PHYSICAL REVIEW D VOLUME 44, NUMBER 6 15 SEPTEMBER 1991

Phase-space structure of the Dirac vacuum

Iwo Bialynicki-Birula,* Paweł Górnicki,* and Johann Rafelski
Department of Physics, University of Arizona, Tucson, Arizona 85721
(Received 1 April 1991)

We study the phase-space correlation function for the Dirac vacuum in the presence of simple field configurations. Our formalism rests on the Wigner transform of the Dirac-Heisenberg correlation function of the Dirac field coupled to the electromagnetic field. A self-consistent set of equations obeyed by the 16 components of the phase-space correlation function and by the electric and magnetic field is derived. Our approach is manifestly gauge invariant. A closed system of integro-differential equations is obtained neglecting the quantum fluctuations of the electromagnetic field as should be appropriate for strong fields. These equations are an extension of the Vlasov equations used in the description of plasma. Our first applications address the production of particles in strong external fields. We set a framework for the inclusion of the back reaction of produced particles and for the description of the local changes of the vacuum state.

I. INTRODUCTION

The structure of the QCD vacuum has been considered an important element for the understanding of strong-interaction physics [1]. The electroweak vacuum Higgs field is believed to be the origin of all elementary-particle masses [2]. The QED vacuum in strong fields has been explored both theoretically and experimentally in heavy-ion collisions and there is a significant discrepancy between the observed particle spectra and the theoretical predictions [3]. All this calls for a renewed effort to develop a systematic framework to describe the structure of the vacuum in a manner which would be to a large degree independent of perturbative quantum field theory. Despite its historical name, the vacuum state is not empty; it is populated with myriads of virtual particles which endow it with a rich structure. The standard, perturbative formulation of quantum field theory deemphasizes this fact by treating the vacuum as just the simplest possible reference state: the one with the lowest possible energy and with all the relevant symmetry properties. All the remaining state vectors that are introduced to model real physical situations are defined *relative* to this unique reference state by building excitations on top of the ground state. This pragmatic approach enables one to calculate effectively the elements of the S matrix with the help of appropriate Feynman diagrams without ever paying attention to the problem of the internal structure of the vacuum and to its time evolution.

In our opinion, in order to obtain an effective description of the local vacuum structure and its time evolution, we should use a single time parameter. This will enable us to treat the QED vacuum as a genuine dynamical system with its fully prescribed dynamics, its conservation laws, and a whole new category of physical phenomena that take place in the phase space of virtual particles. There is an obvious drawback that the correlation functions will not be manifestly covariant, although the theory will remain relativistically invariant. On the other hand, we will gain by introducing a more intuitive picture of the vacuum. In particular, we can introduce the phase-space description with its transparent interpretation. As the main tool we shall use an analog of the Wigner function [4] for the QED vacuum: the Fourier transform with respect to the difference of the coordinates of the Dirac-Heisenberg density matrix. This density matrix, or as we would say today the vacuum expectation value of the product of two field operators, was introduced already in 1934 by Dirac [5] and has been used extensively by Heisenberg [6] and by Heisenberg and Euler [7] in their study of the polarization phenomena in the Dirac vacuum. The spatial Fourier transform of the Dirac-Heisenberg density matrix is a (time-dependent) matrix-valued function of position and momentum. This function is capable of describing many features of the Dirac vacuum in quantum electrodynamics. Since we have combined the ideas of Dirac, Heisenberg, and Wigner, we shall call this object the Dirac-Heisenberg-Wigner (DHW) function.

In this paper we develop the formalism for the phase-space description of the Dirac vacuum in an approximation suitable for strong fields and apply it to several simple cases, setting the stage for future more complex applications. Our formalism has been developed to serve a dual purpose. On the one hand, it gives the description of the field-theoretic vacuum for spin-$\frac{1}{2}$ particles in terms of phase-space concepts that are easier to visualize and understand. On the other hand, it offers a chance to perform the calculations that, within the standard formulation, required cumbersome summations over an infinite set of intermediate states. The realization of both these aims is made possible by our choice of the one-time Wigner function. This function has a much simpler, more direct interpretation and it also enables one to study the time evolution in a more direct way as compared to the Feynman propagator formalism.

Wigner-type transforms of the vacuum expectation values have been used in the past to formulate a kinetic theory of quantum scattering theory [8, 9], in quantum electrodynamics [10, 11] and also in quantum chromodynamics [12]. The main difference between the present paper and those earlier works is that, in accordance with the original idea of Wigner, we use the *one-time* distribution function; i.e., we do not perform the Fourier transformation with respect to the time variable. The four-dimensional Fourier transformation does have the advantage that it secures explicit relativistic covariance but, at the same time, it renders the interpretation of the Wigner function obtained in this manner quite obscure. The situation here is quite similar to that found in the study of the Bethe-Salpeter equation which is also fully relativistic, but one is yet to discover a clear, physical interpretation of the Bethe-Salpeter two-body wave function. Our three-dimensional approach enables us to assign phase-space distributions to quantum states defined *at a given time* and give these distributions a transparent interpretation. It turns out that the lack of explicit relativistic invariance does not hamper our ability to perform effective calculations and even such tasks as renormalization can be presented in a new light.

There are some similarities between our work and the program that is being pursued by Cooper, Mottola, and their collaborators [13–15], who have developed a self-consistent solution of the initial-value problem in QED. The differences between these two approaches are, however, substantial. In the Cooper-Mottola program the charged particles are N species of scalar bosons, the self-consistency is achieved by evaluating the time evolution of the field operator of charged particles, and the extension to spatially inhomogeneous fields has not been implemented. However, their formalism based on the two-point Wightman function does allow for such an extension. In our approach the charged particles are spin-$\frac{1}{2}$ fermions, the field operators are completely eliminated and replaced by the DHW function, and the spatial dependence is fully included in our evolution equations.

Our paper is organized as follows. In Sec. II we derive the basic evolution equations for the DHW function coupled in a self-consistent manner to the Maxwell equations. In Sec. III we study the conservation laws for this system and discuss the physical meaning of various quantities introduced in our description. Examples of solutions of the equations for the DHW function are exhibited in Sec. IV. In particular, we study the pair creation phenomena for homogeneous, time-dependent electric fields. In Sec.V we describe the renormalization of the parameters characterizing the vacuum.

II. THE DHW FUNCTION FOR THE DIRAC FIELD AND ITS TIME EVOLUTION

As our starting point we shall use the standard field equations of quantum electrodynamics in the Heisenberg-Pauli (temporal) gauge [16]. This is the most convenient choice for the study of the time evolution since in this gauge the covariant time derivative in the presence of the electromagnetic field does not contain the potential term ($A_0 = 0$). The equations for the Dirac and Maxwell field operators are

$$i\partial_t \Psi = [\boldsymbol{\alpha} \cdot (-i\nabla - e\mathbf{A}) + \beta m]\Psi(\mathbf{r}, t), \tag{1}$$

$$-i\partial_t \Psi^\dagger = \Psi^\dagger(\mathbf{r}, t)[\boldsymbol{\alpha} \cdot (i\overleftarrow{\nabla} - e\mathbf{A}) + \beta m], \tag{2}$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

$$\partial_t\, \epsilon_0 \mathbf{E} = \nabla \times \mu_0^{-1}\mathbf{B} - \frac{e}{2}[\Psi^\dagger, \boldsymbol{\alpha}\Psi] + \mathbf{j}_{\rm ext}, \tag{5}$$

$$\nabla \cdot \epsilon_0 \mathbf{E} = \frac{e}{2}[\Psi^\dagger, \Psi] + \rho_{\rm ext}. \tag{6}$$

We have set here $\hbar = 1 = c$, but later in some formulas we shall reinsert these constants to identify quantum effects. In order to cover the general case we have assumed that the charge density and the current density are made of two terms. The first term represents the contribution from the quantized Dirac field and the second represents possible additional, external sources of the electromagnetic field. In the second pair of Maxwell equations, we have identified the $\mathbf{D}$ and $\mathbf{H}$ fields with the $\epsilon_0\mathbf{E}$ and $\mu_0^{-1}\mathbf{B}$ fields. At a later stage the distinction between these two pairs $(\mathbf{D}, \mathbf{H})$ and $(\mathbf{E}, \mathbf{B})$ will enable us to treat the problems of vacuum polarization in a more penetrating manner.

