Spectral boundary conditions in one-loop quantum cosmology

Peter D. D'Eath
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, United Kingdom

Giampiero V. M. Esposito*
Department of Applied Mathematics and Theorectical Physics, Silver Street, Cambridge CB3 9EW, United Kingdom and St. John's College, Cambridge CB2 1TP, United Kingdom
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For fermionic fields on a compact Riemannian manifold with a boundary, one has a choice between local and nonlocal (spectral) boundary conditions. The one-loop prefactor in the Hartle-Hawking amplitude in quantum cosmology can then be studied using the generalized Riemann ζ function formed from the squared eigenvalues of the four-dimensional fermionic operators. For a massless Majorana spin-$\frac{1}{2}$ field, the spectral conditions involve setting to zero half of the fermionic field on the boundary, corresponding to harmonics of the intrinsic three-dimensional Dirac operator on the boundary with positive eigenvalues. Remarkably, a detailed calculation for the case of a flat background bounded by a three-sphere yields the same value $\zeta(0)=\frac{11}{360}$ as was found previously by the authors using local boundary conditions. A similar calculation for a spin-$\frac{3}{2}$ field, working only with physical degrees of freedom (and, hence, excluding gauge and ghost modes, which contribute to the full Becchi-Rouet-Stora-Tyutin-invariant amplitude), again gives a value $\zeta(0)=-\frac{289}{360}$ equal to that for the natural local boundary conditions.

I. INTRODUCTION

In recent work by the authors [1,2], one-loop effects in quantum cosmology for fermionic fields have been studied using local boundary conditions. In the case of a massless spin-$\frac{1}{2}$ field $(\psi^A,\tilde{\psi}^{A'})$ in a Riemannian background, which we shall refer to loosely as a Majorana spin-$\frac{1}{2}$ field, [2] the simplest natural local boundary conditions are (using two-component spinors)

$$\sqrt{2}\,{}_e n_A^{\ A'}\psi^A=\epsilon\tilde{\psi}^{A'} , \tag{1.1}$$

where ${}_e n^{AA'}$ is the spinor version of the unit Euclidean normal ${}_e n^\mu$ to the boundary and $\epsilon=\pm1$. Note that the primed field $\tilde{\psi}^{A'}$ is taken to be independent of ψ^A, not related by any conjugation operation. A first-order differential operator for this Riemannian boundary-value problem exists which is symmetric and has self-adjoint extensions. One can then study the generalized Riemann ζ function formed from the squared eigenvalues of the Dirac operator. The value of $\zeta(0)$ yields the one-loop divergence of the quantum amplitude for the Hartle-Hawking quantum state subject to these boundary conditions. Further, $\zeta(0)$ determines the scaling of the one-loop amplitude: in the case of a flat Euclidean four-dimensional background geometry bounded by a three-sphere of radius a, the one-loop amplitude scales as $a^{-\zeta(0)}$ for a fermionic field (in the case of a scale-independent measure). A direct calculation [2] for a massless Majorana spin-$\frac{1}{2}$ field with the boundary conditions (1.1) on a three-sphere in flat four-space gave the value $\zeta(0)=\frac{11}{360}$. Such boundary conditions are of interest because they are part of a supersymmetric family of local boundary conditions for both fermions and bosons [1–5], so that one can check whether or not the one-loop divergences in the Hartle-Hawking amplitude cancel in extended supergravity theories.

Because of the first-order nature of the fermionic operators, one has a choice between local boundary conditions such as Eq. (1.1) and nonlocal (spectral) boundary conditions. While spectral boundary conditions are not in any obvious way related to supersymmetry, they are nevertheless of considerable mathematical interest, and are the subject of this paper. Their mathematical foundations lie in the theory of elliptic equations and in the index theory for the Dirac operator [6]. To illustrate these boundary conditions, consider again the case of a massless Majorana spin-$\frac{1}{2}$ field $(\psi^A,\tilde{\psi}^{A'})$ in the region of flat Euclidean four-space bounded by a three-sphere of radius a. Denote by τ the Euclidean distance from the center of the sphere. Then the field $(\psi^A,\tilde{\psi}^{A'})$ may be expanded in terms of harmonics on the family of spheres centered on the origin, [2,7] as

$$\psi^A=\frac{\tau^{-3/2}}{2\pi}\sum_{npq}\alpha_n^{pq}[m_{np}(\tau)\rho^{nqA}+\tilde{r}_{np}(\tau)\bar{\sigma}^{nqA}] , \tag{1.2}$$

$$\tilde{\psi}^{A'}=\frac{\tau^{-3/2}}{2\pi}\sum_{npq}\alpha_n^{pq}[\tilde{m}_{np}(\tau)\bar{\rho}^{nqA'}+r_{np}(\tau)\sigma^{nqA'}] . \tag{1.3}$$

In the summations, n runs from 0 to ∞, p and q from 1 to $(n+1)(n+2)$. The α_n^{pq} are a collection of matrices introduced for convenience, where, for each n, α_n^{pq} is block diagonal in the indices pq, with blocks $\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$. The harmonics ρ^{nqA} have positive eigenvalues $\frac{1}{2}(n+\frac{3}{3})$ of the intrinsic three-dimensional Dirac operator ${}_e n_{AA'}e^{BA'j\,(3)}D_j$ on the three-sphere, while the harmonics $\bar{\sigma}^{nqA}$ have negative eigenvalues $-\frac{1}{2}(n+\frac{3}{2})$. Here $e^{BA'j}$ is the spinor version of the orthonormal spatial triad on the three-sphere, and ${}^{(3)}D_j$ is the three-dimensional covariant derivative ($j=1,2,3$) [7]. Similarly, the harmonics $\sigma^{nqA'}$ have positive eigenvalues $\frac{1}{2}(n+\frac{3}{2})$ of the corresponding three-dimensional operator on primed spinors, and the harmonics $\bar{\rho}^{nqA'}$ have negative eigenvalues $-\frac{1}{2}(n+\frac{3}{2})$. This expansion can be summarized more simply as

$$\psi^A=\psi^A_{(+)}+\psi^A_{(-)} , \tag{1.4}$$

$$\tilde{\psi}^{A'}=\tilde{\psi}^{A'}_{(-)}+\tilde{\psi}^{A'}_{(+)} , \tag{1.5}$$

where the $(\pm)$ parts correspond to positive and negative eigenvalues, respectively, for the intrinsic three-dimensional Dirac operator.

