two independent functions, δg_1 and δg_2; δg_3, δg_4, and δg_5 are related to these two functions through expressions identical in form to (17).

To evaluate (21) I used a program for index contraction written by Nason [23] for the symbolic manipulation package MACSYMA. Although there are many equivalent channels, $\mathcal{V}_1$ through $\mathcal{V}_5$, because there are only two independent terms δg_1 and δg_2 the redundancy can be used as a check. The result is

$$\delta g_1=\frac{n^2+7}{3}g_1^2+2\frac{(2n^2-3)}{3n}g_1g_2+\frac{n^2+3}{n^2}g_2^2\ ,$$
$$\delta g_2=4g_1g_2+2\frac{(n^2-9)}{3n}g_2^2\ . \tag{22}$$

In $4-\epsilon$ dimensions the momentum integral of the one-loop diagram gives a numerical factor times $1/\epsilon$; this numerical factor $1/(8\pi^2\mu^\epsilon)$ cancels against that in the definition of the coupling constant in (15). The β functions are then given by

$$\beta_i(g_1,g_2)=\frac{\partial g_i}{\partial \ln(\mu)}=\epsilon g_i+\tfrac{1}{2}\delta g_i\ ; \tag{23}$$

the factor of $\frac{1}{2}$ multiplying δg_i is the symmetry factor from the one-loop diagram. From (22) the β functions are

$$\beta_1(g_1,g_2)=-\epsilon g_1+\frac{n^2+7}{6}g_1^2+\frac{2n^2-3}{3n}g_1g_2+\frac{n^2+3}{2n^2}g_2^2\ ,$$
$$\beta_2(g_1,g_2)=-\epsilon g_2+2g_1g_2+\frac{n^2-9}{3n}g_2^2\ . \tag{24}$$

As discussed following (15), there are two limits in which the β functions of (24) can be checked. When $g_2=0$, β_1 is equal to the β function of an $O(n^2-1)$ vector theory with coupling g_1 [21]. Second when $n=2$, $\beta_1+\beta_2/2$ equals the β function for an O(3) vector theory with coupling $g_1+g_2/2$.

The fixed points of the theory are the couplings at which both β functions vanish:

$$\beta_1({}^*g_1,{}^*g_2)=\beta_2({}^*g_1,{}^*g_2)=0\ . \tag{25}$$

In general, there are four fixed points to (24): there is the origin ${}^*g^{\rm UV}$,

$${}^*g_1^{\rm UV}={}^*g_2^{\rm UV}=0\ , \tag{26}$$

the vectorlike fixed point, ${}^*g^{\rm vec}$, which is the fixed point for an $O(n^2-1)$ vector theory,

$${}^*g_1^{\rm vec}=\frac{6\epsilon}{(n^2+7)}\ ,\quad {}^*g_2^{\rm vec}=0\ , \tag{27}$$

and two other fixed points ${}^*g^{\pm}$, at which both couplings are nonzero:

$${}^*g_1^{\pm}=\frac{3(n^4-21n^2-108)\mp 3(n^2-9)\sqrt{-2n^4+18n^2+81}}{n^6-35n^4+315n^2+567}\epsilon,$$
$${}^*g_2^{\pm}=\frac{3n(n^4-20n^2+9)\mp 18n\sqrt{-2n^4+18n^2+81}}{n^6-35n^4+315n^2+567}\epsilon\ . \tag{28}$$

It is also necessary to evaluate the slope of the β functions about the fixed points. This is given by the stability matrix ω [21]:

$$\omega_{ij}=\frac{\partial\beta_i}{\partial g_j}\ . \tag{29}$$

The eigenvalues of ω, which is a two-by-two matrix, are evaluated at a given fixed point. If the real part of both eigenvalues of ω are negative, the fixed point is ultraviolet stable; if the real parts are positive, it is infrared stable.

