Acceleration-free spherically symmetric inhomogeneous cosmological model with shear viscosity

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Some new exact solutions to the Einstein equations with an acceleration-free imperfect-fluid source are obtained. Some physical restrictions on the solutions are discussed. Cosmological models built out of these solutions are found to have increasing entropy per baryon and not possess any flatness problem.

I. INTRODUCTION

This is the second in a series of two papers in which we explore imperfect-fluid cosmological models and obtain some new exact solutions to them. The models that we study have the feature that while the underlying associated geometry is in general an inhomogeneous one, the models nonetheless evolve so that at late times the inhomogeneities die out and the Universe becomes the familiar highly symmetric Friedmann-Robertson-Walker one of today. To a present-day observer our models are indistinguishable from the standard one and must hence be regarded as observationally viable. However, since each of our models has a history which is different from that of the standard model, our models do not all suffer from some of the familiar difficulties (horizon, entropy, and flatness problems) which the standard model possesses. They thus provide for potentially interesting cosmologies. In our accompanying companion paper [1] we study shear-free cosmological models with heat flow and bulk viscosity and in the present paper we study acceleration-free ones with shear viscosity. In all the cases the specific non-perfect-fluid terms are found to lead to some interesting implications for cosmology. For a complete discussion of our motivation, and for our formulation of the problem and its notation we refer the reader to Ref. [1].

In this particular paper we study Einstein's field equations with an acceleration-free imperfect-fluid source with shear viscosity coefficient $\eta(r,t)$ and associated energy-momentum tensor [$\rho(r,t)$ and $p(r,t)$ are the standard energy density and pressure of the fluid]

$$T_{\mu\nu}=\rho U_\mu U_\nu+pH_{\mu\nu}-2\eta\sigma_{\mu\nu}\ , \tag{1.1}$$

where

$$H_{\mu\nu}=g_{\mu\nu}+U_\mu U_\nu\ , \tag{1.2}$$

$$2\sigma_{\mu\nu}=H_\mu{}^\alpha H_\nu{}^\beta(U_{\alpha;\beta}+U_{\beta;\alpha}-\tfrac{2}{3}g_{\alpha\beta}U^\gamma{}_{;\gamma})\ , \tag{1.3}$$

and where U_μ is the four-velocity of the fluid. We take the geometry to be spherically symmetric about a single point, and thus isotropic but not homogeneous at arbitrary times. Further, we take the geometry to be acceleration-free (viz. $U^\alpha U_{\beta;\alpha}=0$), so that the most general admissible metric then takes the form

$$ds^2=-dt^2+e^{2\lambda}dr^2+Y^2(d\theta^2+\sin^2\theta\, d\phi^2)\ , \tag{1.4}$$

where λ and Y are functions of r and t only; while the fluid four-velocity vector itself then simplifies to

$$U_\mu=(-1,0,0,0) \tag{1.5}$$

so that the fluid is comoving with the geometry. In this geometry the Einstein equations take the form

$$\kappa\rho=\frac{1}{Y^2}+2\dot\lambda\frac{\dot Y}{Y}+\frac{\dot Y^2}{Y^2}-e^{-2\lambda}\left[2\frac{Y''}{Y}-2\lambda'\frac{Y'}{Y}+\frac{Y'^2}{Y^2}\right]\ , \tag{1.6}$$

$$\kappa\left[p-\frac{4}{3}\eta\left[\dot\lambda-\frac{\dot Y}{Y}\right]\right]=-\frac{1}{Y^2}-2\frac{\ddot Y}{Y}-\frac{\dot Y^2}{Y^2}+e^{-2\lambda}\frac{Y'^2}{Y^2}\ , \tag{1.7}$$

$$\kappa\left[p+\frac{2}{3}\eta\left[\dot\lambda-\frac{\dot Y}{Y}\right]\right]=-\ddot\lambda-\dot\lambda^2-\dot\lambda\frac{\dot Y}{Y}-\frac{\ddot Y}{Y}+e^{-2\lambda}\left[\frac{Y''}{Y}-\lambda'\frac{Y'}{Y}\right]\ , \tag{1.8}$$

$$\frac{\dot Y'}{Y}-\dot\lambda\frac{Y'}{Y}=0\ , \tag{1.9}$$

where κ denotes the quantity $8\pi G$, so that the Bianchi identities impose the following two constraints on the fluid:

$$\dot\rho+\left[\dot\lambda+2\frac{\dot Y}{Y}\right](\rho+p)=\frac{4}{3}\eta\left[\dot\lambda-\frac{\dot Y}{Y}\right]^2 \tag{1.10}$$

and

$$p'=\frac{4}{3}\frac{\partial}{\partial r}\left[\eta\left[\dot\lambda-\frac{\dot Y}{Y}\right]\right]+4\eta\left[\dot\lambda-\frac{\dot Y}{Y}\right]\frac{Y'}{Y}\ . \tag{1.11}$$

(In passing we note that because of the conservation of the energy-momentum tensor we find that unlike the perfect-fluid case where some acceleration is necessary to support a pressure gradient, for an imperfect fluid this gradient may be supported by the viscosity instead.)

