Topology changes in (2+1)-dimensional quantum gravity

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We study topology-changing processes in (2+1)-dimensional quantum gravity with a negative cosmological constant. By playing the "gluing-many-polyhedra game" for hyperbolic geometry, we explicitly construct an infinite number of different instantonlike solutions. These solutions can be used to evaluate various topology-changing amplitudes in the WKB approximation.

I. INTRODUCTION

Since the pioneering work of topological geons by Wheeler [1], it has been increasingly recognized that topology-changing processes in space-time may play an important role in quantum gravity. For example, we can mention the recent proposal by Hawking [2] and Coleman [3] that the sum over histories of the universe including the wormhole topologies may solve the problem of the cosmological constant. We expect that new physics will be open, if the topology-changing processes in space-time are taken into account.

The topology changes cannot occur classically, provided that (1) there is no closed timelike curve and (2) space-time singularity is not admitted. The latter necessarily appears in the topology-changing processes if matter satisfies the weak energy condition which is generally believed to be physical. However, such topology-changing processes are possible in quantum mechanics, e.g., by quantum tunneling. In Hawking's path-integral approach to quantum gravity, the tunneling transition amplitudes can be semiclassically evaluated by looking for solutions of the Einstein equation in Euclidean signature region with appropriate boundary conditions. Unfortunately very few solutions have been found for such topology-changing processes in (3+1)-dimensional quantum gravity [4].

(2+1)-dimensional gravity is a useful toy model for (3+1)-dimensional gravity. This model is considerably simpler because it contains only the global degrees of freedom. Previously we investigated the nucleation of the universe in the (2+1)-dimensional gravity model [5], which can be regarded as a topology-changing process in the sense that the universe takes a transition from the initial state of nothing to the final state of nontrivial topology. In this paper we shall investigate more general topology-changing processes. For example, a genus-2 universe changes into a genus-3 universe, or into two genus-2 universes.

Our present treatment is based on the recent discussions by Gibbons and Hartle [4], who deduced some rather strong restrictions on the space-time topology and geometry under the "no-boundary" condition. The space-time considered in their paper consists of a certain number of Lorentzian manifolds which are attached to a compact Riemannian manifold. The boundaries of the Lorentzian and Riemannian manifolds must be totally geodesic, i.e., all the components of extrinsic curvature vanish there because the extrinsic curvature is essentially the time derivative of the spatial metric and the spatial metric have to be smooth. This fact can easily be understood in physical terms. In general, the dynamical motion momentarily stops when the system goes in and out of the quantum-mechanical tunnel. In the case of quantum mechanics of geometry in particular, this implies that all the components of the extrinsic curvature vanish at that moment. The result is that the possibilities of topology-changing processes by quantum tunneling are excluded for the space-time having a non-negative Ricci tensor. In particular, topology changes cannot occur in the case of a non-negative cosmological constant without matter. In the present paper we concentrate our investigation on the case of a negative cosmological constant.

We follow the same steps as in the previous paper [5]. In three dimensions, the number of independent components of the Riemann tensor is equal to that of the Ricci tensor. Therefore there are no gravitational-wave modes in (2+1)-dimensional gravity so that we have only to find out the appropriate three-dimensional compact negative constant-curvature spaces with totally geodesic boundaries. Once such three-dimensional manifolds are obtained, we can evaluate the topology-changing amplitude in the WKB approximation. We assume that the amplitude can be formally described by Hawking's Riemannian path integral as

$$T(i,f)=\sum_{M_R}\int \mathcal{D}g\,\exp(-S_E[g])\ , \tag{1}$$

where h_i and h_f are the two-dimensional metrics on the initial spacelike hypersurface $\Sigma^{(i)}$ and the final spacelike hypersurface $\Sigma^{(f)}$, respectively, and S_E is the Euclidean action

