Soliton stars at finite temperature

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We investigate the properties of soliton stars in the Lee-Wick model when a temperature dependence is introduced into this model. It is found that at some critical temperature $T_c \sim 100-200$ MeV, a first-order phase transition occurs leading to the formation of soliton stars with characteristics similar to those considered by Lee and Pang but with a much smaller mass and size. We study the evolution of these soliton stars with the temperature from the early Universe to the present time.

INTRODUCTION

In 1974, Lee and Wick [1] proposed a simple field-theoretical model for interacting scalar bosons or for fermions interacting with scalar bosons. In a subsequent work, Friedberg, Lee, and Sirlin [2] presented a class of solitonic solutions of these models, the nontopological solitons. These soliton properties motivated the use of the Lee-Wick model as an effective theory of interacting quarks in hadron spectroscopy [3]. The scalar field plays the role of an effective field which confines the quarks inside the soliton. The predictions of this Lee-Wick model for the hadron spectra are as satisfactory as those of other effective models for nonperturbative QCD. More recently, in a series of papers Lee and collaborators [4] applied the Lee-Wick model to astrophysics. In particular they addressed the question of the stability of the nontopological solitons when the number of fermions inside the soliton, instead of being a few, becomes extremely large ($>10^{60}$). They found that such cold stellar configurations can exist and they called them soliton stars. They also showed that soliton stars with a mass $\sim 10^{12}\, M_\odot$ and size $\sim 10^{-1}$ light years can be formed. However, the properties of these soliton stars were obtained with a very special choice of the parameters in the model, and, as will be seen, a different choice can lead to soliton stars with very different properties, more like those of neutron stars.

Although the existence of gigantic stars of the type found in Ref. [4] does not seem to be compatible with the present experimental observations, one could contemplate the possibility of the formation of such objects at an earlier epoch of the evolution of the Universe. In the work presented in this paper we explore such a possibility and study the relics of these objects at the present time. Since the early universe was hot we were led to consider the temperature dependence in the Lee-Wick model and investigate the possible formation of a soliton star at finite temperature and then its evolution with temperature from the early Universe to the present time.

The plan of this paper is as follows. We first recall the properties of the Lee-Wick model and study the properties of the soliton solutions, in particular, the stability conditions, in terms of the parameters of the model and in terms of the number of fermions inside the soliton when this number is very large. We also calculate in a simple way the effects of gravity on the stability properties of the soliton stars, especially, the limits where they collapse into black holes. We then make an analysis at finite temperature and show the possibility of a phase transition which leads to a model with parameters similar to those in Ref. [4] but at a nonzero temperature with the consequence of the formation of a soliton star. We finally envisage the evolution of the soliton star to the present time. In the course of this work we derive a very simple method for introducing temperature dependence into the Lee-Wick model which can be also applied more generally to other field-theoretical models. The details of this method as well as the various approximations used in our paper are presented in the appendixes.

SOLITON STARS AT ZERO TEMPERATURE

The Lee-Wick model. This model in the version with fermions [5] is defined by the Lagrangian

$$\mathcal{L}(x)=\tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma-U(\sigma)+\bar{\Psi}\gamma^\mu\partial_\mu\Psi-m\bar{\Psi}\left[1-\frac{\sigma}{\sigma_0}\right]\Psi, \tag{1}$$

where the fermions $\Psi(x)$ interact via the scalar field $\sigma(x)$.

The self-interaction of the σ field is usually given in the form

$$U(\sigma)=\frac{a}{2}\sigma^2+\frac{b}{3}\sigma^3+\frac{c}{4}\sigma^4\ , \tag{2}$$

where a,b,c are free parameters. This form was chosen so that the model is renormalizable. For the purposes of this work we have rewritten it as

$$U(\sigma)=\tfrac{1}{2}m_\sigma^2\sigma^2\left[1-\frac{\sigma}{\sigma_0}\right]^2+B\left[4\left[\frac{\sigma}{\sigma_0}\right]^3-3\left[\frac{\sigma}{\sigma_0}\right]^4\right]. \tag{3}$$

In this form the three parameters σ_0, B, and m_σ have a more direct physical interpretation. For example, with B positive the constant field configuration $\sigma=\sigma_0$ is a local minimum of U which has an energy density (pressure) B greater than the energy density at the absolute minimum at $\sigma=0$. This form allows also a more immediate comparison with the Lagrangian used in Ref. [4].