In attempting to find solutions to the Einstein equations we note first that the integration of Eq. (1.9) is straightforward, yielding

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$$e^{\lambda}=\left|\frac{Y'}{\alpha(r)}\right| , \tag{1.12}$$

where $\alpha(r)$ is an integration function. Following the general approach we described in Ref. [1] we impose the physically motivated boundary condition that the metric asymptotically approach a Robertson-Walker one. Taking this asymptotic metric to be of the form $Y=re^{\lambda}$, $e^{\lambda}=R(t)/(1+kr^2/4)$ (where $k=-1,0,1$) in an isotropic coordinate system then requires the function $\alpha(r)$ to be of the form

$$\alpha_k(r)=\frac{1-kr^2/4}{1+kr^2/4} \tag{1.13}$$

at all times. Next, eliminating λ from Eqs. (1.6)–(1.8) by the use of Eq. (1.12) yields

$$\kappa\rho=\frac{1}{Y^2Y'}\frac{\partial}{\partial r}[Y(1-\alpha^2)+\dot{Y}^2Y] , \tag{1.14}$$

$$\kappa\left[p-\tfrac{4}{3}\eta\left[\frac{\dot{Y}'}{Y'}-\frac{\dot{Y}}{Y}\right]\right]=-\frac{1}{Y^2}[1-\alpha^2+2\ddot{Y}Y+\dot{Y}^2] , \tag{1.15}$$

$$\kappa\left[p+\tfrac{2}{3}\eta\left[\frac{\dot{Y}'}{Y'}-\frac{\dot{Y}}{Y}\right]\right]=-\frac{1}{2YY'}\frac{\partial}{\partial r}[1-\alpha^2+2\ddot{Y}Y+\dot{Y}^2]. \tag{1.16}$$

In order to solve these equations, we need to specify an equation of state. For the early radiation-dominated Universe we take, following the procedure described in Ref. [1], the local version of the standard equation of state, viz.,

$$\rho(r,t)=3p(r,t) , \tag{1.17}$$

while for the matter-dominated Universe we take

$$p(r,t)=0 . \tag{1.18}$$

With the use of Eq. (1.17) or Eq. (1.18), Eqs. (1.14)–(1.16) can then be solved with the functional dependence of $\eta(r,t)$ on space and time then being obtained as part of the solution (as a consequence of satisfying our required asymptotic boundary condition) rather than its being some *a priori* prescribed function of $\rho(r,t)$ and $p(r,t)$. In this paper we shall concentrate on the early Universe and shall study the radiation-dominated case of Eq. (1.17) to find some new exact inhomogeneous solutions which we present in Secs. II and III.

Ideally, to solve our model completely we would need, as already noted in Ref. [1], to study the imperfect-fluid Einstein equations in conjunction with the general-relativistic Boltzmann transport equation in order to obtain closed-form expressions for the transport coefficients which appear in Eq. (1.1). For the moment this is a totally prohibitive problem. In the absence of any such solutions we have opted both here and in Ref. [1] to take a much more limited approach and instead try to find solutions simply by imposing a boundary condition on the Einstein equations, namely, that our models each evolve into the standard cosmology at late times. Thus the models which we present in this series of papers are to be considered as illustrative of the kind of behavior which could possibly be expected in general imperfect-fluid cosmological models. Since the solutions that we present are exact they allow us to study in some detail the time history of the Universe so that we can obtain (Sec. IV) a reasonably clear picture of how it is at least in principle possible to avoid some or even all of the familiar problems of the standard cosmological model.

II. EXACT INHOMOGENEOUS SOLUTIONS

Because of Eq. (1.17) and, because of our simplifying choice of vanishing bulk-viscosity coefficient, the fluid energy-momentum tensor is traceless. This tracelessness condition then leads to

$$\frac{\partial}{\partial r}\left[Y(1-\alpha^2)+Y\frac{\partial}{\partial t}(Y\dot{Y})\right]=0 , \tag{2.1}$$

so that

$$Y\left[1-\alpha^2+\frac{\partial}{\partial t}(Y\dot{Y})\right]=\beta(t) , \tag{2.2}$$

where $\beta(t)$ is an arbitrary function of t. In general it is quite hard to find exact solutions to our model especially given our asymptotic boundary condition that our metric approach a Roberston-Walker form at large times. However, we have found some classes of solutions in certain simplified cases which we shall report on in this paper.

To begin with, we study first the most simple case in which our metric is to asymptotically approach a flat ($k=0$) Robertson-Walker metric. Then the integration function associated with Eq. (1.12) is given by the very simple form $\alpha(r)=1$. Further, in this case Eq. (2.2) reduces to

$$Y\frac{\partial}{\partial t}(Y\dot{Y})=\beta(t) . \tag{2.3}$$

This equation can be solved for some particular choices of $\beta(t)$. We look for solutions of the form

$$Y=[f_1(r)g_1(t)+f_2(r)g_2(t)]^{2/3} , \tag{2.4}$$

where f_1 and f_2 are not proportional to each other and nor are g_1 and g_2. We impose this particular requirement since Y would otherwise be a separable function of r and t which would lead to a metric that is either Robertson-Walker at all times or which never approaches a Roberston-Walker metric at all. Inserting Eq. (2.4) into Eq. (2.3) yields an equation that can be satisfied in a finite number of inequivalent ways with the function $\beta(t)$ being restricted to taking only certain specific forms. With our additional asymptotic requirement that the metric becomes Roberston-Walker at large times we obtain four different solutions for the function $Y(r,t)$ which are of interest to cosmology. (In this paper we only present the solutions to the equations that we have found, and we refer the reader to Ref. [2] for details of the derivation.) Substituting the four respective expressions for $Y(r,t)$ into Eqs. (1.12) and (1.14)–(1.16) we then obtain expressions for e^{λ}, ρ, and η in each of the four cases. The four sets of solutions are listed below.