$$S_E=-\frac{1}{16\pi G}\int (R-2\Lambda)\sqrt{g}\; d^3x\ . \tag{2}$$

The path integral is over a smooth three-metric g on the Riemannian space-time manifold M_R which has appropriate boundaries $\Sigma^{(i)}$ and $\Sigma^{(f)}$ by assumption, and the summation over M_R means that we should also sum over different topologies of space-time M_R. Then we can use the solutions obtained to evaluate the path integral (1). As mentioned in the previous work, the classical action $\bar{S}_E$ is simply proportional to the volume of M_R:

$$\bar{S}_E = \frac{1}{4\pi G}\frac{V}{\sqrt{|\Lambda|}} , \tag{3}$$

where V is a numerical value representing the volume of M_R in the case of $\Lambda=-1$. The transition amplitude (1) is evaluated as

$$T(i,f) \sim \sum_g N_g \exp(-\bar{S}_E[g]) . \tag{4}$$

In Sec. II we explicitly construct an infinite number of different instantonlike solutions by playing the "gluing-many-polyhedra game" on the projective (Klein) model for hyperbolic geometry [6–8], whose rules are given in the previous paper [5]. Section III is devoted to summary and discussions.

II. CONSTRUCTION OF TOPOLOGY-CHANGING SOLUTIONS

In this section, we explicitly construct some topology-changing solutions.

In our previous paper [5], we showed that a compact orientable hyperbolic three-manifold with a totally geodesic boundary describes the nucleation of a universe from nothing in a (2+1)-dimensional gravity model with a negative cosmological constant in the case of no matter fields. In a similar way, we easily see that the topology change in space-time from a spatial surface $\Sigma^{(i)}$ to another spatial surface $\Sigma^{(f)}$ is described by a compact oriented hyperbolic three-manifold M with a totally geodesic boundary $\Sigma^{(i)} \cup \Sigma^{(f)}$. Here we do not restrict our consideration to the case that the boundary surfaces $\Sigma^{(i)}$ and $\Sigma^{(f)}$ are both connected.

Topology-changing solutions can be grouped according to the topology of their boundary. We denote topology-changing configurations with the boundary $\Sigma=\Sigma_{g1}\cup\Sigma_{g2}\cup\cdots\cup\Sigma_{gn}$ as a type $(g_1,g_2,\ldots,g_n)$ solution, where g_i is an integer greater than or equal to 2 and Σ_g represents a Riemann surface with genus $=g$. Note that there are many different solutions of the same type since this grouping pays attention only to the topology of the boundary. More precisely, any two solutions of the same type are distinct if the boundaries have different moduli. Even if the moduli of the boundaries are identical, interpolating three-manifolds can be different. Also note that there is no geometrical distinction between an "initial surface" and a "final surface." We can freely divide the components of the boundary into initial and final surfaces. For example, any type (g_1,g_2,g_3) solution can be interpreted as

$$\Sigma^{(i)} = \text{nothing} \rightarrow \Sigma^{(f)}=\Sigma_{g1}\cup\Sigma_{g2}\cup\Sigma_{g3} ,$$

$$\Sigma^{(i)}=\Sigma_{g1}\rightarrow\Sigma^{(f)}=\Sigma_{g2}\cup\Sigma_{g3} ,$$

$$\Sigma^{(i)}=\Sigma_{g1}\cup\Sigma_{g2}\rightarrow\Sigma^{(f)}=\Sigma_{g3} ,\text{etc.}$$

This gives rise to crossing relations for the topology-changing amplitudes.

We would like to describe the constructions of type (2,3) and type (2,2,2) solutions in detail, using the technique of hyperbolic geometry. Other solutions will be briefly discussed later.

(1) Type (2,3) solution.

The projective model is a model for the hyperbolic space constructed on an open three-disk $D^3 = \{(x,y,z)\in R^3 | x^2+y^2+z^2<1\}$, in which geodesics are Euclidean segments and totally geodesic surfaces are Euclidean planes.

We construct a type (2,3) solution by gluing two regular truncated octahedra embedded in the projective model D^3 appropriately.

First we embed a regular octahedron and the projective model D^3 into R^3 so that both of them would center around the origin. We can expand or contract the octahedron to let the angle between each pair of the faces of the octahedron to be $\pi/4$. See Fig. 1. The octahedron of this size has its vertices out of the sphere at infinity ∂D^3 and its edges intersect with D^3.