In principle we can introduce two different objects from which we can build the components of the DHW function:

$$C^\pm_{\alpha\beta} = \langle\Phi|\Psi_\alpha(\mathbf{r}_1, t)\Psi^\dagger_\beta(\mathbf{r}_2, t)|\Phi\rangle \pm \langle\Phi|\Psi^\dagger_\beta(\mathbf{r}_2, t)\Psi_\alpha(\mathbf{r}_1, t)|\Phi\rangle, \tag{7}$$

where we have chosen for definiteness the expectation value in a pure state Φ, but an arbitrary, mixed state could also be considered. In this paper, we shall be primarily interested in the study of the states that are close to the vacuum state, but our dynamical equations will be valid in a general case.

Owing to canonical commutation relations, C^+ in (7) gives the delta function $\delta_{\alpha\beta}\delta(\mathbf{r}_1 - \mathbf{r}_2)$. Thus, only C^- carries nontrivial information about the system. We shall define, therefore, the DHW function for the Dirac field in terms of the commutator of the field operators. This commutator is odd under charge conjugation, which means that in the absence of external sources particles and antiparticles make the same contributions (except for the difference in sign) to the DHW function. In the presence of the electromagnetic field we should modify the definition of the DHW function in order to make it gauge invariant. The appropriate modification has been already introduced by Dirac [5] and Heisenberg [6] in the form of the exponent of the line integral of the vector potential. Other modifications leading to gauge-invariant objects are also possible, but the line integral is unique since only in this case the argument $\mathbf{p}$ of the Wigner function can be interpreted as an eigenvalue of the kinetic momentum $(\hbar/i)\nabla - e\mathbf{A}$ of the particle [17].

With the inclusion of the line integral, the definition of the DHW function reads

$$W_{\alpha\beta}(\mathbf{r},\mathbf{p},t) = -\tfrac{1}{2}\int d^3s \exp(-i\mathbf{p}\cdot\mathbf{s}) \times \left\langle \Phi \left| \exp\left(-ie\int_{-1/2}^{1/2} d\lambda\, \mathbf{s}\cdot\mathbf{A}(\mathbf{r}+\lambda\mathbf{s},t)\right) [\Psi_\alpha(\mathbf{r}+\mathbf{s}/2,t),\Psi^\dagger_\beta(\mathbf{r}-\mathbf{s}/2,t)] \right| \Phi \right\rangle . \quad (8)$$

The matrix $W_{\alpha\beta}$ is by construction Hermitian and hence, when expanded into a complete set of 4×4 Hermitian matrices, will yield 16 real coefficient functions. We shall choose as the basis for this expansion the set of matrices introduced originally by Dirac. The 16 matrices of this set are built as (external) products from two sets, $(1,\boldsymbol{\sigma})$ and $(1,\boldsymbol{\rho})$, of 2×2 Pauli matrices (including the unit matrix for completeness). The matrices $\boldsymbol{\sigma}$ describe the spin degree of freedom whereas the matrices $\boldsymbol{\rho}$ act on the particle-antiparticle degrees of freedom. The correspondence between the matrices built from the ρ's and σ's and the standard γ matrices runs as follows:

$$\begin{aligned}&\psi^\dagger(1,\rho_1,\rho_2,\rho_3,\sigma_k,\rho_1\sigma_k,\rho_2\sigma_k,\varepsilon^{ijk}\rho_3\sigma_k)\psi,\\ &\bar\psi(\gamma^0,i\gamma^0\gamma_5,\gamma_5,1,-i\gamma_5\gamma^k,\gamma^k,-i\gamma^0\gamma^k,i\gamma^{ij})\psi,\end{aligned} \quad (9)$$

where $\gamma_5=\gamma^0\gamma^1\gamma^2\gamma^3$ and $\gamma^{ij}=\gamma^i\gamma^j$, $(i\neq j)$.

The expansion of the DHW function into the basis set of matrices will be written in the form

$$\mathbf{W}(\mathbf{r},\mathbf{p},t) = \frac{1}{4}\left(f_0+\sum_{i=1}^{3}\rho_i f_i + \boldsymbol{\sigma}\cdot\mathbf{g}_0 + \sum_{i=1}^{3}\rho_i\boldsymbol{\sigma}\cdot\mathbf{g}_i\right), \quad (10)$$

where all the expansion coefficients f and $\mathbf{g}$ are dimensionless functions of $\mathbf{r},\mathbf{p}$, and t. To simplify the notation, we have suppressed the matrix indices α and β. It can be seen from the correspondence table (9) that the functions f_0, f_3 and $\mathbf{g}_1,\mathbf{g}_2$ transform as scalars and vectors, whereas the functions f_1,f_2 and $\mathbf{g}_0,\mathbf{g}_3$ transform as pseudoscalars and pseudovectors.

The time derivative of the DHW function can be expressed, with the help of the field equations (1) and (2), in terms of the DHW function itself and two new expectation values involving an additional electric or magnetic field operator. At this point we shall make *the only approximation* that is needed to derive our self-consistent set of equations: we replace the expectation value of the following products of field operators by the corresponding products of expectation values: i.e.,

$$\left\langle \Phi \left| \mathbf{E}(\mathbf{r},t)\exp\left(ie\int_{\mathbf{r}_1}^{\mathbf{r}_2} d\boldsymbol{\xi}\cdot\mathbf{A}(\boldsymbol{\xi},t)\right)[\Psi(\mathbf{r}_1,t),\Psi^\dagger(\mathbf{r}_2,t)]\right|\Phi\right\rangle \to \langle\Phi|\mathbf{E}(\mathbf{r},t)|\Phi\rangle \left\langle \Phi \left| \exp\left(ie\int_{\mathbf{r}_1}^{\mathbf{r}_2} d\boldsymbol{\xi}\cdot\mathbf{A}(\boldsymbol{\xi},t)\right)[\Psi(\mathbf{r}_1,t),\Psi^\dagger(\mathbf{r}_2,t)]\right|\Phi\right\rangle, \quad (11)$$

$$\left\langle \Phi \left| \mathbf{B}(\mathbf{r},t)\exp\left(ie\int_{\mathbf{r}_1}^{\mathbf{r}_2} d\boldsymbol{\xi}\cdot\mathbf{A}(\boldsymbol{\xi},t)\right)[\Psi(\mathbf{r}_1,t),\Psi^\dagger(\mathbf{r}_2,t)]\right|\Phi\right\rangle \to \langle\Phi|\mathbf{B}(\mathbf{r},t)|\Phi\rangle \left\langle \Phi \left| \exp\left(ie\int_{\mathbf{r}_1}^{\mathbf{r}_2} d\boldsymbol{\xi}\cdot\mathbf{A}(\boldsymbol{\xi},t)\right)[\Psi(\mathbf{r}_1,t),\Psi^\dagger(\mathbf{r}_2,t)]\right|\Phi\right\rangle. \quad (12)$$

This is an approximation of the Hartree type, in which the quantum fluctuations of the electromagnetic field are neglected. We believe that for fields that slowly vary with time this approximation is appropriate in the strong field regime, even when the fields have large spatial gradients. In view of the gauge invariance of our formulation, this approximation may be called the mean-field-strength approximation or the *mean-EB approximation*. Note that we are disregarding only the quantum correlations between the electric and magnetic fields and the Dirac field while keeping the potential in the line integral inside the original matrix element.