Case (i)

$$Y=A^{1/2}[\tfrac{1}{2}(r^{3/2}-\sqrt{r^3+r_0^3}\,)(t+t_0)^{3/4}+\tfrac{1}{2}(r^{3/2}+\sqrt{r^3+r_0^3}\,)t^{3/4}]^{2/3}\,, \tag{2.5}$$

$$e^{\lambda}=\frac{(Ar)^{1/2}}{\sqrt{r^3+r_0^3}}\,\frac{\dfrac{-r^{3/2}+\sqrt{r^3+r_0^3}}{2}(t+t_0)^{3/4}+\dfrac{r^{3/2}+\sqrt{r^3+r_0^3}}{2}t^{3/4}}{\left|\dfrac{r^{3/2}-\sqrt{r^3+r_0^3}}{2}(t+t_0)^{3/4}+\dfrac{r^{3/2}+\sqrt{r^3+r_0^3}}{2}t^{3/4}\right|^{1/3}}\,, \tag{2.6}$$

$$k\rho=\frac{3}{4t^2}\,\frac{(r^{3/2}-\sqrt{r^3+r_0^3}\,)^2\left[1+\dfrac{t_0}{t}\right]^{-1/2}-(r^{3/2}+\sqrt{r^3+r_0^3}\,)^2}{(r^{3/2}-\sqrt{r^3+r_0^3}\,)^2\left[1+\dfrac{t_0}{t}\right]^{3/2}-(r^{3/2}+\sqrt{r^3+r_0^3}\,)^2}\,. \tag{2.7}$$

In this first case we find that there is a hypersurface represented by

$$\left[1+\frac{t_0}{t}\right]^{3/4}=\left[\frac{r^{3/2}+\sqrt{r^3+r_0^3}}{r_0^{3/2}}\right]^2 \tag{2.8}$$

on which Y is zero and e^{λ} is infinite. This surface thus corresponds to a singularity in the metric. The divergence of the energy density on the surface suggests that it is a real physical singularity which can be viewed as a big bang, an inhomogeneous big bang which occurs at different times for different locations. The surface given by Eq. (2.8) is that of a shrinking sphere located where the singularity is. The big bang takes place the earliest at spatial infinity and progresses toward the center. It will not take place in the neighborhood of $r=0$ until t approaches infinity. Thus we have a incomplete physical spacetime. Outside the shrinking "fire ball" the spacetime is regular and physical, on the sphere there is a singularity, and inside the sphere the physical spacetime does not exist (the energy density is negative there). Because of the singularity, an insurmountable barrier, the inside region is unattainable to the physical world outside and, therefore, does not cause us any concern. At any given spatial point the big bang occurs at the time given by Eq. (2.8). Afterwards, the metric approaches the Robertson-Walker one as $t\to\infty$. This is an example of spacetime which is very inhomogeneous at very early times and then gradually becomes homogeneous.

Case (ii)

$$Y=A^{1/2}[\tfrac{1}{2}(r^{3/2}-\sqrt{r^3-r_0^3}\,)(t+t_0)^{3/4}+\tfrac{1}{2}(r^{3/2}+\sqrt{r^3-r_0^3}\,)t^{3/4}]^{2/3}\,, \tag{2.9}$$

$$e^{\lambda}=\frac{(Ar)^{1/2}}{\sqrt{r^3-r_0^3}}\,\frac{\left|\dfrac{-r^{3/2}+\sqrt{r^3-r_0^3}}{2}(t+t_0)^{3/4}+\dfrac{r^{3/2}+\sqrt{r^3-r_0^3}}{2}t^{3/4}\right|}{\left[\dfrac{r^{3/2}-\sqrt{r^3-r_0^3}}{2}(t+t_0)^{3/4}+\dfrac{r^{3/2}+\sqrt{r^3-r_0^3}}{2}t^{3/4}\right]^{1/3}}\,, \tag{2.10}$$

$$k\rho=\frac{3}{4t^2}\,\frac{(r^{3/2}-\sqrt{r^3-r_0^3}\,)^2\left[1+\dfrac{t_0}{t}\right]^{-1/2}-(r^{3/2}+\sqrt{r^3-r_0^3}\,)^2}{(r^{3/2}-\sqrt{r^3-r_0^3}\,)^2\left[1+\dfrac{t_0}{t}\right]^{3/2}-(r^{3/2}+\sqrt{r^3-r_0^3}\,)^2}\,. \tag{2.11}$$

In the second case the coordinate r is restricted to the range $r_0<r<\infty$. Thus the "origin" is at $r=r_0$ instead of at $r=0$. As in case (i) there is a big-bang hypersurface described by

$$\left[1+\frac{t_0}{t}\right]^{3/4}=\left[\frac{r^{3/2}+\sqrt{r^3-r_0^3}}{r_0^{3/2}}\right]^2 \tag{2.12}$$

on which e^{λ} vanishes, giving rise to a big-bang singularity. The behavior of ρ is similar to that of case (i).

For each of the above two cases the function $\beta(t)$ takes the form

$$\beta(t)=\frac{r_0^3t_0^2A^{3/2}}{32t^{5/4}(t+t_0)^{5/4}} \tag{2.13}$$

up to a case-dependent overall plus or minus sign, with the coefficient of shear viscosity η then being obtained from Eqs. (1.15) and (1.16). The resulting η turns out to be independent of the spatial coordinate r and is therefore the same in both the cases being of the form

$$\eta(t)=\frac{1}{8\kappa}\left[\frac{1}{t+t_0}+\frac{1}{t}\right] . \tag{2.14}$$

The shear-viscosity coefficient is thus nicely positive as it should be (see Sec. IV) and asymptotically approaches zero as $t\rightarrow\infty$, a property which we expect $\eta(t)$ to possess since we impose the asymptotic condition that our solutions approach the Roberston-Walker form. These properties are also shared by the expressions for the shear-viscosity coefficient η which follow in the next two sets of solutions.

Case (iii)

$$Y=A^{1/2}(r^{3/2}t^{3/4}+b/r^{3/2})^{2/3} , \tag{2.15}$$

$$e^{\lambda}=\frac{A^{1/2}}{r}\frac{|r^{3/2}t^{3/4}-b/r^{3/2}|}{|r^{3/2}t^{3/4}+b/r^{3/2}|^{1/3}} , \tag{2.16}$$

$$\kappa\rho=\frac{3}{4t^2}\frac{t^{3/2}}{t^{3/2}-b^2/r^6} , \tag{2.17}$$

$$\eta=\frac{1}{8\kappa t} , \tag{2.18}$$

$$\beta(t)=-\frac{b}{8}A^{3/2}t^{-5/4} . \tag{2.19}$$

In the third case there is again a singularity hypersurface,

$$t^{3/2}=\frac{b^2}{r^6} , \tag{2.20}$$

on which either Y (when $b<0$) or e^{λ} (when $b>0$) vanishes and on which ρ diverges. The surface is also a shrinking sphere outside of which, or to the future of which, the physical spacetime lies. The coefficient of shear viscosity is found to be positive and independent of the spatial coordinates.