The energy and mass of the soliton for a large number of fermions. As in Ref. [4] we want to investigate the properties of a system with a very large number of fermions ($N\geq 10^{60}$) interacting through the scalar field σ. In particular we would like to study the possibility that stable solitonic solutions exist. We consider a spherical volume of radius R in which, apart from a surface shell, we take the mean-field approximation: $\sigma=\sigma_0$ inside and $\sigma=0$ outside. In the surface shell, σ changes smoothly from σ_0 to 0 with a profile that is determined by minimizing the energy in the surface region. It is not difficult to show from the Lagrangian of Eqs. (1)–(3) that the shell is of thickness $\sim 1/m_\sigma$ and has a surface energy density $s\sim\frac{1}{6}m_\sigma\sigma_0^2$, and for large R, the σ field energy is given by a volume term and a surface term

$$E(\sigma)=\tfrac{4}{3}\pi BR^3+4\pi sR^2 . \tag{4}$$

In the mean-field approximation, fermions become effectively massless due to their interaction with the σ field inside the sphere but acquire a large mass m outside which keeps them inside. For very large N, the fermions will be uniformly distributed throughout the volume and their energy density can be evaluated in the relativistic Thomas-Fermi approximation. This gives $\frac{1}{4}(9\pi)^{2/3}(N/V)^{4/3}$. The total energy of the interacting system is therefore

$$E=\alpha\left[\frac{N}{R}\right]^{4/3}+4\pi sR^2+\frac{4\pi}{3}BR^3 \tag{5}$$

with $\alpha=\frac{1}{2}(\frac{3}{2})^{5/3}\pi^{1/3}$.

It is easy to see that the minimum energy configuration of this system has a finite radius and this configuration is called the soliton star in Ref. [4]. The expression (5) also shows that the σ field confines the fermions inside the soliton through both a volume and a surface term, although only one such term is necessary to give an energy minimum with a finite radius. The relative importance of these two terms not only depends upon the pressure B and surface tension s but also on the fermion number N. For $B>0$ and N large enough the volume term will dominate the surface term. This can be seen by making a change of scale

$$R=\left[\frac{\alpha}{4\pi B}\right]^{1/4}N^{1/3}x . \tag{6}$$

Then

$$E=(4\pi\alpha^3B)^{1/4}N\left[\frac{1}{x}+\frac{1}{3}x^3+\epsilon x^2\right] \tag{7}$$

with

$$\epsilon=\left[\frac{4\pi}{\alpha}\right]^{1/4}\frac{s}{B^{3/4}}\frac{1}{N^{1/3}} . \tag{8}$$

If $\epsilon\ll 1$ the minimum energy occurs for $x\simeq 1$ and the surface contribution ϵx^2 is negligible. This corresponds to

$$N\gg\left[\frac{4\pi}{\alpha}\right]^{3/4}\left[\frac{s}{B^{3/4}}\right]^3 . \tag{9}$$

We will see that this bound plays an important role in that it governs the properties of the soliton stars.