In studying the classical boundary-value problem for the massless Dirac equation, one finds that classical solutions corresponding to boundary data with a nonzero coefficient $\tilde{r}_{np}(a)$ or $\tilde{m}_{np}(a)$ diverge as a negative power of τ near the origin [7]. Boundary data with a nonzero coefficient of $m_{np}(a)$ or $r_{np}(a)$ yield a regular solution of the massless Dirac equation, proportional to a positive power of τ. Thus the classical boundary-value problem is only well posed if one specifies the $m_{np}(a)$ and $r_{np}(a)$, but not the remaining data. In the case of a general manifold with boundary, knowledge of the spectrum of the intrinsic three-dimensional Dirac operator is necessary if one wishes to compute the η invariant which gives a boundary contribution to the index of the Dirac operator for the manifold with boundary [6]. (In the generic case, the index will be nonzero, and the classical boundary-value problem will not be well posed [6].) In this paper we are however not concerned with this index, but rather, as explained above, with the zeta function $\zeta(s)$ formed from the eigenvalues of the four-dimensional Dirac operator, subject now to boundary conditions in which $\psi^A_{(+)}$ and $\tilde{\psi}^{A'}_{(+)}$ are specified on the boundary in our flat-space example.

Thus, just as one has a well-posed classical problem with these boundary data, one similarly expects that the analogous quantum amplitude, the Hartle-Hawking path integral

$$K_{\rm HH}=\int e^{-I_E}D\psi^A D\tilde{\psi}^{A'} \tag{1.6}$$

for the fermions, is naturally studied by taking spectral boundary conditions in which $\psi^A_{(+)}$ and $\tilde{\psi}^{A'}_{(+)}$, or equivalently the $m_{np}(a)$ and $r_{np}(a)$ in our example, are specified on the boundary. Here the Euclidean action is

$$I_E=\frac{i}{2}\int d^4x\sqrt{g}\,[\tilde{\psi}^{A'}(\nabla_{AA'}\psi^A)-(\nabla_{AA'}\tilde{\psi}^{A'})\psi^A]+I_B . \tag{1.7}$$

The fermionic fields are taken to be anticommuting, and Berezin integration is being used [7]. With our conventions, the Infeld–van der Waerden connection symbols $\sigma_a^{\ AA'}$ are taken to be $\sigma_0=-(i/\sqrt{2})I$, $\sigma_i=\Sigma_i/\sqrt{2}$ ($i=1,2,3$), where Σ_i are the Pauli matrices. A boundary term I_B, discussed in Ref. [7], is needed in general. In our simple example of the three-sphere, the Hartle-Hawking amplitude is then a function $K_{\rm HH}[m_{np}(a),r_{np}(a)]$ of the spectral boundary data. The one-loop properties of this amplitude can be studied without loss of generality by setting the allowed boundary data $m_{np}(a)$, $r_{np}(a)$ to zero, $\forall n,p$, so that the boundary conditions become

$$\psi^A_{(+)}=0,\quad \tilde{\psi}^{A'}_{(+)}=0 . \tag{1.8}$$

(A similar simplification was made in Ref. [2] in choosing the local boundary conditions (1.1), rather than specifying a nonzero spinor field $\sqrt{2}\,{}_en_A^{\ A'}\psi^A-\epsilon\tilde{\psi}^{A'}$ on the boundary.) The boundary term I_B is zero in this case.

In Sec. II the action (1.7) is expanded in harmonics, subject to the spectral boundary conditions (1.8) on the three-sphere of radius a bounding a region of flat four-space. The eigenvalue equation arising in the evaluation of the one-loop functional determinant is derived. The resulting ζ function formed from the squared eigenvalues is related to the heat kernel $G(T)$ for the Laplacian operator on spinors. In Sec. III the Laplace transforms of the corresponding Green's functions are derived. The detailed calculations leading to the asymptotic expansion of $G(T)$ as $T\to 0^+$ are described in Sec. IV. These lead to the value $\zeta(0)=\frac{11}{360}$, which remarkably is the same as that found previously [2] for a massless Majorana spin-$\frac{1}{2}$ field using the local boundary conditions (1.1) on the same manifold with boundary. An analogous calculation for the spin-$\frac{3}{2}$ field is sketched in Sec. V, working only with the physical degrees of freedom in a particular gauge. This of course excludes the contribution of gauge and ghost modes which should appear in the full Becchi-Rouet-Stora-Tyutin- (BRST-) invariant path integral. Nevertheless, it is striking that the value $\zeta(0)=-\frac{289}{360}$ obtained is again identical to that found using the natural local boundary conditions for the physical degrees of freedom [1]. Some comments are included in Sec. VI.

II. EIGENVALUES FOR SPECTRAL BOUNDARY CONDITIONS ON S^3

In the case of a region of four-dimensional flat Euclidean space bounded by a three-sphere of radius a, we decompose the massless spin-$\frac{1}{2}$ field $(\psi^A,\tilde{\psi}^{A'})$ as in Eqs. (1.2) and (1.3) and impose the spectral boundary conditions (1.8), so that

$$m_{np}(a)=0,\quad r_{np}(a)=0,\quad \forall n,p . \tag{2.1}$$

The Hartle-Hawking path integral (1.6), with the Euclidean action I_E given by Eq. (1.7), can then be studied, with $m_{np}(\tau),r_{np}(\tau)$ constrained by Eq. (2.1) at the boundary, but $\tilde{m}_{np}(a)$ and $\tilde{r}_{np}(a)$ unconstrained. The physical fields $(\psi^A,\tilde{\psi}^{A'})$ summed over in the path integral should at least be bounded near the origin $\tau=0$. Because of the factor $\tau^{-3/2}$ in Eqs. (1.2) and (1.3), this implies that

$$m_{np}(0)=r_{np}(0)=\tilde{m}_{np}(0)=\tilde{r}_{np}(0)=0,\quad \forall n,p . \tag{2.2}$$

The action I_E can then be expanded out in terms of harmonics, by analogy with the treatment of Ref. [7], as

$$I_E=\sum_{n=0}^{\infty}\sum_{p=1}^{(n+1)(n+2)}[I_n(m_{np},\tilde{m}_{np})+I_n(r_{np},\tilde{r}_{np})] , \tag{2.3}$$

where

$$I_n(x,\tilde{x})=\int_0^a d\tau\left[\frac{1}{2}(\tilde{x}\dot{x}+x\dot{\tilde{x}})-\frac{n+\frac{3}{2}}{\tau}\tilde{x}x\right] \tag{2.4}$$

and an overdot denotes $d/d\tau$. Note that, as remarked in the Introduction, no boundary term I_B of the type described in Ref. [7] appears in the action, because of the boundary conditions (2.1) and (2.2).