In the mean-EB approximation the equations for $\mathbf{W}_{\alpha\beta}$ in the matrix form read

$$\mathsf{D}_t\mathbf{W} = -\frac{c}{2}\mathsf{D}\cdot\{\rho_1\boldsymbol{\sigma},\mathbf{W}\} - \frac{ic}{\hbar}[\rho_1\boldsymbol{\sigma}\cdot\mathsf{P}+\rho_3 mc,\mathbf{W}], \quad (13)$$

where D_t, D, and P are the following nonlocal operators:

$$\begin{aligned}\mathsf{D}_t &= \partial_t + e\int_{-1/2}^{1/2} d\lambda\,\mathbf{E}(\mathbf{r}+i\lambda\boldsymbol{\partial}_p,t)\cdot\boldsymbol{\partial}_p\\ &= \partial_t + e\mathbf{E}(\mathbf{r},t)\cdot\boldsymbol{\partial}_p - \frac{e\hbar^2}{12}(\nabla\cdot\boldsymbol{\partial}_p)^2\mathbf{E}(\mathbf{r},t)\cdot\boldsymbol{\partial}_p + \cdots,\end{aligned} \quad (14)$$

$$\begin{aligned}\mathsf{D} &= \nabla + e\int_{-1/2}^{1/2} d\lambda\,\mathbf{B}(\mathbf{r}+i\lambda\boldsymbol{\partial}_p,t)\times\boldsymbol{\partial}_p\\ &= \nabla + e\mathbf{B}(\mathbf{r},t)\times\boldsymbol{\partial}_p - \frac{e\hbar^2}{12}(\nabla\cdot\boldsymbol{\partial}_p)^2\mathbf{B}(\mathbf{r},t)\times\boldsymbol{\partial}_p + \cdots,\end{aligned} \quad (15)$$

$$\mathsf{P} = \mathbf{p} - ie\int_{-1/2}^{1/2} d\lambda\,\lambda\,\mathbf{B}(\mathbf{r}+i\lambda\boldsymbol{\partial}_p,t)\times\boldsymbol{\partial}_p$$
$$= \mathbf{p} + \frac{e\hbar}{12}(\nabla\cdot\boldsymbol{\partial}_p)\,\mathbf{B}(\mathbf{r},t)\times\boldsymbol{\partial}_p + \ldots, \qquad (16)$$

where the ellipses involve still higher derivatives of the fields $\mathbf{E}$ and $\mathbf{B}$. For slowly varying fields the terms containing the field derivatives are small; the smallness parameter being equal to the ratio of the Compton radius to the length parameter characterizing the spatial variation of the electromagnetic field. If all the terms containing the derivatives of the electromagnetic field are dropped, the nonlocal operators D_t, $\mathbf{D}$, and P reduce to their local form:

$$\mathsf{D}_t = \partial_t + e\mathbf{E}(\mathbf{r},t)\cdot\boldsymbol{\partial}_p, \qquad (17)$$

$$\mathbf{D} = \nabla + e\mathbf{B}(\mathbf{r},t)\times\boldsymbol{\partial}_p, \qquad (18)$$

$$\mathsf{P} = \mathbf{p}. \qquad (19)$$

This local approximation may also be viewed as a classical limit, since the disregarded terms involve increasing powers of the Planck's constant. For spatially homogeneous fields the local form (19) is exact; no quantum corrections appear in the operators D_t, $\mathbf{D}$, and P. However, in contradistinction to the case of nonrelativistic wave mechanics, the evolution equations for the components of the DHW function in homogeneous electromagnetic fields do contain the Planck's constant and, as we shall show, are capable of describing such characteristic quantum phenomena as the pair production.

By comparing the coefficients that multiply the same matrices, we obtain a set of differential equations for the functions f and $\mathbf{g}$. The equations for the expectation values of the electric and magnetic field are obtained by simply taking the expectation values of the operator Maxwell equations (3)–(6).

The full set of equations for the 16 components of the DHW function describing the Dirac field and 6 functions describing the electromagnetic field has the form

$$\mathsf{D}_t f_0 + c\mathbf{D}\cdot\mathbf{g}_1 = 0, \qquad (20)$$

$$\mathsf{D}_t f_1 + c\mathbf{D}\cdot\mathbf{g}_0 = -2\frac{mc^2}{\hbar}f_2, \qquad (21)$$

$$\mathsf{D}_t f_2 + 2\frac{c}{\hbar}\mathsf{P}\cdot\mathbf{g}_3 = 2\frac{mc^2}{\hbar}f_1, \qquad (22)$$

$$\mathsf{D}_t f_3 - 2\frac{c}{\hbar}\mathsf{P}\cdot\mathbf{g}_2 = 0, \qquad (23)$$

$$\mathsf{D}_t\mathbf{g}_0 + \mathbf{D}f_1 - 2\frac{c}{\hbar}\mathsf{P}\times\mathbf{g}_1 = 0, \qquad (24)$$

$$\mathsf{D}_t\mathbf{g}_1 + \mathbf{D}f_0 - 2\frac{c}{\hbar}\mathsf{P}\times\mathbf{g}_0 = -2\frac{mc^2}{\hbar}\mathbf{g}_2, \qquad (25)$$

$$\mathsf{D}_t\mathbf{g}_2 + \mathbf{D}\times\mathbf{g}_3 + 2\frac{c}{\hbar}\mathsf{P}f_3 = 2\frac{mc^2}{\hbar}\mathbf{g}_1, \qquad (26)$$

$$\mathsf{D}_t\mathbf{g}_3 - \mathbf{D}\times\mathbf{g}_2 - 2\frac{c}{\hbar}\mathsf{P}f_2 = 0, \qquad (27)$$

$$\partial_t\mathbf{B} = -\nabla\times\mathbf{E}, \qquad (28)$$

$$\nabla\cdot\mathbf{B} = 0, \qquad (29)$$

$$\partial_t\,\epsilon_0\mathbf{E} = \nabla\times\mu_0^{-1}\mathbf{B} - \mathbf{j}, \qquad (30)$$

$$\nabla\cdot\epsilon_0\mathbf{E} = \rho. \qquad (31)$$

The charge density and the current density are expressed in the following way by the components of the DHW function:

$$\rho(\mathbf{r},t) = e\int d\tilde{p}\,f_0(\mathbf{r},\mathbf{p},t) + \rho_{\rm ext}(\mathbf{r},t), \qquad (32)$$

$$\mathbf{j}(\mathbf{r},t) = e\int d\tilde{p}\,\mathbf{g}_1(\mathbf{r},\mathbf{p},t) + \mathbf{j}_{\rm ext}(\mathbf{r},t), \qquad (33)$$

where $d\tilde{p} = (2\pi\hbar)^{-3}d^3p$. The equations satisfied by the coefficient functions f and $\mathbf{g}$ exhibit a remarkable symmetric structure, somewhat similar to that of the Maxwell equations, which may serve as a partial justification for our choice of the parametrization of the DHW function.

The set of coupled, nonlinear integro-differential equations (20)–(27) and (28)–(31) will form the basis for our discussion of the self-consistent time evolution of the quantum electrodynamic state under the assumption that the electromagnetic field is macroscopic, slowly varying in space and time. Complicated as these equations may seem, they are still by far simpler than the infinite set of Dyson-Schwinger equations for all the propagators that fully describes all quantum electrodynamic phenomena without any approximations. No exact solutions of the Dyson-Schwinger have ever been found, whereas for our equations we do obtain various nonperturbative, analytic and numerical solutions.