The Lee-Pang soliton stars [4]. In Ref. [4], the constant B is taken to be zero and $m_\sigma=\sigma_0=30$ GeV so that the confinement is due only to the surface tension s. $U(\sigma)$ vanishes for both $\sigma=0$ and $\sigma=\sigma_0$; the vacuum is degenerate. In this case the total energy of the system is

$$E=\alpha\frac{N^{4/3}}{R}+4\pi sR^2 . \tag{10}$$

Minimizing E with respect to R at fixed N leads to a soliton mass of

$$M=\tfrac{3}{2}\alpha^{2/3}(8\pi s)^{1/3}N^{8/9} \tag{11}$$

with a radius

$$R=\left[\frac{\alpha}{8\pi s}\right]^{1/3}N^{4/9} \tag{12}$$

and a fermion number density

$$\rho=\frac{6s}{\alpha}N^{-1/3} . \tag{13}$$

From Eq. (12) the radius increases faster than $N^{1/3}$ for large N so that the fermion density decreases as N increases; also as pointed out in Ref. [4] the exponent of N in Eq. (11) is <1 ensuring, for large N, the stability of the soliton against break up into smaller units. Thus stable soliton star configurations with very large N result from this model.

Effects of gravity. For very large N, however, the effects of gravitation become important and for a critical value $N=N_c$ a soliton star can collapse into a black hole. The effects of gravitation were calculated by Lee and Pang [4] using general relativity. With their values for m_σ and σ_0, $s=\frac{1}{6}m_\sigma\sigma_0^2=\frac{1}{6}$ (30 GeV)^3, this leads to a critical fermion number $N<N_c=10^{76}$ which implies that $M_c=10^{13}$ $M_\odot$ and $R_c\sim 1$ light years.

Actually these results can be understood in the simple approximation of coupling the Newtonian gravitational field to the energy density inside the soliton, treating this as constant throughout. Including the gravitational energy it modifies Eq. (10) so that

$$E(R)=\frac{\alpha N^{4/3}}{R}+4\pi sR^2-\frac{3}{5}\frac{G}{R}\left[\frac{\alpha N^{4/3}}{R}\right]^2 . \tag{14}$$

Making a change of length scale

$$R=\left[\frac{\alpha N^{4/3}}{8\pi s}\right]^{1/3} x \tag{15}$$

and defining

$$\epsilon=\tfrac{6}{5}G(8\pi s)^{2/3}\alpha^{1/3}N^{4/9} \tag{16}$$

one gets

$$E(x)=\tfrac{1}{2}(8\pi s)^{1/3}\alpha^{2/3}N^{8/9}\left[\frac{2}{x}+x^2-\frac{\epsilon}{x^3}\right]. \tag{17}$$

Neglecting gravity ($\epsilon=0$) this has its minimum at $x=1$ and gives the results of Eqs. (11) and (12). For $\epsilon \ll 1$, gravitational effects will only be a small perturbation in the neighborhood of this minimum although the configuration becomes metastable to collapse to small x (black-hole formation). The condition for the metastable minimum to exist is that $-2/x^2+2x-3\epsilon/x^4$ should be zero for some positive x and this is satisfied if $\epsilon<(\tfrac{2}{5})^{5/3}$. This condition is not met for

$$N>\left[\frac{1}{3}\right]^{9/4}\left[\frac{2}{5}\right]^{3/2}\frac{1}{G^{9/4}(8\pi s)^{3/2}\alpha^{3/4}} \tag{18}$$

and the star will collapse to a black hole. Using this condition we get exactly the same results for N_c, M_c, and R_c as in Ref. [4]. A similar conclusion can be reached if one simply adopts the Schwarzschild criterion that for stability, R must be greater than $2GM$.

Thus with $B=0$, a soliton star can reach the mass of a galaxy before becoming unstable to black-hole formation. It is the special choice of parameters of Ref. [4], leading to the vacuum degeneracy, along with the assumption of massless fermions, that gives this result.