Because of the degeneracy $(n+1)(n+2)$ in the label p, and because of the splitting of the action I_E in Eq. (2.3) into identical pieces involving $(m_{np},\tilde{m}_{np})$ and $(r_{np},\tilde{r}_{np})$, the complete path integral (1.6) splits into a product of Berezin integrals:

$$K_{\text{HH}}=\prod_{n=0}^{\infty}\left[\int d[x]d[\tilde{x}]\exp[-I_n(x,\tilde{x})]\right]^{2(n+1)(n+2)} . \tag{2.5}$$

The boundary conditions in each integration are then

$$x(0)=\tilde{x}(0)=0,\quad x(a)=0 , \tag{2.6}$$

following Eqs. (2.1) and (2.2).

Equivalently, one can follow the procedure of Ref. [2] and study the eigenvalue equations

$$\nabla_{AA'}\psi_m^A=\lambda_m\tilde{\psi}_{mA'},\quad \nabla_{AA'}\tilde{\psi}_m^{A'}=\lambda_m\psi_{mA} , \tag{2.7}$$

naturally arising from variation of the action (1.7), subject to the boundary conditions (1.8). The eigenfunctions $(\psi_m^A,\tilde{\psi}_m^{A'})$ are clearly found by separation of variables, being of the form

$$\psi_n^A=\tau^{-3/2}x_{nk}(\tau)\rho^{npA},\quad \tilde{\psi}_n^{A'}=\tau^{-3/2}\tilde{x}_{nk}(\tau)\bar{\rho}^{npA'} , \tag{2.8}$$

or

$$\psi_n^A=\tau^{-3/2}\tilde{x}_{nk}(\tau)\bar{\sigma}^{npA},\quad \tilde{\psi}_n^{A'}=\tau^{-3/2}x_{nk}(\tau)\sigma^{npA'} . \tag{2.9}$$

Here the pair $(x_{nk}(\tau),\tilde{x}_{nk}(\tau))$ is an eigenvector corresponding to variation of the action $I_n(x,\tilde{x})$ of Eqs. (2.4) and (2.5), obeying

$$\left[\frac{d}{d\tau}-\nu\right]x_{nk}=E_{nk}\tilde{x}_{nk} , \tag{2.10}$$

$$\left[-\frac{d}{d\tau}-\nu\right]\tilde{x}_{nk}=E_{nk}x_{nk} , \tag{2.11}$$

subject to the boundary conditions $x_{nk}(0)=\tilde{x}_{nk}(0)=0$, $x_{nk}(a)=0$, where $\nu=(n+\frac{3}{2})/\tau$ and $E_{nk}=i\lambda_{nk}$. For each n, the index k labels the countable set of corresponding eigenfunctions and -values.

The Gaussian fermionic path integral (2.5) is then formally proportional to the product of the eigenvalues $\prod_{n,k}(\lambda_{nk}/\tilde{\mu})^{2(n+1)(n+2)}$. Here the constant $\tilde{\mu}$ with dimensions of mass has been introduced in order to make the product dimensionless [2]. As shown below, the λ_{nk} are purely imaginary, or equivalently the E_{nk} are real. Further, the E_{nk} occur in equal and opposite pairs, since if the pair $(x_{nk}(\tau),\tilde{x}_{nk}(\tau))$ corresponds to an eigenvalue E_{nk} in Eq. (2.10), then clearly the pair $(x_{nk}(\tau),-\tilde{x}_{nk}(\tau))$ corresponds to the eigenvalue $-E_{nk}$. Hence the path integral can also be written as $\prod_{n,k}(|\lambda_{nk}|/\tilde{\mu})^{2(n+1)(n+2)}$. The Berezin integration rules imply that one should only include those values of k, say $k=1,2,\ldots$, which correspond to positive values of E_{nk}. This formal expression must then be regularized using ζ-function methods.

The coupled first-order equations (2.10) and (2.11) lead to the second-order equation

$$\left[\frac{d^2}{d\tau^2}-\frac{(n+1)^2-\frac{1}{4}}{\tau^2}+E_{nk}^2\right]x_{nk}=0 , \tag{2.12}$$

together with a corresponding equation for $\tilde{x}_{nk}(\tau)$. The solutions $(x_{nk}(\tau),\tilde{x}_{nk}(\tau))$ obeying the boundary conditions (2.6) are

$$x_{nk}=A_{nk}\sqrt{\tau}J_{n+1}(E_{nk}\tau) , \tag{2.13}$$

$$\tilde{x}_{nk}=-A_{nk}\sqrt{\tau}J_{n+2}(E_{nk}\tau) , \tag{2.14}$$

subject to the eigenvalue condition

$$J_{n+1}(E_{nk}a)=0,\quad n=0,1,2,\ldots . \tag{2.15}$$

The quantities E_{nk}^2 are clearly real and positive since they are eigenvalues for the self-adjoint problem (2.12), subject to boundary conditions $x_{nk}(0)=x_{nk}(a)=0$, and since Eq. (2.12) involves a positive operator. For a given n, the coefficients A_{nk} in Eqs. (2.13) and (2.14) can be chosen such that the eigenfunctions $x_{nk}(\tau)$ are orthonormal in the inner product $(u,v)=\int_0^a d\tau\, u(\tau)v(\tau)$, as are the corresponding $\tilde{x}_{nk}(\tau)$. For each n, the action $I_n(x,\tilde{x})$ of Eq. (2.4) becomes a diagonal sum over the eigenfunctions $x_{nk},\tilde{x}_{nk}$ for $k=1,2,\ldots$. Performing the Berezin integrations in Eq. (2.5), one arrives at the formal expression quoted in the previous paragraph:

$$K_{\text{HH}}=\prod_{n=0}^{\infty}\prod_{k=1}^{\infty}\left[\frac{E_{nk}}{\tilde{\mu}}\right]^{2(n+1)(n+2)} , \tag{2.16}$$

where the $E_{nk}(k=1,2,\ldots)$ are the positive eigenvalues obeying Eq. (2.15).