III. INTERPRETATION OF THE DHW FUNCTION

A. Conservation laws

In order to gain a better understanding of the physical meaning of various components of the DHW function, we shall now express in terms of these components all the conservation laws that are valid for our theory in the absence of external sources. The conservation laws can be expressed either in the integral form or in the form of continuity equations (continuity equation for the energy-momentum tensor for this theory has been already noted by Heisenberg [6]). Here we shall write down only the simpler, integral forms of the conservation laws.

The most fundamental and also the simplest of the conservation laws is the total charge conservation

$$\frac{dQ}{dt} = 0, \qquad (34)$$

$$Q = e \int d\Gamma \; f_0(\mathbf{r}, \mathbf{p}, t), \tag{35}$$

where $d\Gamma$ is the volume element of the phase space, $d\Gamma = d\tilde{r}\, d\tilde{p} = (2\pi\hbar)^{-3}\, d^3r\, d^3p$.

Total energy E, momentum $\mathbf{P}$, and angular momentum $\mathbf{M}$ of the system are also conserved;

$$\frac{dE}{dt} = 0, \tag{36}$$

$$E = \int d\Gamma \, [c\mathbf{p} \cdot \mathbf{g}_1(\mathbf{r}, \mathbf{p}, t) + mc^2 f_3(\mathbf{r}, \mathbf{p}, t)] + \tfrac{1}{2} \int d\mathbf{r} \, [\epsilon_0 \mathbf{E}^2(\mathbf{r}, t) + \mu_0^{-1} \mathbf{B}^2(\mathbf{r}, t)], \tag{37}$$

$$\frac{d\mathbf{P}}{dt} = 0, \tag{38}$$

$$\mathbf{P} = \int d\Gamma \, \mathbf{p} f_0(\mathbf{r}, \mathbf{p}, t) + \int d\tilde{r} \, [\epsilon_0 \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)], \tag{39}$$

$$\frac{d\mathbf{M}}{dt} = 0, \tag{40}$$

$$\mathbf{M} = \int d\Gamma [\mathbf{r} \times \mathbf{p} f_0(\mathbf{r}, \mathbf{p}, t) + \frac{\hbar}{2} \mathbf{g}_0(\mathbf{r}, \mathbf{p}, t)] + \int d\tilde{r} \, \mathbf{r} \times [\epsilon_0 \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)]. \tag{41}$$

Relativistic invariance of the theory requires that the time derivative of the first moment of the energy distribution is equal to the total momentum. The corresponding equation is indeed satisfied in our theory:

$$\frac{d\mathbf{N}}{dt} = 0, \tag{42}$$

$$\mathbf{N} = \int d\Gamma \, \mathbf{r} \, [c\mathbf{p} \cdot \mathbf{g}_1(\mathbf{r}, \mathbf{p}, t) + mc^2 f_3(\mathbf{r}, \mathbf{p}, t)] + \tfrac{1}{2} \int d\tilde{r} \, \mathbf{r} \, [\epsilon_0 \mathbf{E}^2(\mathbf{r}, t) + \mu_0^{-1} \mathbf{B}^2(\mathbf{r}, t)] - t \, \mathbf{P}. \tag{43}$$

Finally, there is a conservation law that results from the unitarity of the time evolution of the Dirac field. It is an analog of the conservation of the scalar product of two Wigner functions obeying the same equation in nonrelativistic quantum mechanics:

$$\frac{d}{dt} \int d\Gamma \, W_1(\mathbf{r}, \mathbf{p}, t) \, W_2(\mathbf{r}, \mathbf{p}, t) = 0. \tag{44}$$

In our case, this conservation law takes on the form of the conservation of the scalar product of two 16-dimensional vectors built from two sets of functions f and $\mathbf{g}$, both satisfying Eqs. (20)–(27) with the same electromagnetic field. When these two vectors are taken to be the same, we obtain the conservation of the norm,

$$\frac{d}{dt} \int d\Gamma \left(f_0^2 + \sum_{i=1}^{3} f_i^2 + \mathbf{g}_0^2 + \sum_{i=1}^{3} \mathbf{g}_i^2 \right) = 0. \tag{45}$$

The easiest method to prove all the conservation laws listed above is to derive them first in the local approximation, for slowly varying fields, and then observe that they are also valid without this simplifying assumption, because all the additional terms in the evolution equations for the functions f and $\mathbf{g}$ involve higher derivatives with respect to $\mathbf{p}$. Therefore, the contributions to conserved quantities due to the nonlocal terms will always vanish upon the integration over $\mathbf{p}$.

B. The classical limit

In order to obtain the classical limit of the evolution equations (20)–(31) we observe first that the following ansatz will yield a solution of all these equations,

$$f_0 = 0, \quad f_1 = 0, \quad f_2 = 0, \quad f_3 = \frac{mf}{E_p}, \qquad \mathbf{g}_0 = 0, \quad \mathbf{g}_1 = \mathsf{P}\left(\frac{f}{E_p}\right), \quad \mathbf{g}_2 = 0, \quad \mathbf{g}_3 = 0, \tag{46}$$

provided the function f obeys the equation

$$\mathsf{D}_t f + c\mathbf{D} \cdot \mathsf{P}(f/E_p) = 0. \tag{47}$$

In the local limit (19), when all the quantum corrections to the derivatives D_t and $\mathbf{D}$ and to the operator P are dropped, the equation (47) for f reduces to the relativistic Vlasov equation of plasma theory:

$$\partial_t f + \mathbf{v} \cdot \nabla f + e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_p f = 0, \tag{48}$$

where the velocity vector $\mathbf{v}$ is related to the kinetic momentum by the relativistic formula $\mathbf{v} = \mathbf{p}/E_p$.

We would like to point out that in the classical limit the one-time Wigner function leads *directly* to the classical kinetic equation in contrast with the situation for the four-dimensional (nonlocal in time) Wigner function. Even though our evolution equations are similar in form to the set of equations for the four-dimensional Wigner function obtained by Vasak, Gyulassy, and Elze [11], there is an essential difference between the two sets of equations. Namely, their equations are subject to 16 constraint conditions, while in our case no constraints are present. This difference is due to the fact that from the very beginning the one-time Wigner function describes particles on the mass shell while for the four-dimensional Wigner function the projection to the mass shell is achieved only with the help of the constraint conditions. As a result, the classical kinetic equation for the four-dimensional Wigner function contains an additional term $\mathbf{p} \cdot \mathbf{E}\partial/\partial p_0$ which should be eliminated with the help of the constraint conditions.

The fact that for slowly varying fields our equations for the components of the DHW function can be solved by the classical distribution function in phase space obeying the Vlasov equation shows that our formulation conveniently bridges the quantum and the classical modes of description. This does not mean, of course, that every classical distribution function can be obtained as an expectation value (8) of Dirac field operators. We know very well that even in nonrelativistic quantum mechanics not every distribution function can be obtained as a Wigner transform of some wave function or a mixture of such transforms. The easiest way to guarantee that the distribution function is an admissible one, is to solve the initial-value problem with the initial condition representing a genuine quantum state.

C. The meaning of the f and g functions

The correspondence table (9), the conservation laws, and also the time-evolution equations enable us to establish the physical meaning of various components of the DHW function.

(i) $(f_0, \mathbf{g}_1)$. The functions f_0 and $\mathbf{g}_1$ determine the charge density and the current density in the phase space. It is worth noting that Eq. (20) is the phase-space continuity equation for these two quantities.

(ii) (f_3) . The function f_3 determines the mass density.

(iii) $(\mathbf{g}_0)$. In the expression for the angular momentum we can clearly distinguish the orbital part and the spin part. This enables us to identify $\mathbf{g}_0$ as the spin density.