Next we truncate each vertex of the octahedron (Fig. 2). We pay attention to the four faces having a vertex in common. A remarkable property of the projective model guarantees the existence of a unique two-plane which is perpendicular to all of them. We cut the six vertices of the octahedron along these planes to get a regular truncated octahedron embedded completely in the projective model. This embedding induces a metric on the regular truncated octahedron.

We take two such regular truncated octahedra and identify each pair of their faces as Fig. 3 to obtain a space M. The edges bearing the same arrow are identified so that three distinct edges remain.

Now we verify that this gives a hyperbolic three-manifold with a totally geodesic boundary. First we claim that a neighborhood of any point of M is isometric to a ball in the hyperbolic space. This is of course true for an interior point of the regular truncated octahedra. Also for a point on the faces, it is true since the neighbor-

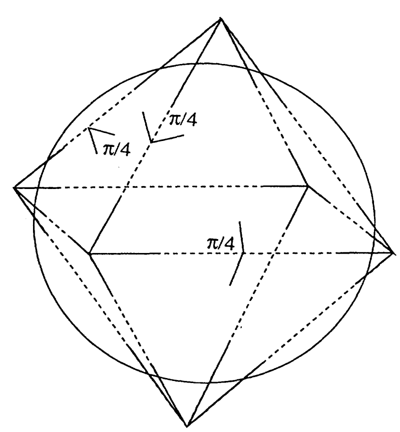


FIG. 1. A regular octahedron embedded in the projectile model D^3 so that all the dihedral angles are $\pi/4$. Note that its vertices are out of the sphere at infinity ∂D^3.

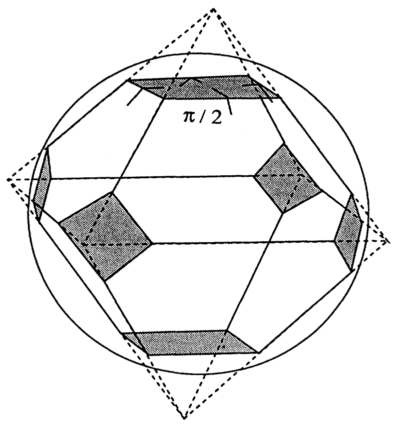


FIG. 2. Truncation of the regular octahedron in Fig. 1. Each vertex is truncated along a two-plane which is perpendicular to the four faces around the common vertex. This regular truncated octahedron entirely resides in D^3.

hood is divided into half spaces by the face. For the point on the edges, it is not so trivial. Note that the angles around the edge must add up to 2π. In fact, eight dihedral pieces with each dihedral angle being $\pi/4$ meet at each identified edge consistently as illustrated in Fig. 4. So the neighborhood of any point on the edges is also isometric to a ball in the hyperbolic space.

Next we claim that the boundary is totally geodesic. Since each square of the truncation is totally geodesically embedded in the projective model, we have only to check the gluing consistency condition along the edges and also around the vertices of the squares. Gluing along the edge causes no problem since the boundary intersects with the face of the octahedra perpendicularly at the edge. Gluing

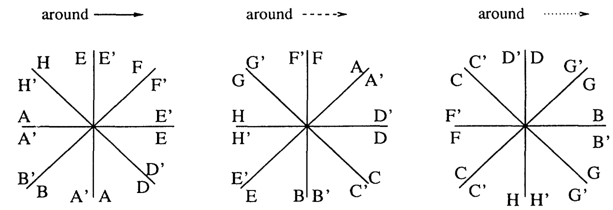


FIG. 4. Eight dihedral pieces with each dihedral angle $\pi/4$ meet at each identified edge consistently. The capital letters A and H and A' through H' correspond to the faces in Fig. 3.

around the vertex is harmless by the same reason as the gluing around the edge of the octahedra. It is also seen from the fact that near the vertex the boundary is a hypersurface which is perpendicular to the edge of the octahedra as shown in Fig. 5.