Following the standard procedure, as, for example, in Ref. [2], the formally divergent infinite product (2.16) is regularized by studying the ζ function for the squared eigenvalues

$$\zeta(s)=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}d_k(n)(E_{nk})^{-2s} . \tag{2.17}$$

Here the degeneracy $d_k(n)=2(n+1)(n+2)$ is in fact independent of k. The series (2.17) converges for $\text{Re}(s)>2$, and can be analytically continued to a meromorphic function with poles only at $s=\frac{1}{2},1,\frac{3}{2},2$. The formal expression $\ln(K_{\text{HH}})$, with K_{HH} given by Eq. (2.16), is then evaluated as $-\frac{1}{2}\zeta'(0)-\zeta(0)\ln\tilde{\mu}$.

III. GREEN'S FUNCTIONS AND THE HEAT KERNEL

The quantity $\zeta(0)$, which gives the divergence and scaling properties of the one-loop amplitude, is evaluated by studying the heat kernel, defined for $T>0$ by

$$G(T)=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}2(n+1)(n+2)e^{-(E_{nk})^2T} . \quad (3.1)$$

This is related to the zeta function $\zeta(s)$ of Eq. (2.17) by

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}dT\,T^{s-1}G(T) \quad (3.2)$$

for $\mathrm{Re}(s)>2$. The heat kernel $G(T)$ has the standard asymptotic expansion

$$G(T)\sim\sum_{i=0}^{\infty}B_iT^{i/2-2} , \quad (3.3)$$

as $T\to 0^+$, where in particular $B_4=\zeta(0)$ [8].

In the present case of spectral boundary conditions, the eigenvalue condition

$$J_{n+1}(E_{nk}a)=0,\quad n=0,1,2,\ldots$$

[Eq. (2.15)], with degeneracy $2(n+1)(n+2)$, is considerably simpler than the eigenvalue condition $[J_{n+1}(E_{nk}a)]^2-[J_{n+2}(E_{nk}a)]^2=0$, $n=0,1,2,\ldots$, with degeneracy $(n+1)(n+2)$, found in Ref. [2] for a spin-$\frac{1}{2}$ field with local boundary conditions on S^3. As a consequence, we can use a more straightforward treatment, following (among others) Schleich [8] and Stewartson and Waechter [9], which involves the Green's function for the heat equation for each n. The value $\zeta(0)=\frac{11}{360}$ which results is, perhaps surprisingly, equal to that found in Ref. [2] for local boundary conditions.

One proceeds by considering, for each $n=0,1,2,\ldots$, the Green's function defined for $T>0$ by

$$G_n(\tau,\tau',T)=\sum_{k=1}^{\infty}x_{nk}(\tau)x_{nk}(\tau')e^{-(E_{nk})^2T} , \quad (3.4)$$

with $G_n(\tau,\tau',T)=0$ for $T\le 0$. Here the $x_{nk}(\tau)$ are the eigenfunctions of Eq. (2.12), obeying Eq. (2.12) and $x_{nk}(0)=x_{nk}(a)=0$, and normalized according to $\int_0^a d\tau\,x_{nk}(\tau)x_{nl}(\tau)=\delta_{kl}$, as described in Sec. II. Here $G_n(\tau,\tau',T)$ is the Green's function for the heat equation

$$\left[\frac{\partial}{\partial T}-\frac{\partial^2}{\partial\tau^2}+\frac{(n+1)^2-\frac{1}{4}}{\tau^2}\right]G_n(\tau,\tau',T)=\delta(\tau-\tau')\delta(T) . \quad (3.5)$$

It obeys the boundary conditions

$$G_n(a,\tau',T)=G_n(\tau,a,T)=G_n(0,\tau',T)=G_n(\tau,0,T)=0 . \quad (3.6)$$

By setting $\tau=\tau'$ and integrating, one recovers the contribution

$$G_n(T)=\int_0^a d\tau\,G_n(\tau,\tau,T)=\sum_{k=1}^{\infty}e^{-(E_{nk})^2T} \quad (3.7)$$

to the heat kernel

$$G(T)=\sum_{n=0}^{\infty}2(n+1)(n+2)G_n(T) . \quad (3.8)$$

The Laplace transform of the Green's function,

$$\hat{G}_n(\tau,\tau',\sigma^2)=\int_0^{\infty}dT\,e^{-\sigma^2T}G_n(\tau,\tau',T) , \quad (3.9)$$

obeys the differential equation

$$\left[\frac{\partial^2}{\partial\tau^2}-\sigma^2-\frac{(n+1)^2-\frac{1}{4}}{\tau^2}\right]\hat{G}_n(\tau,\tau',\sigma^2)=-\delta(\tau-\tau') . \quad (3.10)$$

Following Eq. (3.6), $\hat{G}_n(\tau,\tau',\sigma^2)$ is zero whenever either τ or τ' is 0 or a. It can be found explicitly in terms of modified Bessel functions (cf. Refs. [8–11]) as

$$\hat{G}_n(\tau,\tau',\sigma^2)=(\tau_<)^{1/2}(\tau_>)^{1/2}\frac{I_{n+1}(\sigma\tau_<)}{I_{n+1}(\sigma a)}\times[I_{n+1}(\sigma a)K_{n+1}(\sigma\tau_>)-I_{n+1}(\sigma\tau_>)K_{n+1}(\sigma a)] , \quad (3.11)$$

where $\tau_>$ ($\tau_<$) is the larger (smaller) of τ and τ'. This gives a splitting of the Laplace transform $\hat{G}(\tau,\tau',\sigma^2)$ of the function $G(\tau,\tau',T)$, defined as

$$G(\tau,\tau',T)=\sum_{n=0}^{\infty}2(n+1)(n+2)G_n(\tau,\tau',T) , \quad (3.12)$$

in the form

$$\hat{G}(\tau,\tau',\sigma^2)=\hat{G}^F(\tau,\tau',\sigma^2)+\hat{G}^I(\tau,\tau',\sigma^2) . \quad (3.13)$$

Here

$$\hat{G}^F(\tau,\tau',\sigma^2)=\sum_{n=0}^{\infty}2(n+1)(n+2)(\tau_<)^{1/2}(\tau_>)^{1/2}I_{n+1}(\sigma\tau_<)K_{n+1}(\sigma\tau_>) \quad (3.14)$$

is the "free contribution," which corresponds to the boundary conditions of vanishing at the origin and at infinity. The "interacting contribution" is

$$\hat{G}^I(\tau,\tau',\sigma^2)=-\sum_{n=0}^{\infty}2(n+1)(n+2)(\tau_<)^{1/2}(\tau_>)^{1/2}\frac{K_{n+1}(\sigma a)}{I_{n+1}(\sigma a)}I_{n+1}(\sigma\tau_<)I_{n+1}(\sigma\tau_>) . \quad (3.15)$$

By studying the large-σ^2 behavior of these functions, or the corresponding small-T behavior of $G(T)$ as in Eq. (3.3), one finds $\zeta(0)$.