(iv) $(\mathbf{g}_3)$. The difference between the function $\mathbf{g}_3$ and $\mathbf{g}_0$ is the same as between the mass density and the charge density; the antiparticles contribute with the opposite sign. Therefore, the function $\mathbf{g}_3$ determines the magnetic moment density.

(v) (f_1, f_2) . While a direct characterization of the functions f_1, f_2 in terms of classical concepts does not seem to be possible, they can be given some meaning with the use of the evolution equations for the DHW function. Thus, according to Eq. (24), f_1 determines the rate of change of the helicity density $\mathbf{p}\cdot\mathbf{g}_0/|\mathbf{p}|$, while f_2 plays a somewhat similar role for the magnetic moment. The functions f_1 and f_2 are mutually coupled by the *Zitterbewegung* oscillations as seen from Eqs. (21) and (22).

(vi) $(\mathbf{g}_2)$. A similar mutual coupling occurs for the function $\mathbf{g}_2$ in relation to the current density $\mathbf{g}_1$, as seen from Eqs. (25) and (26). We can also see from Eq. (23) that the projection of the function $\mathbf{g}_2$ in the direction of momentum determines the rate of change of the mass density f_3.

D. Connection between the DHW function and the Feynman propagator

A direct relation between the DHW function and the one-electron Feynman propagator G (defined with the appropriate line integral),

$$-iG(\mathbf{r},t;\mathbf{r}',t') = \left\langle 0^{\rm out} \left| \exp\left(ie \int_{\mathbf{r},t}^{\mathbf{r}',t'} d\xi^\mu A_\mu(\xi) \right) T(\psi(\mathbf{r},t)\,\bar{\psi}(\mathbf{r}',t')) \right| 0^{\rm in} \right\rangle , \tag{49}$$

exists only when the external field does not produce particles. Only then we can identify the in vacuum state and the out vacuum state,

$$|0^{\rm in}\rangle = |0^{\rm out}\rangle = |0\rangle, \tag{50}$$

and the transition matrix element becomes equal to the vacuum expectation value in the definition of the DHW function. Next, we have to take the symmetric limit $t \to t'$ in order to obtain the commutator. Finally, we perform the Fourier transformation with respect to the difference of the space coordinates to obtain

$$W_{\alpha\beta}(\mathbf{r},\mathbf{p},t) = \frac{i}{2}\int d^3s \exp(-i\mathbf{p}\cdot\mathbf{s})G(\mathbf{r}+\mathbf{s}/2,t,\mathbf{r}-\mathbf{s}/2,t). \tag{51}$$

This formula will be used in the next section to find the DHW function in a static magnetic field.

IV. SIMPLE SOLUTIONS OF THE EQUATIONS FOR THE DHW FUNCTION

A. The free vacuum solution

Before we turn to the properties of the DHW function in the presence of electromagnetic fields we will write down the coefficient functions for the free-field vacuum. These functions can be obtained from the vacuum expectation values of the free Dirac field operators. The only nonvanishing functions in this case are f_3 and $\mathbf{g}_1$ and they have the values

$$f_3^0(\mathbf{p}) = -\frac{2m}{E_p}, \quad \mathbf{g}_1^0(\mathbf{p}) = -\frac{2\mathbf{p}}{E_p}, \tag{52}$$

where $E_p = \sqrt{m^2+\mathbf{p}^2}$. Because of the cancellation of the contributions from positive and negative values of $\mathbf{p}$, the net current in the free vacuum is zero. The functions (52) form the simplest solution of Eqs. (20)–(27), but more complicated solutions can be obtained by solving the initial-value problem starting with the free vacuum solution.

The components f_3 and $\mathbf{g}_1$ determine the energy density in the phase space [cf. Eq. (37)]. By subtracting the free vacuum values (52) from the values of these components obtained in the presence of material walls, we shall obtain a convenient description of the Casimir effect. The advantage of this approach is that if we postpone the integration over momenta until after the subtraction, we will not encounter any infinities.

B. Solution in a homogeneous magnetic field

Since pure magnetic fields do not produce pairs, we can extract the DHW function for the homogeneous, constant magnetic from the Feynman propagator, as shown in Eq. (51). The appropriate Feynman propagator has been given by Tsai [18]. Only the components $f_1, f_3, \mathbf{g}_1$, and g_3 turn out to be different from zero and they can be expressed as the integrals

$$(f_1, f_3, \mathbf{g}_1, \mathbf{g}_3) = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\kappa \, \exp[-\kappa^2 m_{\|}^2 - \tanh(\kappa^2 \mathcal{B}) p_{\perp}^2/\mathcal{B}] \times (p_{\|} \tanh(\kappa^2\mathcal{B}),\ m,\ \mathbf{p} - \mathbf{p}_{\perp} \tanh^2(\kappa^2\mathcal{B}),\ m \tanh(\kappa^2\mathcal{B})\mathbf{B}/|\mathcal{B}|), \tag{53}$$

where $\mathcal{B} = |e\mathbf{B}|$ and $p_{\|}$ and $\mathbf{p}_{\perp}$ are the components of the momentum vector in the direction of the $\mathbf{B}$ field and in the direction perpendicular to the field, respectively, and $m_{\|} = \sqrt{m^2 + p_{\|}^2}$. One can check by a direct substitution that these expressions are a solution of Eqs. (20)–(27). For a weak magnetic field they reduce to

$$(f_1, f_3, \mathbf{g}_1, \mathbf{g}_3) \approx -(\ e\mathbf{p}\cdot\mathbf{B}/E_p^3,\ 2m/E_p + 5m(eB)^2 p_{\perp}^2/4E_p^7, 2\mathbf{p}/E_p + 5\mathbf{p}\,(eB)^2 p_{\perp}^2/4E_p^7 - 3\mathbf{p}_{\perp}(eB)^2/2E_p^5,\ me\mathbf{B}/E_p^3). \tag{54}$$

C. Solutions in a homogeneous electric field

The simplest example that already includes pair production is that of the homogeneous electric field. In this case we cannot use the Feynman propagator to calculate the DHW function since we need the expectation value of the Dirac field operators and not the matrix element between two (different) state vectors, $|0^{\rm in}\rangle$ and $|0^{\rm out}\rangle$. Instead, we shall use our time evolution equations.

Because of the planar symmetry of the homogeneous field, the full set of equations for the 16 components of the DHW function can be reduced to a set of 3 ordinary differential equations. In order to perform this reduction, we observe that only the components $\mathbf{g}_0$ and $\mathbf{g}_2$ couple to the vacuum components f_3 and $\mathbf{g}_1$ initially present. Thus, we can seek the solution in the subspace of 10-dimensional vectors $W_{10} = (f_3,\ \mathbf{g}_0,\ \mathbf{g}_1,\ \mathbf{g}_2)$. The evolution equation in this subspace can be written as

$$(\partial_t + e\mathbf{E}\cdot\boldsymbol{\partial}_p)\, W_{10} + M(\mathbf{p})\, W_{10} = 0, \tag{55}$$

where M denotes the following 10×10 submatrix of the original 16×16 matrix:

$$\begin{pmatrix} 0 & 0 & 0 & -2\mathbf{p} \\ 0 & 0 & -2\times\mathbf{p} & 0 \\ 0 & -2\times\mathbf{p} & 0 & 2m \\ 2\mathbf{p} & 0 & -2m & 0 \end{pmatrix}. \tag{56}$$