Another type of soliton star. If the condition for a degenerate vacuum is relaxed and one takes $B>0$, then for large N, the surface term is negligible. It can be easily seen that this occurs for very realistic values of B. For example, with $B=(100\ \text{MeV})^4$, a value generally used in hadron spectroscopy, and $s=\tfrac{1}{6}(30\ \text{GeV})^3$ the lower bound in the relation (9) is as low as 10^{22}. (The value for s is not accurately known but we have checked that all our conclusions are insensitive to the actual value for s in the range $(1\ \text{GeV})^3$–$(30\ \text{GeV})^3$. In the following we will therefore take $s=\tfrac{1}{6}(30\ \text{GeV})^3$ as in Ref. [4].)

In this case minimizing the energy

$$E=\alpha\frac{N^{4/3}}{R}+\frac{4}{3}\pi BR^3, \tag{19}$$

one gets a soliton mass of

$$M=\tfrac{4}{3}\alpha^{3/4}(4\pi B)^{1/4}N \tag{20}$$

with a radius

$$R=\left[\frac{\alpha}{4\pi B}\right]^{1/4}N^{1/3}. \tag{21}$$

In contrast to Eq. (11) the mass is now proportional to N and gravity is now necessary to ensure stability against break up.

Again we estimate the gravitational effects by coupling the Newtonian gravitational field to the energy density treating this as constant throughout the star. Including gravitational energy it modifies Eq. (19) so that, in contrast with Eq. (14),

$$E(R)=\frac{\alpha N^{4/3}}{R}+\frac{4}{3}\pi BR^3-\frac{3}{5}\frac{G}{R}\left[\frac{\alpha N^{4/3}}{R}+\frac{4}{3}\pi BR^3\right]^2. \tag{22}$$

Making a change of length scale $R=(\alpha/4\pi B)^{1/4}N^{1/3}x$ and defining $\epsilon=\tfrac{1}{5}GN^{2/3}(4\pi\alpha B)^{1/2}$ gives

$$E(x)=\tfrac{1}{3}\alpha^{3/4}(4\pi B)^{1/4}N\left[\frac{3}{x}+x^3-\frac{\epsilon}{x}\left[\frac{3}{x}+x^3\right]^2\right]. \tag{23}$$

Neglecting gravity this has a minimum at $x=1$ and gives the results of Eqs. (20) and (21). Including gravity the energy minimum becomes metastable, but for $\epsilon \ll 1$ gravitational effects in the neighborhood of the minimum are small. We find that the minimum becomes unstable to gravitational collapse if $\epsilon>0.046$ and taking $B=(100\ \text{MeV})^4$ one gets a limit for N, $N_c \sim 10^{58}$ which in turn yields $M_c \sim 3M_\odot$, and $R_c \sim 25$ km. These characteristics are more like those of neutron stars.

Thus, the properties of soliton stars depend dramatically on the choice of the parameters for the self-interaction $U(\sigma)$ in the Lee-Wick model. The soliton stars can have a galactic mass in one case, and a few solar mass in the other case. No stars of the Lee-Pang type have been identified, so we presume that they do not exist, at least at the present time. There is still a possibility that within the Lee-Wick model such stars could have been formed at an earlier epoch. As seen above, the two types of soliton stars are a consequence of the choice of two different sets of parameters. One could envisage a transition between these two sets of parameters when another variable, such as temperature, is introduced. Since the analysis has been made so far at zero temperature, we are led to introduce finite temperatures into the Lee-Wick model.

SOLITON STARS AT FINITE TEMPERATURE

Introduction of the temperature dependence into the Lee-Wick model. We are, of course, interested in the situation where the number of fermions N exceeds by far the limit 10^{22} at which the volume term dominates the surface term at zero temperature. Thus, in contrast with Ref. [4] we adopt the case where $B>0$.