IV. DETAILED CALCULATION OF THE INFINITE SUMS FOR SPIN $\frac{1}{2}$

Following the calculation of Schleich [8], taking the inverse Laplace transform term by term and integrating with respect to τ as in Eqs. (3.7) and (3.8), the free part of the heat kernel is found to be

$$G^F(T)=\int_0^{a^2/2T}\sum_{n=1}^{\infty} n(n+1)I_n(y)e^{-y}dy \ . \qquad (4.1)$$

Using the integral representation of the Bessel functions, [11] one obtains the identity

$$\sum_{n=1}^{\infty} n^2 I_n(y)=\frac{y}{2}e^y \ , \qquad (4.2)$$

which implies

$$\sum_{n=1}^{\infty}\int_0^{a^2/2T} n^2 I_n(y)e^{-y}dy=\frac{a^4}{16T^2} \ . \qquad (4.3)$$

Moreover, using the relations [11]

$$I_0'(y)=I_1(y) \ , \qquad (4.4)$$

$$nI_n(y)=\frac{y}{2}[I_{n-1}(y)-I_{n+1}(y)] \ , \qquad (4.5)$$

one has

$$\begin{aligned}\int_0^{a^2/2T}\sum_{n=1}^{\infty} nI_n(y)e^{-y}dy &=\int_0^{a^2/2T}\sum_{n=1}^{\infty}[I_{n-1}(y)-I_{n+1}(y)]\frac{y}{2}e^{-y}dy \\ &=\tfrac{1}{2}\int_0^{a^2/2T} ye^{-y}I_0(y)dy+\tfrac{1}{2}\int_0^{a^2/2T} ye^{-y}\frac{dI_0}{dy}dy \\ &=\frac{1}{2}\frac{a^2}{2T}e^{-a^2/2T}I_0\left[\frac{a^2}{2T}\right] \\ &\quad +\tfrac{1}{2}\int_0^{a^2/2T} e^{-y}(2y-1)I_0(y)dy \ . \end{aligned} \qquad (4.6)$$

The problem of computing $G^F(T)$ is thus reduced to that of computing the right-hand side of Eq. (4.6). Using again the identity $I_0'(y)=I_1(y)$, together with $I_1'(y)=I_0(y)-y^{-1}I_1(y)$, one has the relations among the indefinite integrals:

$$\begin{aligned}\int ye^{-y}I_0(y)dy &=\int\left[ye^{-y}(I_0+I_1)-y^2e^{-y}(I_0+I_1)+y^2e^{-y}\left[I_1+I_0-\frac{I_1}{y}\right]\right]dy \\ &=y^2e^{-y}(I_0+I_1)-\int ye^{-y}(I_0+I_1)dy \ , \end{aligned} \qquad (4.7)$$

giving

$$3\int ye^{-y}I_0(y)dy=y^2e^{-y}(I_0+I_1)-ye^{-y}I_0+\int I_0e^{-y}dy \ . \qquad (4.8)$$

In addition,

$$\int I_0e^{-y}dy=\int\left[ye^{-y}\left[I_1+I_0-\frac{I_1}{y}\right]+(I_0+I_1)(e^{-y}-ye^{-y})\right]dy=ye^{-y}(I_0+I_1) \ , \qquad (4.9)$$

Hence the integral on the right-hand side of Eq. (4.6) is found to be

$$\int e^{-y}(2y-1)I_0(y)dy =e^{-y}\{\tfrac{2}{3}[y^2I_0+(y^2+y)I_1]-y(I_0+I_1)\} \ . \qquad (4.10)$$

The relations (4.1)–(4.6) and (4.10) imply that, as $T\to 0^+$,

$$G^F(T)\sim\frac{a^4}{16}T^{-2}+\frac{a^3}{6\sqrt{\pi}}T^{-3/2}-\frac{a}{8\sqrt{\pi}}T^{-1/2} +O(\sqrt{T}) \ . \qquad (4.11)$$

In deriving Eq. (4.11) we have used the following asymptotic relations [11] valid as $z\to\infty$:

$$e^{-z}z^2I_0(z)\sim\frac{z^{3/2}}{\sqrt{2\pi}}+\frac{1}{8}\frac{\sqrt{z}}{\sqrt{2\pi}}+O\left[\frac{1}{\sqrt{z}}\right] \ , \qquad (4.12)$$

$$e^{-z}z^2I_1(z)\sim\frac{z^{3/2}}{\sqrt{2\pi}}-\frac{3}{8}\frac{\sqrt{z}}{\sqrt{2\pi}}+O\left[\frac{1}{\sqrt{z}}\right] \ . \qquad (4.13)$$

The Laplace transform of the kernel of the interacting part is given by

$$G^I(\sigma^2)=-a^2\sum_{n=1}^{\infty} n^2 f(n;\sigma a)-a^2\sum_{n=1}^{\infty} nf(n;\sigma a) \ , \qquad (4.14)$$

where

$$f(n;\sigma a)=\left[1+\frac{n^2}{\sigma^2a^2}\right]I_n(\sigma a)K_n(\sigma a) -I_n'(\sigma a)K_n'(\sigma a)-\frac{I_n'(\sigma a)}{\sigma a I_n(\sigma a)} \qquad (4.15)$$

is the function defined in Eq. (5.9) of Ref. [8], and in Ref. [9]. In fact (see Kennedy [12]) the sums $\sum_{n=1}^{\infty}$ in Eq. (4.14) diverge because of the factors of n^2 and n. This occurs because we are attempting to take the Laplace transform of a function $G(T)$ which is singular as $T \rightarrow 0$ [see Eq. (3.3)]. This difficulty can be avoided by first computing the sums $\sum_{n=1}^{N}$ for large σ^2, using the asymptotic expansion of $f(n;\sigma a)$ valid uniformly [8–10] with respect to n at large σa, and then taking the inverse Laplace transform before taking the limit $N \rightarrow \infty$. The first series in Eq. (4.14) has already been studied in the case of scalar fields [9]. In the case of the second series, the Watson transform used in Refs. [8,9] is a source of complications, because $nf(n;\sigma a)$ is not an even function of n. Instead, we take the inverse Laplace transform of the large-σ^2 expansion of $nf(n;\sigma a)$, and compute the sum (as an asymptotic series valid as $T \rightarrow 0^+$) with the help of the Euler-Maclaurin formula [13].