We may convert Eq. (55) into a set of ordinary differential equations by separating out the classical time evolution. This procedure is very similar to solving first-order partial differential equations by the method of characteristics. To this end we substitute for each component W^i of W in Eq. (55) the expression

$$W^i(\mathbf{p}, t) = \int d\tilde{\mathbf{p}}\, w^i(\mathbf{p}_0, t)\, \delta(\mathbf{p} - \mathbf{p}(\mathbf{p}_0|t)), \tag{57}$$

where $\mathbf{p}(\mathbf{p}_0|t)$ is the solution of the classical equation of motion in the homogeneous electric field,

$$\frac{d\mathbf{p}(t)}{dt} = e\mathbf{E}(t), \tag{58}$$

obeying the initial condition $\mathbf{p}(\mathbf{p}_0|t=0) = \mathbf{p}_0$,

$$\mathbf{p}(\mathbf{p}_0|t) = \mathbf{p}_0 + e\int dt\, \mathbf{E}(t). \tag{59}$$

The time dependence of w is due to quantum effects that come on top of the classical flow in phase space. This flow has been taken care of by the time-dependent shift of the argument of the δ function. As a result of the substitution (57), the derivative with respect to momentum disappears and we are left with ordinary differential equations for the new vector w. The price to be paid for this simplification is that the matrix $M(\mathbf{p})$ becomes time dependent when the vector $\mathbf{p}$ is expressed in terms of its initial value $\mathbf{p}_0$:

$$\partial_t w^i(\mathbf{p}_0, t) + M^i_j(\mathbf{p}(\mathbf{p}_0|t))w^j(\mathbf{p}_0, t) = 0. \tag{60}$$

Since the functions w^i depend not only on time but also on the parameters $\mathbf{p}_0$, we shall continue to denote the time derivative by the symbol ∂_t. We will seek that solution of Eq. (60) which at $t = 0$ starts as the free vacuum. It turns out that in this case the following set of only three orthonormal vectors is sufficient to span the complete space of solutions of the initial-value problem:

$$F_0^i = \frac{1}{E_p}\begin{pmatrix} m \\ 0 \\ \mathbf{p} \\ 0 \end{pmatrix}, \quad F_1^i = \frac{1}{m_{\perp} E_p}\begin{pmatrix} -m(\mathbf{n}\cdot\mathbf{p}) \\ 0 \\ \mathbf{n}E_p^2 - \mathbf{p}(\mathbf{n}\cdot\mathbf{p}) \\ 0 \end{pmatrix}, \quad F_2^i = \frac{1}{m_{\perp}}\begin{pmatrix} 0 \\ \mathbf{n}\times\mathbf{p} \\ 0 \\ m\mathbf{n} \end{pmatrix}, \tag{61}$$

where $\mathbf{n}$ is a unit vector chosen in the direction of the electric field and $m_\perp = \sqrt{m^2 + \mathbf{p}_\perp^2}$ is the transverse mass (with respect to the field direction). The solution of (60) can be written in the form

$$w^i = -2(a_0 F_0^i + a_1 F_1^i + a_2 F_2^i). \tag{62}$$

The factor of (-2) has been separated out to have the following unit-vector representation of the free vacuum state:

$$(a_0, a_1, a_2)_{\rm vac} = (1, 0, 0). \tag{63}$$

In order to obtain the equations for the three coefficient functions a, we use the following relations for the vectors F:

$$\begin{aligned} M_j^i(\mathbf{p}) F_0^j &= 0, \\ M_j^i(\mathbf{p}) F_1^j &= -2E_p F_2^i, \\ M_j^i(\mathbf{p}) F_2^j &= 2E_p F_1^i; \end{aligned} \tag{64}$$

$$\begin{aligned} \mathbf{n}\cdot\partial_p\, F_0^i &= \frac{m_\perp}{E_p^2} F_1^i, \\ \mathbf{n}\cdot\partial_p\, F_1^i &= -\frac{m_\perp}{E_p^2} F_0^i, \\ \mathbf{n}\cdot\partial_p\, F_2^i &= 0. \end{aligned} \tag{65}$$

With the use of these relations, the equation (60) yields the following three equations for the functions a:

$$\partial_t a_0 = \frac{\mathcal{E} m_\perp}{E_p^2} a_1, \tag{66}$$

$$\partial_t a_1 = -\frac{\mathcal{E} m_\perp}{E_p^2} a_0 - 2E_p a_2, \tag{67}$$

$$\partial_t a_2 = 2E_p a_1, \tag{68}$$

where $\mathcal{E} = |e\mathbf{E}|$ and $\mathbf{p}$ is a function of t with its initial value $\mathbf{p}_0$ treated as a parameter. In the derivation of Eqs. (66)–(68) we have taken into account the dependence of the vectors F on time through $\mathbf{p}$. To satisfy the initial conditions, we set vector a at $t = 0$ equal to (63). As a result of the conservation law (45) the sum of the squares of the a's is a constant of motion. Thus, the time evolution of the vector (a_0, a_1, a_2) is a pure rotation of a unit vector. This formal similarity between the behavior of the Dirac particle in a homogeneous electric field and the spin precession has been pointed out before [19]. For large values of t, when E_p is very large, the component a_0 takes on a constant value and Eq. (66) for a_0 decouples from the remaining two equations. Since a_1 and a_2 oscillate rapidly with frequency $2E_p$, all time averages involve only the function a_0. Smooth and oscillatory behaviors also characterize the dependence of the coefficient a_0, on one hand, and the coefficients a_1 and a_2, on the other hand, when they are treated as functions of the momentum component parallel to the electric field (cf. Figs. 1, 2, and 3). The large variations of the coefficient a_0 at both ends are caused by the sudden switching of the field. These transient effects do not influence the average value of a_0 for large t. In Fig. 4 we show this coefficient calculated for $t = 30$. The only difference between this result and the one obtained for $t = 20$ is a larger width of the trough, the depth and the shape of the transients are exactly the same.

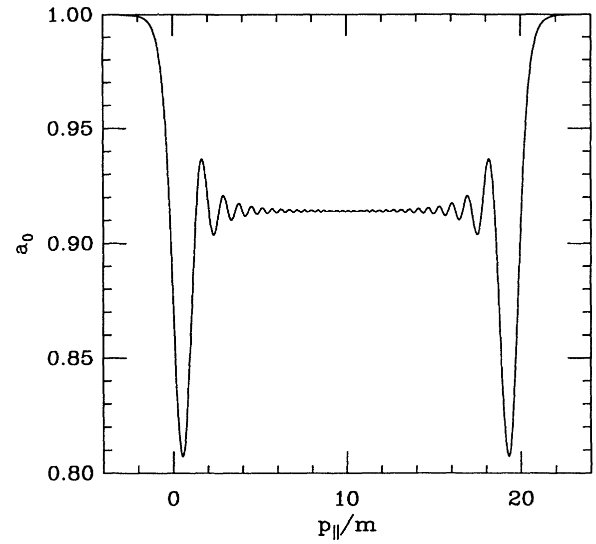

FIG. 1. The coefficient a_0 as a function of the component of momentum parallel to the electric field for $t = 20\hbar/mc^2$. The value 1 corresponds to free vacuum.

D. Pair production and the vacuum-persistence probability

We shall show now that the solutions of Eqs.(66)–(68) reproduce, in the limit when $t \to \infty$, the well-known results [20] for the rate of pair production in a constant electric field. As seen from Fig. 1, except for transient effects at both ends, the homogeneous electric field pro-

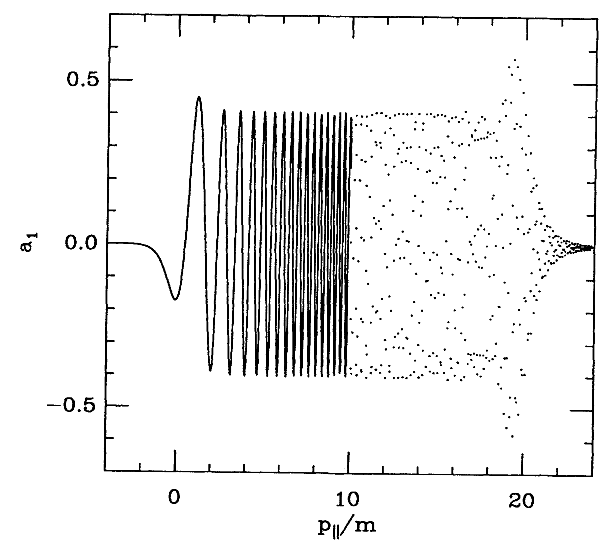

FIG. 2. The coefficient a_1 as a function of the component of momentum parallel to the electric field for $t = 20\hbar/mc^2$. In view of very fast oscillations of this function for greater values of p, the calculated points have not been connected.