Case (iv)

$$Y=A^{1/2}(r^{3/2}t^{3/4}+b/t^{1/4})^{2/3} , \tag{2.21}$$

$$e^{\lambda}=\frac{(Ar)^{1/2}t^{3/4}}{|r^{3/2}t^{3/4}+b/t^{1/4}|^{1/3}} , \tag{2.22}$$

$$\kappa\rho=\frac{3}{4t^2}\frac{r^{3/2}-b/3t}{r^{3/2}+b/t} , \tag{2.23}$$

$$\eta=\frac{1}{4\kappa t} , \tag{2.24}$$

$$\beta(t)=\tfrac{2}{9}b^2A^4(At)^{-5/2} . \tag{2.25}$$

In the fourth case we see that for $b>0$ there is no singularity for $t>0$ and ρ is negative for small t. This is thus an unphysical case. For $b<0$ however this case becomes similar to cases (i), (ii), and (iii). Then a physical singularity is found on the surface

$$r^{3/2}=-\frac{b}{t}=\frac{|b|}{t} \tag{2.26}$$

and represents an inhomogeneous big bang.

With regard to our entire set of solutions we note that in the region outside of the singularity sphere all four of our solutions not only have positive energy density, which is essentially the requirement of the "weak energy condition," but our solutions actually satisfy the stronger "dominant energy condition" wherein the scalar quantity $W_\mu T^{\mu\nu}W_\nu$ is always positive and the four-vector $T^{\mu\nu}W_\nu$ is always nonspacelike for any timelike vector W_μ we may choose. Moreover, the contribution of the shear viscosity to the energy-momentum tensor attenuates faster than the energy density itself, indicating that the viscosity contribution dies out as $t\rightarrow\infty$. For example, in solution (iv) we have the relation

$$\eta\left[\dot{\lambda}-\frac{\dot{Y}}{Y}\right]=\frac{b}{4\kappa t^3}\frac{1}{r^{3/2}+b/t} . \tag{2.27}$$

Thus both sides of Eq. (2.27) behave like t^{-3} even while ρ goes like t^{-2} as $t\rightarrow\infty$. The components of the energy-momentum tensor $T_1{}^1$ and $T_2{}^2$ can also be obtained:

$$T_1{}^1=p-\tfrac{4}{3}\eta\left[\frac{\dot{Y}'}{Y'}-\frac{\dot{Y}}{Y}\right]=\frac{1}{4\kappa t^2}\frac{r^{3/2}-2b/t}{r^{3/2}+b/t} , \tag{2.28}$$

$$T_2{}^2=p+\tfrac{2}{3}\eta\left[\frac{\dot{Y}'}{Y'}-\frac{\dot{Y}}{Y}\right]=\frac{1}{4\kappa t^2}\frac{r^{3/2}+b/3t}{r^{3/2}+b/t} . \tag{2.29}$$

With these expressions it can readily be checked that the "dominant energy condition," which can be written as $\rho>|T_1{}^1|$ and $\rho>|T_2{}^2|$ in our case, is indeed satisfied. Therefore, these solutions can be considered as candidate physical models. All these solutions share a common feature, namely, that they have a center which is unreachable from the physical region. Thus, the Universe is very inhomogeneous and there is an inhomogeneous big bang. The observed Universe, which is a small part of the complete spacetime manifold, thus has to locate far enough from the center so that the observable region is presently homogeneous.

To conclude this section we note one further feature of our solutions. Specifically, all of our metrics not only approach the Robertson-Walker one as $t\rightarrow\infty$ but they also do so as $r\rightarrow\infty$. In other words, they are spatially asymptotically of the Robertson-Walker type just as the Schwarzschild solution is spatially asymptotically of the Minkowski type.

III. INHOMOGENEOUS SOLUTIONS WITH $\beta(t)$ BEING ZERO

In the preceding two sections solutions were obtained for various choices of the integration function $\beta(t)$. The solutions that we found all possess singularities near the center of coordinates so that the center has to be excluded from the physical spacetime. In each case the entire spacetime metric never completely evolves into a Roberston-Walker geometry, only the region far enough away from the center actually attains such uniformity. In this section we return again to Eq. (2.2) and study another soluble case, the one where $\beta(t)=0$. This time we are able to obtain exact solutions for each of the three spatial geometries ($k=-1,0,1$) associated with Eq. (1.13). In contrast with our previous solutions this time we will find exact solutions which do uniformly approach the Roberston-Walker one with there now being no spatially singular point at all except at very early times.

To obtain these additional solutions we note first that with the use of Eq. (1.13), Eq. (2.2) may now be written as

$$\frac{kr^2}{(1+kr^2/4)^2}+\frac{\partial}{\partial t}(Y\dot{Y})=0 \tag{3.1}$$

when $\beta(t)$ is zero. The most general solution to Eq. (3.1) is

$$Y=\frac{r}{1+kr^2/4}[-kt^2+a(r)t+b(r)]^{1/2}\ . \tag{3.2}$$

Note that in general a and b are functions of r. If a and b were both independent of r the solution would simply be the Roberston-Walker one at all times. We are thus interested in cases where at least one of them is not constant. We now study such cases for the various different values of k.