Setting $r = n/\sqrt{n^2+\sigma^2 a^2}$, one has the asymptotic series [8,9]

$$nf(n;\sigma a) \sim \frac{n\sqrt{n^2+\sigma^2 a^2}}{\sigma^2 a^2}\left[\frac{r}{2}\frac{1-r^2}{n} - \frac{r^4}{2}\frac{1-r^2}{n^2} + \frac{r^3}{8}\frac{(1-r^2)(1-12r^2+15r^4)}{n^3} + \frac{r^4}{16}\frac{(1-r^2)(2-53r^2+168r^4-125r^6)}{n^4} + \cdots\right], \tag{4.16}$$

valid as $\sigma \rightarrow \infty$, uniformly in n. This is derived from the uniform asymptotic expansions of $I_\nu(z), K_\nu(z), I'_\nu(z)$, and $K'_\nu(z)$ described in Refs. [10,11]. Denoting by L_I the inverse Laplace transform, one thus has the asymptotic series

$$L_I[nf(n;\sigma a)] \sim \frac{1}{2a^2} n e^{-n^2T/a^2} - \frac{2T^{3/2}}{3\sqrt{\pi}a^5} n^3 e^{-n^2T/a^2} + \frac{T}{8a^4} n e^{-n^2T/4a^2} - \frac{3T^2}{4a^6} n^3 e^{-n^2T/a^2} + \frac{5}{16}\frac{T^3}{a^8} n^5 e^{-n^2T/a^2}$$
$$+ \frac{1}{6\sqrt{\pi}}\frac{T^{3/2}}{a^5} n e^{-n^2T/a^2} - \frac{53}{30}\frac{T^{5/2}}{\sqrt{\pi}a^7} n^3 e^{-n^2T/a^2} + \frac{168}{105}\frac{T^{7/2}}{\sqrt{\pi}a^9} n^5 e^{-n^2T/a^2}$$
$$- \frac{50}{189}\frac{T^{9/2}}{\sqrt{\pi}a^{11}} n^7 e^{-n^2T/a^2} + \cdots . \tag{4.17}$$

Note that, when each term on the right-hand side of Eq. (4.17) is summed from $n=1$ to N, the resulting function of T does not always converge uniformly to the sum $\sum_{n=1}^{\infty}(\ )$ in a neighborhood $T \in (0,\delta)$ [see Eqs. (4.19)–(4.21)]. Nevertheless, a study of the error terms shows that it is valid to take the limit $N \rightarrow \infty$ as in Eqs. (4.19)–(4.21), and then examine the small-T behavior of the resulting contributions to $G(T)$.

In order to compute sums of the type $\sum_{n=1}^{\infty} n^{(2m+1)} e^{-n^2T/a^2}$ where $m = 0,1,2,\ldots$, we can use the Euler-Maclaurin formula [13]

$$\tfrac{1}{2}F(0)+F(1)+F(2)+\cdots - \int_0^\infty F(y)dy = -\tfrac{1}{2}\tilde{B}_2F'(0) - \frac{\tilde{B}_4}{4!}F'''(0) - \frac{\tilde{B}_6}{6!}F'''''(0)\cdots , \tag{4.18}$$

for the function $F(y) = ye^{-y^2T/a^2}$. In Eq. (4.18), the $\tilde{B}_i$ denote the Bernoulli numbers. Thus we get

$$\sum_{n=1}^{\infty} ne^{-n^2T/a^2} = \frac{a^2}{2T} - \frac{1}{12} - \frac{T}{120a^2} - \frac{T^2}{504a^4} + \cdots . \tag{4.19}$$

The other sums arising from Eq. (4.17) are obtained by differentiating Eq. (4.19) with respect to T. This yields

$$\sum_{n=1}^{\infty} n^3 e^{-n^2T/a^2} = \frac{a^4}{2T^2} + \frac{1}{120} + \frac{T}{252a^2} + \cdots , \tag{4.20}$$

$$\sum_{n=1}^{\infty} n^5 e^{-n^2T/a^2} = \frac{a^6}{T^3} - \frac{1}{252} + \cdots , \tag{4.21}$$

and so on.

Combining the resulting contribution to the asymptotic expansion of $G^I(T)$ as $T \rightarrow 0^+$ with the other piece, arising from the first term in Eq. (4.14), given in Ref. [9], as well as the expansion (4.11) of $G^F(T)$, one finds the asymptotic expansion of the heat kernel:

$$G(T) \sim \frac{a^4}{16}T^{-2} + a^3\left[\frac{1}{6\sqrt{\pi}} - \frac{\sqrt{\pi}}{8}\right]T^{-3/2} + a\left[\frac{5}{24\sqrt{\pi}} - \frac{11\sqrt{\pi}}{256}\right]T^{-1/2} + \frac{11}{360} + O(\sqrt{T}) , \tag{4.22}$$

valid as $T \rightarrow 0^+$. In particular, this yields

$$\zeta(0) = \tfrac{11}{360} \tag{4.23}$$

for the spin-$\frac{1}{2}$ field with spectral boundary conditions on the sphere.

V. CALCULATION OF $\zeta(0)$ FOR THE SPIN-$\frac{3}{2}$ FIELD WITH SPECTRAL BOUNDARY CONDITIONS

In this section we sketch the corresponding calculation of $\zeta(0)$ for a linearized spin-$\frac{3}{2}$ field subject to spectral boundary conditions on a three-sphere of radius a. As in the original quantum-gravity calculation of Schleich [8], we work only with physical degrees of freedom by imposing a gauge condition and constraints. This will exclude the contribution of gauge and ghost modes which appear in the full BRST-invariant path integral. (For a BRST-invariant approach to computing $\zeta(0)$ for a spin-$\frac{3}{2}$ field with local boundary conditions, see Ref. [14].) The value $\zeta(0)=-\frac{289}{360}$ found here for spectral boundary conditions is again identical to that found using the natural local boundary conditions [1], working with the same physical degrees of freedom inside the three-sphere.