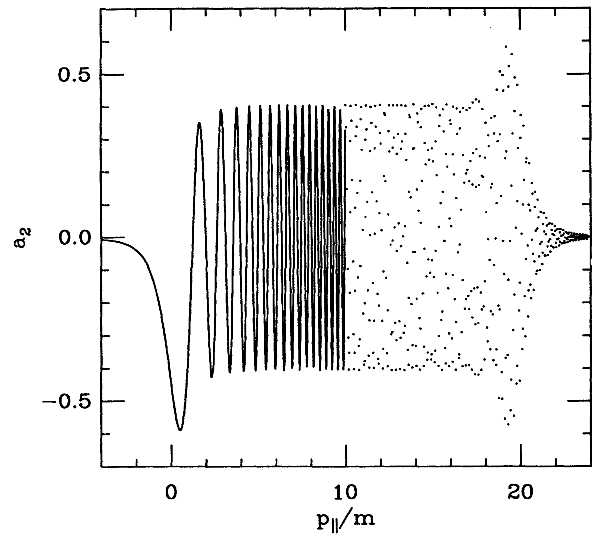

FIG. 3. The coefficient a_2 as a function of the component of momentum parallel to the electric field for $t = 20\hbar/mc^2$, shown as in Fig. 2.

duces a rectangular trough in the state occupancy of the Dirac sea. The front of this trough, as seen from Figs. 1 and 4, is moving with constant velocity $e\mathbf{E}$. The depth of the trough $(1 - a_0)/2$ is a nonlinear function of the electric field intensity. Figure 5 based on the numerical solutions shows beyond any doubt that this dependence is described by the exponential formula

$$(1 - a_0)/2 = \theta(\mathcal{E}t - p_\parallel)\theta(p_\parallel)\exp(-\pi m_\perp^2/\mathcal{E}). \quad (69)$$

This formula leads to the following value of the current produced by the field:

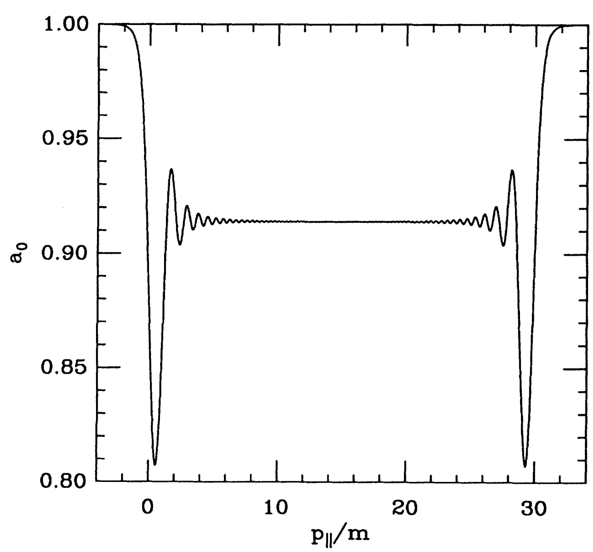

FIG. 4. The coefficient a_0 as a function of the component of momentum parallel to the electric field for $t = 30\hbar/mc^2$. It is seen that except for the width of the trough the shape is the same as for $t = 20$.

$$\mathbf{j} = e\int d\tilde{\mathrm{p}}(\mathbf{g}_1 - \mathbf{g}_1^0)$$
$$= 4(2\pi)^{-3}\int d^2p_\perp \int_0^{\mathcal{E}t} dp_\parallel \exp(-\pi m_\perp^2/\mathcal{E})\ \mathbf{p}/E_p. \quad (70)$$

Owing to the symmetry of the problem, only the component of the current parallel to the field is different from zero. From the form of the current we can determine the number of pairs $n(\mathbf{p}, T)$ produced in time T per unit volume of phase space using the following relation for the current density in phase space:

$$\mathbf{j} = 2e\mathbf{v}\, n(\mathbf{p}, T). \quad (71)$$

The factor of 2 arises because both the particle and the antiparticle contribute to the current. By comparing the formulas (70) and (71), we see that, apart from the factor of $(2\pi)^3$, the expression $1 - a_0$ gives the density of pairs in phase space:

$$n(\mathbf{p}, T) = \frac{1}{4\pi^3}\theta(\mathcal{E}T - p_\parallel)\theta(p_\parallel)\exp(-\pi m^2/|e\mathbf{E}|). \quad (72)$$

This expression is in full agreement with the formula for the vacuum persistence probability [20]. All we have to do to establish this connection is to take into account the relation between the number of fermion pairs and the vacuum persistence probability p_v, given by Feynman [21, 22]:

$$p_v = |\langle 0^t|0\rangle|^2 = \exp\left(\int d^3r \frac{d^3p}{(2\pi)^3}\ln[1 - n(\mathbf{p}, t)]\right). \quad (73)$$

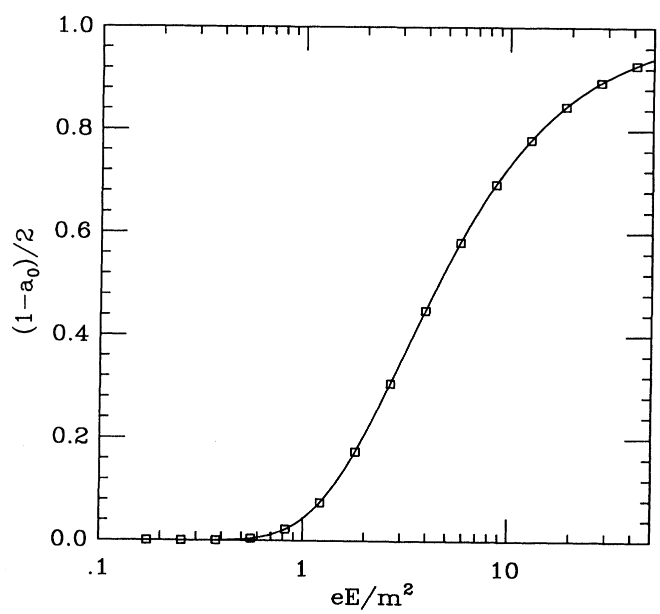

FIG. 5. Comparison between the numerical solutions (plotted as squares) of the differential equations for the coefficient a_0 at the center of the trough in momentum space for different values of the electric field with the analytical formula $\exp(-\pi m^2/|e\mathbf{E}|)$ obtained by Schwinger (continuous line).

After expanding the logarithm in the exponent, we can perform all the integrations over p. The Gaussian integration over $p_\perp$ gives the factor of $|e\mathbf{E}|/n$ and the integration over $p_\parallel$ gives the factor of $T|e\mathbf{E}|$. In this way, the exponent in Eq. (73) reduces exactly to the famous Schwinger expression:

$$p_v = \exp\left(-VT\frac{\alpha}{\pi^2}\mathbf{E}^2\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n\pi m^2/|e\mathbf{E}|}\right), \tag{74}$$

where α is the fine-structure constant.