When $\epsilon>0$ the fixed point at the origin ${}^*g^{\rm UV}$ is ultraviolet stable. To illustrate what happens in the infrared limit, consider the stability matrix for the vectorlike fixed point. When $g_1={}^*g_1^{\rm vec}$ and $g_2=0$,

$$\omega_{11}=-\epsilon+\frac{n^2+7}{3}{}^*g_1^{\rm vec}=\epsilon\ ,$$
$$\omega_{12}=\frac{2n^2-3}{3n}{}^*g_1^{\rm vec}=\frac{2(2n^2-3)}{n(n^2+7)}\epsilon\ ,$$
$$\omega_{21}=0\ ,$$
$$\omega_{22}=-\epsilon+2{}^*g_1^{\rm vec}=-\frac{(n^2-5)}{(n^2+7)}\epsilon\ . \tag{30}$$

Since one of the off-diagonal elements of ω vanishes, its eigenvalues are equal to the diagonal elements. The first element ω_{11} is always positive, but the second element ω_{22} is only positive for $n<\sqrt{5}$.

Thus when $n<\sqrt{5}$, to leading order in ϵ the critical behavior of an adjoint SU(n) field is in the same universality class as an $O(n^2-1)$ vector field.

For $n>\sqrt{5}$, the vectorlike fixed point is not infrared stable, and if there is any infrared stable fixed point, it must be either ${}^*g^+$ or ${}^*g^-$. From (28), because of the factor of $\sqrt{-2n^4+18n^2+81}$, if n is too large both ${}^*g^+$ and ${}^*g^-$ are complex, and so unacceptable as fixed points. This happens when $n>n_c=3[(1+\sqrt{3})/2]^{1/2}$. In the range $n_c\geq n\geq\sqrt{5}$, an evaluation of the stability matrix shows that the infrared stable fixed point is ${}^*g^+$. Nevertheless, in this range of n the theory still does not have a well-defined critical point. This is most easily seen by considering the theory about $n=\sqrt{5}$. For $n=\sqrt{5}+\kappa$, to leading order in κ for small κ,

$${}^*g_1^+=\left[\frac{1}{2}-\frac{\sqrt{5}}{33}\kappa+\cdots\right]\epsilon\ ,\quad {}^*g_2^+=\left[-\frac{5}{32}\kappa+\cdots\right]\epsilon\ . \tag{31}$$

At $n=\sqrt{5}$, ${}^*g^+$ coincides with the vector fixed point ${}^*g^{\rm vec}$. They differ as κ increases, but when κ is nonzero and positive, ${}^*g_2^+$ is negative. It can be shown that ${}^*g_2^+$ is negative for all between $\sqrt{5}$ and n_c. By (15), however, the potential is bounded from below only if g_2 is positive.

Thus for $n_c \geq n \geq \sqrt{5}$, the theory does not have an infrared stable fixed point in the domain of stability.

The above analysis assumes that the fermions lie in a complex representation of the local gauge group. If the fermions lie in a real representation, then the global flavor symmetry is O(2*N*) for $m=0$, and $O(N)\times O(N)$ when $m\neq 0$ [9]. The effective theory is constructed from a field Φ which is a real, symmetric *N*-by-*N* matrix; the effective Lagrangian for the parity-even theory remains that of (15), and involves just the part of Φ with zero trace. The β functions for O(2*N*) are surely different than U(2*N*), but I expect that the results are similar: there is no infrared stable fixed point for *N* larger than some small value.

IV. CONCLUSIONS

From the preceding section, to leading order in ϵ the critical behavior of an adjoint SU(2*N*) field is well defined only when

$$N < \frac{\sqrt{5}}{2} \tag{32}$$

(remember $n=2N$). For such *N*, the critical indices are those of a vector field with symmetry $O(4N^2-1)$. For values of $N>\sqrt{5}/2$, there is no infrared stable fixed point at which the potential is bounded from below. As for scalar electrodynamics in four dimensions, in this instance there is a fluctuation-induced first-order phase transition [20]. In other words, even if one tries to tune the mass to zero, one is generated dynamically.

It is worth emphasizing that for $N<\sqrt{5}/2$, the renormalization group does *not* predict that the phase transition is of second order; only what the universality class is if it is of second order. In contrast, for $N>\sqrt{5}/2$, the renormalization group predicts that the transition is inescapably of first order.