The spin-$\frac{3}{2}$ field, as appearing, e.g., in $N=1$ supergravity, is described by a potential $(\psi^A{}_\mu,\tilde{\psi}^{A'}{}_\mu)$ in the Euclidean regime ($\mu=0,1,2,3$). In a Hamiltonian treatment [15], the quantities $(\psi^A{}_i,\tilde{\psi}^{A'}{}_i)$ are the dynamical variables ($i=1,2,3$), while $(\psi^A{}_0,\tilde{\psi}^{A'}{}_0)$ appear as Lagrange multipliers. The gravitational field is correspondingly described by the tetrad $e^{AA'}{}_\mu$. Taking the geometry to be flat, and $x^0=\tau$ to be the radial distance from the origin, while x^i ($i=1,2,3$) are coordinates on the three-sphere, we impose the gauge conditions

$$e_{AA'}{}^j\psi^A{}_j=0,\quad e_{AA'}{}^j\tilde{\psi}^{A'}{}_j=0\ . \tag{5.1}$$

If in addition we require that the dynamical variables $(\psi^A{}_i,\tilde{\psi}^{A'}{}_i)$ obey the linearized supersymmetry constraint equations [15] on the family of three-spheres centered on the origin, then the expansion of $(\psi^A{}_i,\tilde{\psi}^{A'}{}_i)$ in harmonics [16], analogous to Eqs. (1.2) and (1.3) for the spin-$\frac{1}{2}$ field, takes the simplified form

$$\psi^A{}_i=\frac{\tau^{-3/2}}{2\pi}\sum_{npq}\alpha_n^{pq}[m_{np}(\tau)\beta^{nqABB'}+\tilde{r}_{np}(\tau)\bar{\mu}^{nqABB'}]e_{BB'i}\ , \tag{5.2}$$

$$\tilde{\psi}^{A'}{}_i=\frac{\tau^{-3/2}}{2\pi}\sum_{npq}\alpha_n^{pq}[\tilde{m}_{np}(\tau)\bar{\beta}^{nqBA'B'}+r_{np}(\tau)\mu^{nqBA'B'}]e_{BB'i}\ . \tag{5.3}$$

In the summations, n runs from 0 to ∞, and p and q now run from 1 to $(n+1)(n+4)$. For each n, α_n^{pq} is a matrix again block diagonal in the indices pq, with blocks $\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$. Here

$$\beta^{nqABB'}=-\rho^{nq(ABC)}n_C{}^{B'}\ , \tag{5.4}$$

where $n^{AA'}=i_e n^{AA'}$ is the Lorentzian normal [2,7], and the harmonic $\rho^{nq(ABC)}$, symmetric in its three unprimed indices, has positive eigenvalue $\frac{1}{2}(n+\frac{5}{2})$ of the appropriate three-dimensional Dirac operator. Similarly

$$\mu^{nqBA'B'}=-\sigma^{nq(A'B'C')}n^B{}_{C'}\ , \tag{5.5}$$

where the harmonic $\sigma^{nq(A'B'C')}$, symmetric in its three primed indices, also has positive eigenvalue $\frac{1}{2}(n+\frac{5}{2})$. The harmonics $\bar{\mu}^{nqABB'}$ and $\bar{\beta}^{nqBA'B'}$ are given similarly in terms of harmonics $\bar{\sigma}^{nq(ABC)}$ and $\bar{\rho}^{nq(A'B'C')}$, which have negative eigenvalues $-\frac{1}{2}(n+\frac{5}{2})$ of the three-dimensional operator.

Just as in the spin-$\frac{1}{2}$ case, the simplest natural spectral boundary conditions for spin $\frac{3}{2}$ are

$$m_{np}(a)=0,\quad r_{np}(a)=0,\quad \forall n,p\ . \tag{5.6}$$

The requirement that the physical fields be bounded near the origin $\tau=0$, in the Hartle-Hawking path integral for the linearized spin-$\frac{3}{2}$ theory, implies that $m_{np},r_{np},\tilde{m}_{np},\tilde{r}_{np}$ vanish $\forall n,p$ at $\tau=0$. The Euclidean action I_E for the linearized spin-$\frac{3}{2}$ field, working only with the physical degrees of freedom given in Eqs. (5.2) and (5.3), is analogous to the spin-$\frac{1}{2}$ expression in Eqs. (2.3) and (2.4):

$$I_E=\sum_{n=0}^{\infty}\sum_{p=1}^{(n+1)(n+4)}[\hat{I}_n(m_{np},\tilde{m}_{np})+\hat{I}_n(r_{np},\tilde{r}_{np})]\ , \tag{5.7}$$

where

$$\hat{I}_n(x,\tilde{x})=\int_0^a d\tau\left[\tfrac{1}{2}(\tilde{x}\dot{x}+x\dot{\tilde{x}})-\frac{n+\frac{5}{2}}{\tau}\tilde{x}x\right]\ . \tag{5.8}$$

The resulting Berezin integration for the Hartle-Hawking amplitude $K_{\rm HH}$ gives the formal expression

$$K_{\rm HH}=\prod_{n=0}^{\infty}\prod_{k=1}^{\infty}\left[\frac{\hat{E}_{nk}}{\tilde{\mu}}\right]^{2(n+1)(n+4)}\ . \tag{5.9}$$

The $\hat{E}_{nk}$ ($k=1,2,\ldots$) are the positive eigenvalues of the system

$$\left[\frac{d}{d\tau}-\hat{\nu}\right]x_{nk}=\hat{E}_{nk}\tilde{x}_{nk}\ , \tag{5.10}$$

$$\left[-\frac{d}{d\tau}-\hat{\nu}\right]\tilde{x}_{nk}=\hat{E}_{nk}x_{nk}\ , \tag{5.11}$$

subject to $x_{nk}(0)=\tilde{x}_{nk}(0)=x_{nk}(a)=0$, where $\hat{\nu}=(n+\frac{5}{2})/\tau$. The solutions are of the form

$$x_{nk}=B_{nk}\sqrt{\tau}J_{n+2}(\hat{E}_{nk}\tau)\ , \tag{5.12}$$

$$\tilde{x}_{nk}=-B_{nk}\sqrt{\tau}J_{n+3}(\hat{E}_{nk}\tau)\ , \tag{5.13}$$

with the eigenvalue condition

$$J_{n+2}(\hat{E}_{nk}a)=0,\quad n=0,1,2,\ldots\ , \tag{5.14}$$

and degeneracy $2(n+1)(n+4)$.

The formal expression (5.9) for $K_{\rm HH}$ is then evaluated by studying the corresponding ζ function and heat kernel, by analogy with Secs. II–IV. The formulas in Sec. III can be straightforwardly modified to allow for the different degeneracy $2(n+1)(n+4)$, different differential equations (5.10) and (5.11), and eigenvalue condition (5.14). In particular, Eqs. (3.14) and (3.15) should be modified by replacing the factor of $(n+2)$ by $(n+4)$, and by changing the order $n+1$ of each modified Bessel function I_{n+1} or K_{n+1} to the order $n+2$.