V. RENORMALIZATION

In the phase-space description of the Dirac theory the ultraviolet divergences occur when the components of the DHW function do not fall off sufficiently fast for large momenta. The integration over p leads then to the well-known, infinite charge renormalization. The (almost) classical interpretation of the components of the DHW function enables us to interpret the renormalization procedure in terms of easily understood concepts of the electrodynamics in material media. We shall start from the following two sets of expressions for the components of the DHW function in a weak, static, slowly varying electric and magnetic field:

$$f_0 = -\frac{e}{2E_p^3}\left(\nabla\cdot\mathbf{E} - \frac{(\mathbf{p}\cdot\nabla)(\mathbf{p}\cdot\mathbf{E})}{E_p^2}\right), \tag{75}$$

$$f_1 = -\frac{e}{E_p^3}\mathbf{p}\cdot\mathbf{B}, \tag{76}$$

$$f_2 = 0, \tag{77}$$

$$f_3 = -\frac{2m}{E_p} + \frac{em}{2E_p^5}\mathbf{p}\cdot(\nabla\times\mathbf{B}), \tag{78}$$

$$\mathbf{g}_0 = \frac{e}{E_p^3}\mathbf{p}\times\mathbf{E}, \tag{79}$$

$$\mathbf{g}_1 = -\frac{2\mathbf{p}}{E_p} - \frac{e}{2E_p^3}\left(\frac{4}{3}\nabla\times\mathbf{B} - \frac{(\mathbf{p}\cdot\nabla)(\mathbf{p}\times\mathbf{B})}{E_p^2} - \frac{\mathbf{p}\;\mathbf{p}\cdot(\nabla\times\mathbf{B})}{E_p^2}\right), \tag{80}$$

$$\mathbf{g}_2 = \frac{em\mathbf{E}}{E_p^3}, \tag{81}$$

$$\mathbf{g}_3 = -\frac{em\mathbf{B}}{E_p^3}. \tag{82}$$

These expressions can be derived in a systematic manner from the set of integral equations for the DHW function obtained by the adiabatic switching on of the field. This will be done in a future publication. Here, it will suffice to state that the expressions (75)–(82) are solutions of our differential equations (20)–(27) without proving the uniqueness. Upon substituting the expressions (75) and (80) into the equations (32) and (33) and after performing the angular integrations in momentum space, we obtain the following expressions for the polarization charge $\rho_{\rm pol}$ and for the polarization current $\mathbf{j}_{\rm pol}$:

$$\rho_{\rm pol} = -\frac{e^2}{4\pi^2}\int_0^\Lambda dp\,\frac{p^2}{E_p^3}\left(1-\frac{p^2}{3E_p^2}\right)\nabla\cdot\mathbf{E}, \tag{83}$$

$$\mathbf{j}_{\rm pol} = -\frac{e^2}{4\pi^2}\int_0^\Lambda dp\,\frac{p^2}{E_p^3}\left(\frac{4}{3}-\frac{2p^2}{3E_p^2}\right)\nabla\times\mathbf{B}. \tag{84}$$

Both integrals are, of course, logarithmically divergent and require a momentum cutoff Λ, but the terms dependent on Λ can be absorbed by redefining the permittivity ϵ_0 and the permeability of the vacuum μ_0. To this end, we substitute the polarization charge and the polarization current into the Maxwell equations (30) and (31) and in the static case under consideration we obtain

$$\nabla\cdot\mathbf{E}\left(\epsilon_0 + \frac{e^2}{24\pi^2}\ln(\Lambda/m)\right) = \rho_{\rm ext}, \tag{85}$$

$$\nabla\times\mathbf{B}\left(\mu_0^{-1} + \frac{e^2}{24\pi^2}\ln(\Lambda/m)\right) = \mathbf{j}_{\rm ext}. \tag{86}$$

We can restore the standard form of these equations by absorbing the (divergent) terms due to the polarizability of the vacuum into the following observable permittivity and the permeability of the vacuum ϵ_v and μ_v,

$$\epsilon_v = \epsilon_0\left(1 + \frac{\alpha}{6\pi}\ln(\Lambda/m)\right), \tag{87}$$

$$\mu_v = \mu_0\left(1 + \frac{\alpha}{6\pi}\ln(\Lambda/m)\right)^{-1}. \tag{88}$$

The change of the permittivity of the vacuum leads to the standard renormalization of the fine-structure constant, the physical value being given by

$$\alpha_{\rm phys} = \frac{e^2}{4\pi\hbar c\epsilon_v}. \tag{89}$$

We note that the product of ϵ and μ, that determines the speed of light, does not change under the renormalization.

The expression of renormalization constants in terms of momentum integrals involving the Wigner function, obtained in this section, is meant only as an illustration of the versatility of our approach. We would like to stress that our discussion of renormalization is far from being complete. In particular, we did not solve the difficult problem of the stability of the renormalization procedure under the time evolution [14].

VI. CONCLUSIONS

We have shown that the phase-space distribution function could be a powerful tool in the study of the vacuum structure. Different components of the DHW function contain a wealth of detailed information about the time evolution in the presence of electromagnetic fields. This has been illustrated by considering pair production in a homogenous external electric field, where we have exhibited the rearrangement of the state occupancy of the vacuum induced by the field. In this case, not only have we shown that for a complex set of partial differential equations a solution can be found, but we have also demonstrated a remarkable degree of agreement between the elements of our numerical solutions and the corresponding physical quantities obtained in the past by completely different, analytical calculations. The expanding hole in the vacuum seen in the solutions of our equations represents a simple and picturesque description of pair-creation phenomena.

We believe that as a consequence of the present work a number of old problems becomes tractable within our scheme; the most important of them being the feedback problem in the sparking of the vacuum. The framework that we have developed may in particular be useful in the description of local changes in the vacuum structure under the presence of adiabatically changing supercritical Coulomb fields generated in heavy-ion collisions [23].

Despite the fact that we have remained within the realm of standard quantum field theory, our approach leads to novel mathematical structures as exemplified by the evolution equations for the DHW function. The approximation we have introduced allowed us to close the system of equations of quantum field theory within a physically justified truncation of the hierarchy of these equations, while maintaining all the symmetries of the theory, including explicit gauge invariance. Our formulation unifies several hitherto disconnected theories, because as special limiting cases we can obtain: the one-particle Dirac theory of electrons in external fields, the Maxwell theory of electromagnetic fields, and the relativistic kinetic theory of charged particles. It was this very connection with kinetic theory that led us to study the Wigner function rather than the Wightman function used by Cooper and Mottola [14]. Mathematically these two approaches are not much different, but our (classical) physical intuition is working better when the phase-space description is used.

An interesting aspect of our approach is a new insight into the renormalization. It is natural in our approach to interpret the renormalization procedure as a change in the values of the permittivity ϵ_0 and the permeability μ_0 of the vacuum. This point of view may be also important for applications in QCD as well as in problems in which the renormalization constants contain information about the local structure of the vacuum. We note in passing that within our gauge-invariant formulation the quadratic divergence of the vacuum polarization does not appear. In that regard our approach is similar to the gauge-invariant formulation of Schwinger [20].

In order to study the permanently existing structures of the interacting QED vacuum arising from persistent interactions, we will have to move away from the (adiabatic) switching-on hypothesis. This will require a new insight into the initial conditions for the vacuum. It seems to us that one may be able to approach this problem treating the DHW function for the physical vacuum as a function of temperature, assuming that the high-T vacuum state can be described by the perturbative solution. Our formalism seems to be particularly well suited to perform this task. After all, the main motivation for Wigner to introduce his function [4] was to study quantum corrections to the thermal distribution function.

ACKNOWLEDGMENT

I.B.B. and P.G. would like to acknowledge the hospitality of the Physics Department of the University of Arizona.

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* Permanent address: Institute for Theoretical Physics, Polish Academy of Sciences, Lotników 32/46, 02-668 Warsaw, Poland.

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