Finally we would like to show that the boundary is topologically a disjoint union of a double-torus ($g=2$ closed surface) and a triple-torus ($g=3$), i.e., this solution belongs to type (2,3). It can be verified that the boundary of M has two topological components Σ_{g1} and Σ_{g2}. The smaller component Σ_{g1} consists of four squares indicated by α in Fig. 3. The other component Σ_{g2} consists of the remaining eight squares indicated by β. We easily see that the smaller component is topologically a double torus by a "patch work," but the determination of the topology of the larger component is rather a tedious task. Here we take a shortcut. We have only to count the Euler characteristic χ of each component since we have already known that each component is a closed surface. In fact, it is easy to count χ because each component is naturally regarded as a polyhedron whose faces are squares. The smaller component has two vertices (the head of the plain arrow and the tail of the dotted arrow), eight edges (with each pair of sixteen edges being identified) and four faces. So

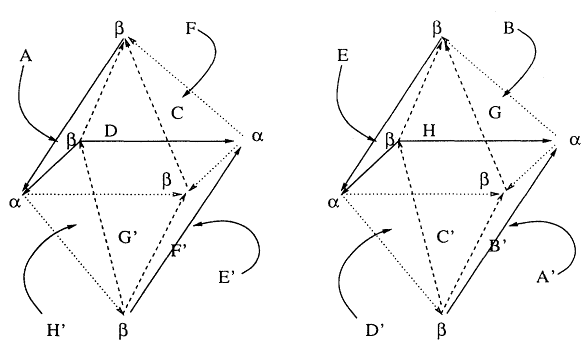


FIG. 3. Gluing two regular truncated octahedra of the same shape. Each face of an octahedron (e.g., A) is identified with another face of the same or the other octahedron (A') so as to match the three arrows on the face. The squares of truncation with the same Greek letter (α,β) gather to from a boundary component diffeomorphic to a double torus and a triple torus, respectively.

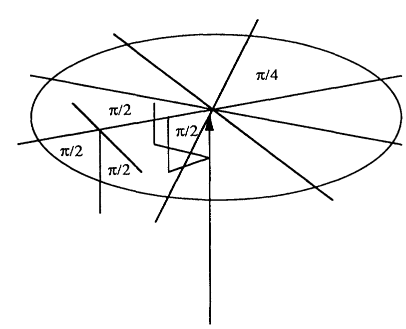


FIG. 5. Twelve squares of truncation of the two octahedra are glued to form two boundaries of a three-manifold. Gluing causes no problem at the boundary since each square of truncation is perpendicular to the corresponding four faces and four edges of the octahedron.

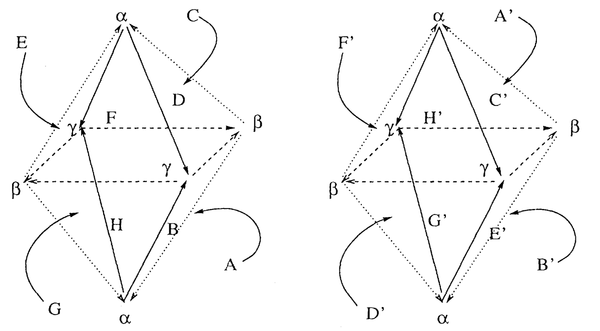


FIG. 6. Gluing two regular truncated octahedra of the same shape. Each face of an octahedron (e.g., A) is identified with another face of the same or the other octahedron (A') so as to match the three arrows on the face. The squares of truncation with the same Greek letter (α,β,γ) gather to form a boundary component diffeomorphic to a double torus.

$$\chi(\Sigma_{g1})=2+(-1)\times 8+(-1)^2\times 4=-2 . \quad (5)$$

In a similar way, we see

$$\chi(\Sigma_{g2})=4+(-1)\times 16+(-1)\times 8=-4 . \quad (6)$$

We conclude that $g_1=2$ and $g_2=3$.

This result is consistent with the Gauss-Bonnet theorem. The curvature of the boundary is constant and the larger component is twice as large as the smaller one in hyperbolic area.