For the sake of brevity we consider only the constant part $B_4=\zeta(0)$ in the expansion (3.3) of the spin-$\frac{3}{2}$ heat kernel $G(T)$ as $T\to 0^+$. The results of Eqs. (4.2)–(4.13) show that the free part $G^F(T)$ of the heat kernel for spin $\frac{3}{2}$, given by Eq. (4.1) with the factor $(n+1)$ replaced by $(n+3)$, gives no contribution to $\zeta(0)$. The only contribution to $\zeta(0)$ for spin $\frac{3}{2}$ arises from the interacting part

$$\hat{G}^{I}(\sigma^2) = -a^2 \sum_{n=2}^{\infty} (n^2-2) f(n;\sigma a) - a^2 \sum_{n=2}^{\infty} n f(n;\sigma a) , \tag{5.15}$$

where $f(n;\sigma a)$ is given by Eq. (4.15). As in Sec. IV, these sums diverge, and the calculation should strictly proceed by first computing the sums $\sum_{n=1}^{N}$ for large σ^2, taking the inverse Laplace transform, and finally taking the limit $N \to \infty$. The contribution from the first term in Eq. (5.15) is found, following Refs. [8,9,12], by using a Watson transform:

$$-a^2 \sum_{n=2}^{\infty} (n^2-2) f(n;\sigma a) = -\frac{a^2}{4i} \int_{C'-Q} d\nu (\nu^2-2) f(\nu;\sigma a) \cot \pi\nu . \tag{5.16}$$

Here the contour of integration $C'-Q$ (compare with Fig. 1 of Ref. [8]) encloses all poles along the real axis, except for those at $\nu = 0, \pm 1$. Using the uniform asymptotic expansion of $f(\nu;\sigma a)$, and computing the contribution from the poles at $\nu = 0, \pm 1$, one finds from the large-σ behavior of Eq. (5.16) that the first term in Eq. (5.15) contributes $-\frac{121}{90}$ to $\zeta(0)$. Using the results of Sec. IV, the second term in Eq. (5.15) is found to contribute $\frac{13}{24}$ to $\zeta(0)$.

Combining these results we find

$$\zeta(0) = -\frac{289}{360} \tag{5.17}$$

for the linearized spin-$\frac{3}{2}$ field subject to spectral boundary conditions on the three-sphere, working only with physical degrees of freedom in the gauge (5.1). Just as in the spin-$\frac{1}{2}$ case of Sec. IV, the value of $\zeta(0)$ is equal to that found previously for the spin-$\frac{3}{2}$ field subject to the natural local boundary conditions [1,2]. Specifically, the value $\zeta(0) = -\frac{289}{360}$ was also found for spin $\frac{3}{2}$, working only with physical degrees of freedom subject again to the gauge conditions (5.1), with the local boundary conditions

$$\sqrt{2}\, {}_e n_A{}^{A'} \psi^A{}_i = \epsilon \tilde{\psi}^{A'}{}_i , \tag{5.18}$$

where $\epsilon = \pm 1$. These local boundary conditions are part of a locally supersymmetric family of boundary conditions for different spins [1,2,5,14], which includes the local boundary conditions (1.1) for spin $\frac{1}{2}$. Of course, as remarked earlier, the results of $\zeta(0)$ calculations for gauge fields such as spin $\frac{3}{2}$ will be modified by the contribution of gauge and ghost modes in a complete BRST-invariant calculation.

VI. COMMENTS

It is striking that the same value $\zeta(0) = \frac{11}{360}$ is obtained for a massless Majorana spin-$\frac{1}{2}$ field on a ball in Euclidean four-space, bounded by a sphere of radius a, whether local boundary conditions (1.1) or spectral boundary conditions are imposed. The different eigenvalue conditions, respectively,

$$[J_{n+1}(Ea)]^2 - [J_{n+2}(Ea)]^2 = 0 , \quad n = 0,1,2,\dots, \quad \text{degeneracy } (n+1)(n+2) , \tag{6.1}$$

and

$$J_{n+1}(Ea) = 0, \quad n = 0,1,2,\dots, \quad \text{degeneracy } 2(n+1)(n+2) , \tag{6.2}$$

offer no obvious explanation of this equality—although they suggest an alternative approach through studying the asymptotic distribution of eigenvalues. The same holds for the value $\zeta(0) = -\frac{289}{360}$ found for the spin-$\frac{3}{2}$ field (taking only physical degrees of freedom), both for local and spectral boundary conditions. There the eigenvalue conditions are, respectively,

$$[J_{n+2}(Ea)]^2 - [J_{n+3}(Ea)]^2 = 0 , \quad n = 0,1,2,\dots, \quad \text{degeneracy } (n+1)(n+4) , \tag{6.3}$$

and

$$J_{n+2}(Ea) = 0, \quad n = 0,1,2,\dots, \quad \text{degeneracy } 2(n+1)(n+4) . \tag{6.4}$$

The eigenvalue conditions for fermions, subject to spectral boundary conditions on the three-sphere, are in fact similar, but not identical, to the eigenvalue conditions for bosons, subject to local (Dirichlet) conditions on the three-sphere. For example, in the case of a scalar field [9,12], Dirichlet conditions give

$$J_{n+1}(Ea) = 0, \quad n = 0,1,2,\dots, \quad \text{degeneracy } (n+1)^2 . \tag{6.5}$$

For a Maxwell field, taking only physical degrees of freedom [17], Dirichlet (magnetic) boundary conditions give

$$J_{n+2}(Ea) = 0, \quad n = 0,1,2,\dots, \quad \text{degeneracy } 2(n+1)(n+3) . \tag{6.6}$$

Analogous equations hold in the spin-2 case [8]. Because of the equality of the $\zeta(0)$ values for fermions with both local and spectral boundary conditions, one might then ask if there is some connection between $\zeta(0)$ values for fermions and bosons with adjacent spins (taking local boundary conditions), reminiscent of supersymmetry. One would also like to understand whether the equality of the local and spectral values for $\zeta(0)$ is a feature peculiar to the highly symmetrical example of a three-sphere surrounding a region of flat four-space, or whether there is an extension of this result to a more general context.

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*Present address: INFN, Gruppo IV Sezione di Napoli, Italy.

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