A. Zero-curvature case

In this case $k=0$ and $\alpha_0(r)=1$. In order for Y to approach the Roberston-Walker form asymptotically $a(r)$ must be a constant. Thus we have in general

$$Y(r,t)=r[A(t-B(r))]^{1/2}\ , \tag{3.3}$$

where A is a constant. From Eqs. (1.12) and (1.14)–(1.16) we obtain

$$e^{\lambda}=\frac{A^{1/2}}{\sqrt{t-B(r)}}\left|t-B(r)-\tfrac{1}{2}rB'(r)\right|\ , \tag{3.4}$$

$$\kappa\rho=\frac{3}{4}\frac{1}{[t-B(r)]^2}\frac{t-B(r)+\dfrac{rB'(r)}{6}}{t-B(r)-\dfrac{rB'(r)}{2}}\ , \tag{3.5}$$

$$\eta=\frac{1}{4\kappa}\frac{1}{t-B(r)}\ . \tag{3.6}$$

Equation (3.5) shows that the energy density has a double pole at $t=B(r)$, a single pole at $t=B(r)+rB'(r)/2$ and also a zero at $t=B(r)-rB'(r)/6$. To have a physical model we need to choose the pole at the later time to be the big-bang singularity so that there is then no singularity after the big bang. If $rB'(r)$ is less than zero the big bang should occur at time $t=B(r)$. But the numerator could then be negative at a t near $B(r)$. This case is thus ruled out. Rather, we have to set

$$rB'(r)\geq 0 \tag{3.7}$$

and the big bang occurs at

$$t=t_0(r)\equiv B(r)+\tfrac{1}{2}rB'(r)\geq B(r)\ . \tag{3.8}$$

With Eqs. (3.7) and (3.8) it can again be shown that the dominant energy condition is satisfied.

Just as in Sec. II we again have an inhomogeneous big bang. So far though no specification of $B(r)$ has been given other than Eq. (3.7) which only requires that $B(r)$ be monotonically increasing. Thus practically one could choose almost any function $B(r)$. The following represent some typical purely illustrative examples of physically motivated forms for the function $B(r)$ that one might actually choose, and their significance lies in the fact that our ability to choose this broad class of functions at all indicates that, unlike the situation in the standard model, our model requires no fine-tuning of parameters or initial conditions.

(i) We might want to choose B such that the big bang takes place uniformly, i.e., so that $t_0(r)=0$. Then we could take

$$B(r)+\tfrac{1}{2}rB'(r)=0\ , \tag{3.9}$$

so that we obtain

$$B(r)=-\frac{b}{r^2}\ , \tag{3.10}$$

where b has to be positive according to Eq. (3.7) which then makes η nicely positive according to Eq. (3.6). With the use of Eq. (3.10), Eqs. (3.3)–(3.5) then reduce to

$$Y(r,t)=r[A(t+b/r^2)]^{1/2}\ , \tag{3.11}$$

$$e^{\lambda}=\frac{A^{1/2}t}{(t+b/r^2)^{1/2}}\ , \tag{3.12}$$

$$\kappa\rho=\frac{3}{4t}\frac{t+4b/3r^2}{(t+b/r^2)^2}\ . \tag{3.13}$$

The big bang thus takes place uniformly at $t=0$ as required. However, we note that the metric is also singular at $r=0$ where it never approaches the Robertson-Walker form. This is then a situation similar to the ones previously discussed in Sec. II.

(ii) A second possibility is to try to find a $B(r)$ that is bounded and nowhere singular throughout the entire three-space. The metric would then uniformly approach the Roberston-Walker one as $t\to\infty$. But the big bang would now have to occur nonuniformly. Accordingly, we would have a big bang which is space dependent. After enough time has elapsed to allow the big bang to take place at every possible spatial point, the metric would then become nonsingular everywhere. A simple example of such $B(r)$ which actually does all this is given by

$$B(r)=\frac{br^2}{r^2+r_0^2} \tag{3.14}$$

which satisfies Eq. (3.7) provided the constant b is positive. With the use of Eq. (3.14) we find

$$Y=A^{1/2}r\left[t-\frac{br^2}{r^2+r_0^2}\right]^{1/2}\ , \tag{3.15}$$

$$e^{\lambda}=\frac{A^{1/2}}{\left[t-\dfrac{br^2}{(r^2+r_0^2)}\right]^{1/2}}\left[t-\frac{br^2(r^2+2r_0^2)}{(r^2+r_0^2)^2}\right]\ , \tag{3.16}$$

$$\kappa\rho=\frac{3}{4}\frac{1}{\left[t-\dfrac{br^2}{(r^2+r_0^2)}\right]^2}\frac{t-\dfrac{br^2(r^2+\frac{2}{3}r_0^2)}{(r^2+r_0^2)^2}}{t-\dfrac{br^2(r^2+2r_0^2)}{(r^2+r_0^2)^2}} . \tag{3.17}$$

The big bang thus takes place at

$$t=t_0(r)\equiv\frac{br^2(r^2+2r_0^2)}{(r^2+r_0^2)^2} \tag{3.18}$$

with both ρ and η being positive at all times $t>t_0(r)$ as required. On the surface of Eq. (3.18) e^λ vanishes. After t becomes greater than $b=\max t_0(r)$ the metric is then completely regular. When $t\gg b$ the metric becomes the Robertson-Walker one throughout the space.

There is thus an infinite number of choices of $B(r)$ that satisfy the physical conditions we impose. This means we can have many different kinds of initial Universe and nonetheless still end up with the same final Robertson-Walker state, with no fine-tuning of initial conditions apparently being necessary. This is indeed the kind of result that one might like to see with viscous processes smoothing out any inhomogeneities present in the early Universe.