(2) Type (2,2,2) solution.

We use the same regular truncated octahedra as building blocks but a different identification rule to get a type (2,2,2) solution. The new identification rule is shown in Fig. 6. Each eight edges of the twenty-four edges is identified and therefore there remains three distinct edges.

In this identification, the boundary has three components, which are illustrated in Fig. 7. Each square is cut into two triangles along the diagonal line (an arrow with a number) and the triangles are relocated to make an octagon. This means that each component is topologically a double torus. It is also seen that the gluing is consistent since the central points in the octagon correspond to the heads of three arrows. Moreover β and γ in Fig. 7 proved to be isometric. We are not certain whether α has the same moduli as β and γ. Though the diagram for α in Fig. 7 is different from those of β and γ, it is not easy to show that there is no isometry from α to β or γ.

We can obtain several other solutions by changing the identification rule.

(3) Other solutions.

We also find other tunneling solutions which express topology change or nucleation of universes. We will show only the outline of construction of these solutions.

First, we can find new solutions by using more general building blocks, i.e., an appropriate number of truncated polyhedra, while we used two octahedra in the previous subsections. In the previous paper [5], we already found that the "nothing to a double torus" [type (2)] solution is constructed by two regular truncated tetrahedra. Here we report "a double torus to a double torus" [type (2,2)] and "scattering of two double tori" [type (2,2,2,2)] solutions as illustrated in Fig. 8. They are constructed by two regular truncated hexahedra whose dihedral angle is $\pi/6$, and four regular truncated hexahedra whose dihedral angle is also $\pi/6$, respectively.

Second, we show an inductive procedure to get another solution with boundaries consisting of more general Riemann surfaces using particular solutions obtained by the "gluing-many-polyhedra game." (1) We prepare $n\times$ (one of solutions composed of several polyhedra). (2) We make an expansion so that the dihedral angle of the regular polyhedra becomes an nth of the dihedral angle of the original regular polyhedra. (3) Let us pick up two of such manifolds as obtained by the steps (1) and (2) and connect them in the following way. Find a face which satisfied the following two conditions.

Condition 1: The face contains all kinds of edges.

Condition 2: Any pair of edges in the face is not identified.

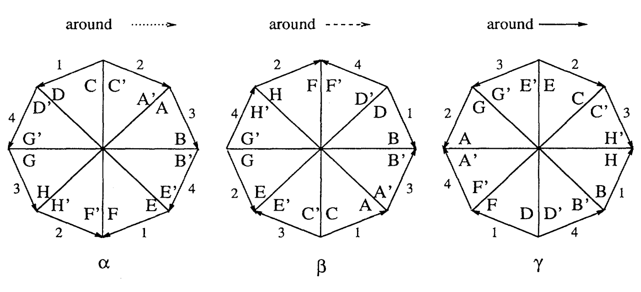


FIG. 7. Eight dihedral pieces with each dihedral angle $\pi/4$ meet at each identified edge consistently. The capital letters A through H and A' through H' correspond to the faces in Fig. 6. Three octagons correspond to three boundary surfaces α,β,γ. Each square of truncation is cut into two triangles along the diagonal line (an arrow with a number) and the triangles are relocated to make an octagon. This means that each component is topologically a double torus.

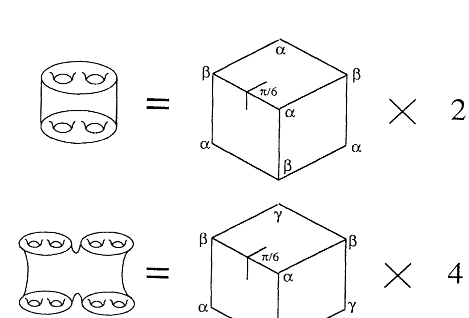


FIG. 8. A type (2,2) solution (illustrated schematically) is constructed by two regular truncated hexahedra whose dihedral angle is $\pi/6$. A type (2,2,2,2) solution is constructed by four regular truncated hexahedra whose dihedral angle is also $\pi/6$.