I compare these results to those of dynamical studies of QED [5–8, 11–19]. First, all studies appear to agree that there is symmetry breaking for a single flavor $N=1$. This in itself is surprising: it must be possible to find some model with $N=1$ that exhibits a second-order; phase transition as $m\rightarrow 0$. Presumably continuum QED is not such a model. But in lattice QED with a nonstandard action it is possible to look for the critical point at the end of a line of first-order transitions. Since an adjoint SU(2) field is equivalent to an O(3) vector, (15), this critical point must be in the universality class of an O(3) vector.

For more than a single flavor the dynamical studies differ. From the Schwinger-Dyson equations, Appelquist, Nash, and Wijewardhana [11] find spontaneous symmetry breaking only for $N\leq 3$; the numerical simulations of Dagotto, Kocic, and Kogut [17] support this. The analysis of Pennington *et al.* [13] supports symmetry breaking for all $N\geq 2$. The renormalization-group analysis here supports symmetry breaking whenever $N\geq 2$.

To be fair, it could be that extrapolating the renormalization-group analysis from small ϵ to $\epsilon=1$ fails. While certainly possible, I note that a similar analysis of the chiral phase transition [24], for the four-dimensional theory at finite temperature, does seem to correctly predict a first-order, fluctuation-induced phase transition when the number of flavors exceeds the value given by the ϵ expansion (there it is $N>\sqrt{2}$).

I close by reviewing the arguments of Ref. [5]. Consider a gauge theory in three dimensions, coupled to *N* flavors of massless fermions. At large *N* the fine-structure constant $\alpha=e^2N$ is of order 1. Introduce a (parity-even) mass *m*, and assume that $m\ll\alpha$. At one-loop order, the fermion self-energy is

$$\Sigma=-m\frac{c}{N}\ln\left[\frac{m}{\alpha}\right] . \tag{33}$$

The constant $c=8/\pi^2$, and is independent of the condition for gauge fixing. To look for a dynamically generated mass in the limit of zero bare mass, equate the self-energy in (33) to *m*. This has two solutions: $m=0$ and

$$m=\alpha\exp\left[-\frac{N}{c}\right] . \tag{34}$$

Because this solution is exponentially small at large *N*, it satisfies that assumption of $m\ll\alpha$ that went into (33).

Contrary to my conclusions in Ref. [5], it is premature to conclude from (33) that flavor-symmetry breaking occurs. Over mass scales as in (34), the effects of nontrivial anomalous dimensions must also be included. For example, up to terms of order 1/N the fermion wave-function renormalization is $Z(p)=1+(d/N)\ln(p/\alpha)$, for some constant *d*, with *p* the momenta. For $p\sim\alpha$, $Z\sim 1$ up to corrections which are strictly of order $1/N$, but for *p* on the order of (34), the corrections to *Z* are of order one. The correct manner in which to treat momenta on the order of (34) would be to use the renormalization group to resum $Z(p)\sim(p/\alpha)^{d/N}$. Thus even at large *N*, to correctly compute (33) it is necessary to include not just mass renormalization, but wave-function and vertex renormalization as well. Since these calculations are considerably more involved than those done to date, the question of flavor-symmetry breaking in QED at large *N* remains open.

Nevertheless, it is safe to conclude that *if* symmetry breaking occurs, then it *must* be over mass scales which are exponentially small in $1/N$: whatever the value of the constant *c* in (34), it is of order one at large *N*, $c\geq 0$ ($c=0$ is no symmetry breaking). This is simply because over mass scales which are any larger than (34), (33) is of order $1/N$, and so cannot produce a dynamical $m\neq 0$ by setting $m=\Sigma$.

To return to the discussion of the Introduction, it is perhaps misleading to claim that flavor-symmetry breaking in QED at large *N* contradicts the concept of a critical coupling. The naive dimensionless coupling constant is $1/N$, but for momenta *p*, a better measure is the "running" coupling, $(1/N)\ln(p/\alpha)$. While both the naive and the running coupling are of order $1/N$ for $p\sim\alpha$, at mass scales as in (34), the running coupling is of order one. From another view, at infinite *N* the theory is surely just free, massless fields. But (34) does not contradict this, for this scale is nonperturbative in $1/N$, and vanishes smoothly at $N \to \infty$.

In closing, I suggest that we cannot presume to say that we understand chiral-symmetry breaking in a theory as complicated as QCD in four dimensions, if we do not even understand flavor-symmetry breaking in a theory as simple as QED in three dimensions.

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