B. Positive-curvature case

In the $k=1$ case the function $\alpha_1(r)$ is given by $(1-r^2/4)/(1+r^2/4)$. Equation (3.2) can be written as

$$Y=\frac{r}{1+r^2/4}[(A(r)-t)(t-B(r))]^{1/2} \tag{3.19}$$

provided

$$A(r)>t>B(r) \tag{3.20}$$

and thus yields a recollapsing closed Universe. Equations (1.15) and (1.16) give for η the expression

$$\eta=\frac{-1}{4\kappa}\frac{\dfrac{\partial}{\partial r}\left[\dfrac{1-\alpha^2+2\ddot{Y}Y+\dot{Y}^2}{Y^2}\right]}{\dfrac{\partial}{\partial r}\left[\dfrac{\dot{Y}}{Y}\right]}=\frac{1}{4\kappa}\frac{A(r)-B(r)}{[A(r)-t][t-B(r)]}\frac{B'(r)[A(r)-t]^2-A'(r)[t-B(r)]^2}{B'(r)[A(r)-t]^2+A'(r)[t-B(r)]^2} . \tag{3.21}$$

To ensure the positivity of η we need to set

$$A'(r)=0,\text{ i.e., } A=\text{const} . \tag{3.22}$$

Consequently we have

$$\eta=\frac{1}{4\kappa}\frac{A-B(r)}{(A-t)[t-B(r)]} . \tag{3.23}$$

With the use of Eqs. (3.19) and (3.22), we can now write Eqs. (1.12) and (1.14) as

$$e^\lambda=\frac{[(A-t)(t-B(r))]^{1/2}}{1+r^2/4}\frac{\left|t-B(r)-\dfrac{rB'(r)}{2\alpha_1(r)}\right|}{t-B(r)} , \tag{3.24}$$

$$\kappa\rho=\frac{3}{4}\frac{[A-B(r)]^2}{[(A-t)(t-B(r))]^2}\frac{t-B(r)-\dfrac{rB'(r)}{2\alpha_1}+\dfrac{2}{3}\dfrac{rB'(r)}{\alpha_1}\dfrac{A-t}{A-B(r)}}{t-B(r)-rB'(r)/2\alpha_1} . \tag{3.25}$$

Similar to the $k=0$ case the big bang occurs at

$$t=t_0(r)\equiv B(r)+\frac{rB'(r)}{2\alpha_1(r)} \tag{3.26}$$

and $B(r)$ thus has to satisfy

$$\frac{rB'(r)}{2\alpha_1(r)}>0 . \tag{3.27}$$

According to observations the Universe today is very homogeneous. Consequently $B(r)$ has to be small compared to the present time t_{present} (and thus small compared to A) if the present homogeneity is to be considered as a global property. This means $B(r)$ has to be bounded. Of course, analogously to the $k=0$ case, $B(r)$ would not need to be bounded if the present homogeneity of the observed Universe were only to be a local property (i.e., if there are presently unobservable parts of the Universe which are inhomogeneous in analogy to the situation discussed in Sec. II).

An example of a bounded $B(r)$ which satisfies Eq. (3.27) is given by

$$B(r)=b\left[\frac{r}{1+r^2/4}\right]^{2c} , \tag{3.28}$$

where the constants b and c are both greater than zero. The initial time $t_0(r)$ as defined in Eq. (3.26) is thus given by

$$t_0(r)=(1+c)B(r)=(1+c)b\left[\frac{r}{1+r^2/4}\right]^{2c}. \tag{3.29}$$

The points $r=0$ and $r=\infty$ (corresponding to the north and south poles in a closed Robertson-Walker space) start the big bang at $t=0$ and their neighboring points then follow them. The surface $r=2$ (corresponding to the equator of the space) is the last to start the big bang, and this occurs at a time at $t=(1+c)b=\max t_0(r)$. Subsequent to this time the metric becomes completely regular and asymptotically approaches the Roberston-Walker form.

C. Negative-curvature case

For the $k=-1$ case the function $\alpha_{-1}(r)$ takes the form $(1+r^2/4)/(1-r^2/4)$. Equation (3.2) can be written as

$$Y=\frac{r}{1-r^2/4}[(t-A(r))(t-B(r))]^{1/2}, \tag{3.30}$$

where $A(r)>B(r)$ is assumed this time. Note that the coordinate r is restricted to the range $0\le r\le 2$ ($r=2$ corresponds to spatial infinity). The coefficient of shear viscosity is given by

$$\eta=\frac{1}{4\kappa}\frac{A(r)-B(r)}{[t-A(r)][t-B(r)]}\times\frac{A'(r)[t-B(r)]^2-B'(r)[t-A(r)]^2}{A'(r)[t-B(r)]^2+B'(r)[t-A(r)]^2}. \tag{3.31}$$

As $t\to\infty$ we find

$$\eta\to\frac{1}{4\kappa}\frac{A(r)-B(r)}{[t-A(r)][t-B(r)]}\frac{A'(r)-B'(r)}{A'(r)+B'(r)}.$$

Thus we require

$$|B'(r)|\le|A'(r)| \tag{3.32}$$

to ensure that $\eta\ge 0$ at all times after the big bang. Equation (3.32) implies $A'(r)\neq 0$ [since otherwise $B'(r)$ would also be zero and thus yield a Robertson-Walker metric at all times]. The energy density is given by

$$\kappa\rho=\frac{3[A(r)-B(r)]^2}{4[(t-A(r))(t-B(r))]^2}\frac{1+\dfrac{2r}{3\alpha_{-1}}\dfrac{A'(r)-B'(r)}{A(r)-B(r)}+\dfrac{r}{6\alpha_{-1}}\left[\dfrac{A'(r)}{t-A(r)}+\dfrac{B'(r)}{t-B(r)}\right]}{1-\dfrac{r}{2\alpha_{-1}}\left[\dfrac{A'(r)}{t-A(r)}+\dfrac{B'(r)}{t-B(r)}\right]}. \tag{3.33}$$

As $t\to\infty$ it approaches

$$\kappa\rho\to\frac{3}{4}\frac{[A(r)-B(r)]^2}{[(t-A(r))(t-B(r))]^2}\left[1+\frac{2r}{3\alpha_{-1}}\frac{A'(r)-B'(r)}{A(r)-B(r)}\right]. \tag{3.34}$$