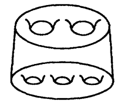
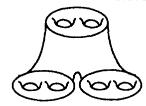
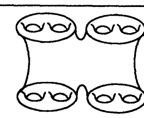
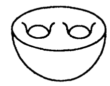
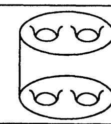
Suppose there exists such a face. This face has been identified to another face in a single solution by a specified gluing rule. Unglue the previously identified two faces to obtain a new object (this is in fact a hyperbolic manifold with a boundary). Prepare two such objects. Then we have two pairs of unglued faces. Interchange the partners and glue the new pairs to get a double covering space. (4) Obviously there exist at least two such faces in the double covering space that satisfy the conditions 1 and 2. See Fig. 9. Therefore we can proceed further to successively combine $3,4,\ldots,n$ such objects.

Recalling that the dihedral angle of a truncated polyhedra for this new solution is an nth of that of the original polyhedra, we can see that the sum of the angles meeting at an identified edge becomes 2π. This procedure gives a hyperbolic three-manifold with totally geodesic boundaries as we shall see below. The number of its boundary components is the same as the one of the original single solution. By computing the Euler characteristics, the genus of its boundaries can be found as shown later. We note that our procedure gives an n-fold covering. The conditions 1 and 2 constitute a sufficient condition for the n-fold covering to admit a hyperbolic structure.

At present, all the solutions that we have found contain at least one face which satisfies the conditions 1 and 2 except for the cases of type (2) and type (2,2) solutions. In these two cases there are no such faces that satisfy the condition 2. By this procedure, for even m ($m=2n, n=1,2,\ldots$), the type $(m,2m-1)$, type (m,m,m), and type (m,m,m,m) solutions are constructed from the type (2,3), type (2,2,2), and type (2,2,2,2) solutions, respectively.

We can also obtain solutions for odd m by preparing solutions composed of "1.5×(polyhedra composing the single solutions)." For example, we can construct type

TABLE I. We classify our solutions. In the first column, the solutions are schematically illustrated. In the other columns, the types of the descendant solutions are shown.

special solution	type of covering space	"×1.5" solution	type of generalized solution
Type (2,3) 	(m,2m-1)	(3,5)	(n+1,2n+1)
Type (2,2,2) 	(m,m,m)	(3,3,3)	(n+1,n+1,n+1)
Type (2,2,2,2) 	(m,m,m,m)	(3,3,3,3)	(n+1,n+1 ,n+1,n+1)
Type (2) 	(m)	(3)	(n+1)
Type (2,2) 	(4n-2,4n-2)	Impossible	(4n-2,4n-2)

m=2n, n=1,2,3....

(a)

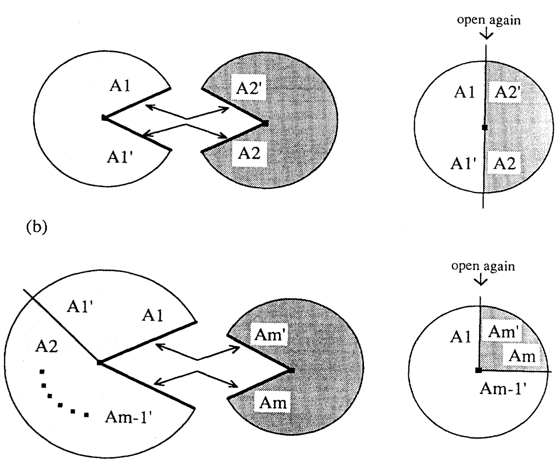


FIG. 9. (a) We take two solutions and unglue the two faces (which should satisfy the conditions 1 and 2). Then we interchange the partners and glue the new pairs to get a double covering space. (b) We can continue the above procedure to get an n-fold covering space of the original solution. This covering is naturally furnished with a hyperbolic structure.