Thus we need to set

$$A(r)-B(r)=C=\text{const} \tag{3.35}$$

in order to get the model to approach the standard cosmology at late times. Using Eq. (3.35) we get

$$Y=\frac{r}{(1-r^2/4)}[(t-B(r)-C)(t-B(r))]^{1/2}, \tag{3.36}$$

$$e^{\lambda}=\frac{1}{(1-r^2/4)}\frac{[t-D_1(r)][t-D_2(r)]}{[(t-B(r)-C)(t-B(r))]^{1/2}}, \tag{3.37}$$

$$\kappa\rho=\frac{3C^2}{4[(t-B(r)-C)(t-B(r))]^2[t-D_1(r)][t-D_2(r)]}\left[t^2-\left[2B(r)+C-\frac{rB'(r)}{3\alpha_{-1}}\right]t+B(r)[B(r)+C]-\frac{rB'(r)}{6\alpha_{-1}}[2B(r)+C]\right], \tag{3.38}$$

$$\eta=\frac{1}{4\kappa}\frac{C^2}{[t-B(r)-C][t-B(r)]}\frac{2t-2B(r)-C}{[t-B(r)-C]^2+[t-B(r)]^2}, \tag{3.39}$$

where

$$D_1(r)=\frac{1}{2}\left\{2B(r)+C+\frac{rB'(r)}{\alpha_{-1}}+\left[C^2+\left[\frac{rB'(r)}{\alpha_{-1}}\right]^2\right]^{1/2}\right\}, \tag{3.40}$$

$$D_2(r)=\frac{1}{2}\left\{2B(r)+C+\frac{rB'(r)}{\alpha_{-1}}-\left[C^2+\left[\frac{rB'(r)}{\alpha_{-1}}\right]^2\right]^{1/2}\right\}. \tag{3.41}$$

Similar to the previous cases, the requirement that ρ be positive and regular after the big bang leads to

$$\frac{rB'(r)}{\alpha_{-1}(r)}>0\ , \tag{3.42}$$

with the big bang taking place at

$$t=t_0(r)\equiv D_1(r)\ . \tag{3.43}$$

It is also instructive to examine the contribution of the shear viscosity to the energy-momentum tensor. For the $k=-1$ case we note that

$$\eta\left[\dot{\lambda}-\frac{\dot{Y}}{Y}\right]=\frac{1}{4\kappa}\frac{C^2[2t-2B(r)-C]\dfrac{rB'(r)}{2\alpha_{-1}}}{[(t-B(r)-C)(t-B(r))]^2[t-D_1(r)][t-D_2(r)]}\ . \tag{3.44}$$

As $t\rightarrow\infty$ this expression goes like t^{-5} while the ρ- and p-dependent terms go like t^{-4}. Therefore, the shear-viscosity contribution does indeed vanish faster than the perfect-fluid piece which is due to the energy density ρ and the pressure p. It is also easy to see that the dominant energy condition is satisfied. In passing we note that the same conclusions also hold for both the $k=0$ and $k=1$ cases that we have just discussed.

To conclude this section we note again that we have obtained a class of exact solutions for each of the three different spatial geometries. The Universe starts with an inhomogeneous big bang, gradually approaches homogeneity and gradually becomes a Robertson-Walker Universe. Unlike the solutions presented in Sec. II which do not approach the Robertson-Walker one globally, the solutions here do in fact homogenize uniformly and globally if $B(r)$ is properly chosen. This particular feature of the solutions given in this section of our present paper is especially interesting since the solution presented in our companion paper [1] approached a Robertson-Walker form only locally and not globally. Even though current observation only requires a local Robertson-Walker metric at late times it is nonetheless interesting to see that there are in fact inhomogeneous cosmologies whose late time behavior is both locally and globally of the standard Robertson-Walker form.

IV. ENTROPY PRODUCTION AND THE FLATNESS PROBLEM

Any physical system has to satisfy the following three thermodynamic conditions.

(i) Conservation of baryon number:

$$(nU^\mu)_{;\mu}=0\ , \tag{4.1}$$

where n is the particle number density and U^μ is the fluid four-velocity vector.

(ii) Gibbs' relation:

$$Td\left[\frac{S}{n}\right]=d\left[\frac{\rho}{n}\right]+pd\left[\frac{1}{n}\right]\ , \tag{4.2}$$

where T is the temperature and S is the entropy density.

(iii) Positivity of entropy production:

$$S^\mu{}_{;\mu}\geq 0\ , \tag{4.3}$$

where $S^\mu=SU^\mu$ is the entropy flow vector of the fluid. In the acceleration-free geometry of Eq. (1.4) this last condition reduces to

$$S^\mu{}_{;\mu}=\frac{4\eta}{3T}\left[\dot{\lambda}-\frac{\dot{Y}}{Y}\right]^2\geq 0 \tag{4.4}$$

and thus the coefficient of shear viscosity, η, has to be non-negative, with this in fact being the case in all of the solutions that we have presented in this paper.

In order to be able to obtain an expression for the entropy density S from Eq. (4.2) we need an expression for the temperature. For the radiation-dominated Universe the most natural and simple assumption is that (see Ref. [1]) of a local blackbody, viz.,

$$\rho(r,t)=aT^4(r,t)\ , \tag{4.5}$$

where a is a constant. Substituting Eqs. (4.5) and (1.17) into Eq. (4.2) then yields

$$d\left[\frac{S}{n}\right]=\frac{4}{3}a^{1/4}d\left[\frac{\rho^{3/4}}{n}\right]=\frac{4}{3}ad\left[\frac{T^3}{n}\right]\ , \tag{4.6}$$

or

$$\frac{S}{n}=\frac{4}{3}a\frac{T^3}{n}=\frac{4}{3}a^{1/4}\frac{\rho^{3/4}}{n} \tag{4.7}$$