TABLE II. The volumes of some simple solutions are shown. These numerical values are calculated in the case of $\Lambda=-1$.

type of solution	building blocks	dihedral angle	volume
(2)	two tetrahedra	$\pi/6$	6.452
(3)	three tetrahedra	$\pi/9$	10.429
(2,3)	two octahedra	$\pi/4$	21.304
(3,5)	three octahedra	$\pi/6$	34.346
(2,2,2)	two octahedra	$\pi/4$	21.304
(3,3,3)	three octahedra	$\pi/6$	34.346
(2,2,2,2)	four hexahedra	$\pi/6$	43.263
(3,3,3,3)	six hexahedra	$\pi/9$	69.100
(2,2)	two hexahedra	$\pi/6$	21.632
(6,6)	six hexahedra	$\pi/12$	70.509

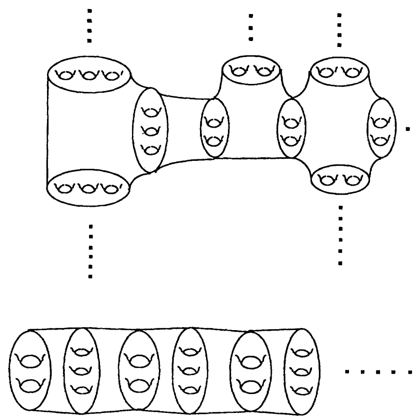


FIG. 10. Plumbing the solutions.

(3,3,3) by three octahedra knowing that two octahedra give type (2,2,2). Again this method does not work for the type (2,2) solution.

Using less systematic prescriptions, we have constructed type (n) and type $(4n-2,4n-2)$ solutions. We do not give details of the constructions. Table I summarizes all our results.

Finally, by plumbing the solutions in Table I, we can construct more and more complicated solutions (Fig. 10) with increasingly large volumes. There are infinite number of solutions of the same type with the same moduli of boundaries. This plumbing is performed by joining the solutions across a pair of their boundaries which have the same moduli. This is possible because the boundaries have vanishing extrinsic curvature.

For the volumes of typical solutions, see Table II [9].

III. SUMMARY AND DISCUSSIONS

We have studied topology-changing processes in the $(2+1)$-dimensional quantum gravity with a negative cosmological constant in the WKB approximation.

Following the recent work by Gibbons and Hartle, we have described the cosmological tunneling process as a transition from the Euclidean to Lorentzian signature regions. The boundary surface turns out to be totally geodesic.

We have shown various explicit examples of hyperbolic three-manifolds with totally geodesic boundaries which can be physically interpreted as topology-changing processes or multiple production of universes by quantum tunneling. More precisely, we have constructed the three-manifolds with the boundaries being three $g=2$ surface, $g=2$ and $g=3$ surfaces, etc., by gluing polyhedra in the projective model of hyperbolic geometry. Further we have shown a definite prescription to construct n-fold coverings of the hyperbolic three-manifolds obtained above, if certain conditions are met. By plumbing obtained solutions, we can find an infinite number of hyperbolic three-manifolds with totally geodesic boundaries, though we have not exhausted all the solutions.

These solutions describe various tunneling processes which exhibit topology change or multiple production of universes. The exponential of the minus of volume of the constructed hyperbolic three-manifold (Table II) essentially gives the corresponding tunneling amplitude in the WKB approximation [Eqs. (3) and (4)]. It is interesting to point out that in general the dynamical degrees of freedom (number of moduli parameters) change in the tunneling processes.

What is left for future investigation is the moduli structure of the boundary surfaces. The Mostow rigidity theorem [6] implies that there should be a kind of selection rule for the topology-changing transitions of universes or restriction on the topology and geometry of the universes born from nothing. The last statement may suggest that the Hartle-Hawking "no boundary condition" [10] strongly restrict the variety of the initial universes which partially solve the initial condition problem in the $(2+1)$-dimensional cosmology.

However, as remarked in our previous paper [5], the rigidity theorem heavily relies on the three-dimensional hyperbolic geometry and therefore it is highly nontrivial to generalize our discussion in the $(2+1)$-dimensional space-time to the one in the physical $(3+1)$-dimensional space-time.

The stability of our instanton solutions is an open question. At least we can say that the rigidity of hyperbolic three-manifolds implies the absence of zero modes.

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