with the integration constant being set equal to zero as required by the third law of thermodynamics. Equation (4.7) is then the required expression for the entropy per baryon. The number density n can be obtained from Eq. (4.1), i.e., from

$$\frac{1}{(-g)^{1/2}}\partial_t[(-g)^{1/2}n]=0\ , \tag{4.8}$$

so that in our geometry the number density obeys

$$n(r,t)=\frac{n_0(r)}{e^\lambda Y^2}\ , \tag{4.9}$$

where $n_0(r)$ is a convenient integration function for Eq. (4.8). Equations (4.7) and (4.9) then give

$$S/n=\tfrac{4}{3}a^{1/4}e^\lambda Y^2\rho^{3/4}/n_0\ . \tag{4.10}$$

Substituting into Eq. (4.10) the expressions for the energy density and the metric coefficients associated with the solutions obtained in Secs. II and III then gives an expression for S/n in each of the solutions. As an example we may take the solution of case (iv), i.e., Eqs. (2.21)–(2.24), of Sec. II. For this particular solution we find that

$$\frac{S}{n}=\left[\frac{S}{n}\right]_{\mathrm{RW}}\left[1+\frac{b}{tr^{3/2}}\right]^{1/4}\left[1-\frac{b}{3tr^{3/2}}\right]^{3/4}, \tag{4.11}$$

where

$$\left[\frac{S}{n}\right]_{\mathrm{RW}}=\left[\frac{4a}{3\kappa^3}\right]^{1/4}\frac{A^{3/2}r^2}{n_0(r)} \tag{4.12}$$

is the corresponding expression for a Robertson-Walker spacetime. The product of the last two factors in Eq. (4.11) is an increasing function of t which is zero initially on the big-bang surface given by Eq. (2.26) where the factor $(1+b/tr^{3/2})^{1/4}$ vanishes (recall that the parameter b that appears here is negative in this solution). Thus the entropy per baryon is indeed zero at the big bang and increases steadily as time increases. When $t\to\infty$ we find that $S/n\to(S/n)_{\mathrm{RW}}$. In other words, the entropy per baryon of the Universe increases smoothly from zero to its present large value as the Universe evolves from the big bang to the present time. The same conclusions can also be made for the other physical solutions that we presented in Secs. II and III. Therefore, the entropy problem that exists in the standard Friedmann big-bang model is absent here.

In addition to being able to resolve the entropy problem, our models also do not appear to suffer from any flatness problem either. Specifically, we may rewrite Eq. (1.14) in the form

$$\kappa\rho=\frac{1}{3}\Theta^2-\frac{1}{3}\left[\frac{\dot{Y}'}{Y'}-\frac{\dot{Y}}{Y}\right]^2+\frac{1}{Y^2Y'}\frac{\partial}{\partial r}[Y(1-\alpha^2)]\ , \tag{4.13}$$

where Θ is the expansion parameter defined by

$$\Theta\equiv U^{\alpha}{}_{;\alpha}=\dot{\lambda}+2\frac{\dot{Y}}{Y}\ . \tag{4.14}$$

In terms of the language of the standard cosmology we can thus identify a present-day critical density

$$\rho_c=\frac{1}{3\kappa}\Theta^2=\frac{1}{3\kappa}\left[\frac{\dot{Y}'}{Y'}+2\frac{\dot{Y}}{Y}\right]^2 . \tag{4.15}$$

Because of the presence of the explicit shear tensor term in Eq. (4.13), we find that, unlike the familiar situation which occurs in the standard model, the Einstein equations no longer require us to have to fix the early Universe magnitude of ρ/ρ_c to incredibly high accuracy so that it can then evolve to a value which is close to one today. In fact, it turns out that in all of the models presented in this paper ρ/ρ_c is actually either zero or infinite at the time of the (inhomogeneous) big bang. Thus our model requires no fine-tuning of the quantity ρ/ρ_c at early times. In a sense then, the flatness problem is only a problem in models where the entire history of the radiation-dominated Universe is the standard perfect-fluid one. This is not the case in any of the models discussed in this series of papers.

With regard to the horizon problem situation in the models considered in this paper, we are unfortunately unable to make as definitive a statement as we did in the model of Ref. [1] (where we actually found a model cosmology with no horizon problem at all) since we can no longer integrate the light-cone geodesic equations exactly, and we cannot even integrate them reliably numerically either because of the singularity structure of the inhomogeneous big bang. We can however explore our models qualitatively to see whether they could have better causal structure than the standard Robertson-Walker model. Thus for the model of case (iii) of Sec. II for instance we find that the light-cone equation takes the form

$$\frac{dr}{dt}=\frac{1}{(At)^{1/2}}\frac{\left[1+\dfrac{b}{r^3t^{3/4}}\right]^{1/3}}{1-\dfrac{b}{r^3t^{3/4}}}\ . \tag{4.16}$$

When the parameter b in Eq. (4.16) is taken to be negative (corresponding to the case where e^{λ} vanishes at the big bang) the right-hand side of Eq. (4.16) is smaller than the corresponding Robertson-Walker expression. The light trajectories are then less causal then those associated with a Robertson-Walker geometry. Cases (i) and (iv) of Sec. II are also in this class. However, when the parameter b is positive (corresponding to the case where Y vanishes at the big bang) the right-hand side of Eq. (4.16) is larger than the corresponding expression in the standard model. Moreover, case (ii) of Sec. II and all the cases considered in Sec. III also behave this way. In these cases then the light-cone trajectories are more causal than those associated with a Robertson-Walker geometry. Unfortunately though, because of the singular nature of the geodesics at early times we have not been able to explore whether this improvement is significant enough to resolve the horizon problem.

To conclude this paper we would like to state that even though there is still a need for further analysis of the solutions presented both here and in our companion paper, we feel that the wide variety of exact solutions that we have found is suggestive of the possibility that all of the difficulties of the standard model may be completely resolvable by inhomogeneous cosmology.

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