At zero temperature, the equilibrium configuration of the soliton star corresponds to a minimum of the energy but at finite temperature the equilibrium is given by the minimum of the free energy. When the methods exposed in the appendixes are applied to the Lee-Wick Lagrangian, the contribution of the σ field to the free energy density is given by Eq. (A13):

$$f(\sigma)=U(\sigma)+\frac{1}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln(1-e^{-\beta\sqrt{k^2+M^2}}) \tag{24}$$

with $\beta=1/(k_BT)$, and

$$M^2=\frac{\partial^2 U}{\partial\sigma^2}=m_\sigma^2\left[1+6\frac{\sigma}{\sigma_0}\left[\frac{\sigma}{\sigma_0}-1\right]\right]+\frac{12B}{\sigma_0^2}\frac{\sigma}{\sigma_0}\left[2-\frac{3\sigma}{\sigma_0}\right]. \tag{25}$$

At zero temperature, $f(\sigma)=U(\sigma)$ and the free energy density has two minima, one at $\sigma=0$ (outside the soliton) and the other at $\sigma=\sigma_0$ (inside). At finite temperature but $k_BT \ll M$, the temperature-dependent contribution to $f(\sigma)$, the second term in Eq. (24), is exponentially small because βM is large both inside and outside the soliton, and the σ field free energy density is little changed from its zero temperature values, namely, zero outside the soliton and B inside.

It is the thermal excitation of the fermion field that provides the important temperature-dependent contributions to the free energy density, in particular inside the soliton. Outside we have seen that the σ field is frozen near to its zero temperature value $\sigma=0$ and the fermion acquires a very large mass. As can be seen from Eq. (A25) of Appendix A, because of this large mass and unless the temperature is very high, the fermion field, like the boson field, makes only exponentially small contributions to the free energy outside the soliton. In contrast, inside the soliton the σ field is frozen near to $\sigma=\sigma_0$ and the fermion mass is very small; hence, the temperature dependence becomes dramatic even at relatively low temperatures. It is shown in Appendix B that when mass can be neglected the fermion free energy density is β^{-4} times a function of one dimensionless parameter $\beta^3\rho$, ρ is the fermion number density. It can be anticipated that at increased temperatures the internal fermion pressure will also increase and inflate the soliton so that ρ will decrease. Hence at high enough temperatures, $\beta^3\rho$ will be small and a power-series expansion of the free energy density will be appropriate. Keeping in Eq. (B11) the first two terms of this expansion, one gets the fermion contribution to the free energy density:

$$f=\frac{1}{\beta^4}\left[-\frac{7\pi^2}{180}+\frac{3}{2}(\beta^3\rho)^2\right]. \tag{26}$$

We will see in the next section that at the time of soliton star formation, the parameter $\beta^3\rho$ is indeed small, and this truncated expansion gives an accurate representation for f.

Thus for a spherical soliton with fermion number N, radius R, and at high enough temperature T, Eqs. (24) and (26) give a total free energy:

$$F=\frac{9}{2}\beta^2\frac{N^2}{4\pi R^3}+4\pi sR^2+\frac{4}{3}\pi R^3\left[B-\frac{7\pi^2}{180\beta^4}\right]. \tag{27}$$

Formation of soliton stars. The term proportional to volume in Eq. (27) becomes negative at temperatures greater than a critical value:

$$k_BT_c=1/\beta_c=\left[\frac{180B}{7\pi^2}\right]^{1/4}, \qquad T_c=127\text{ MeV for } B^{1/4}=100\text{ MeV}. \tag{28}$$

Hence for temperature greater than T_c, the soliton phase would fill the whole of space. The universe would be homogeneous with the scalar field σ having the value σ_0 and the fermions dispersed. As the universe cools, at $T=T_c$ a phase transition occurs. Bubbles of the phase $\sigma=0$ and no fermions, will be formed. During the phase transition the temperature remains constant while the energy for the expansion of the universe is provided by the latent heat liberated as the bubbles expand. It can be shown that this latent heat density has a contribution equal to B from the σ field and $3B$ from the fermion-antifermion pairs. The bubbles of the phase $\sigma=0$ and no fermions will eventually coalesce and the phase $\sigma=\sigma_0$ and with fermions will be left as the isolated bubbles. The soliton stars are formed. This phase transition is similar to the cosmic separation of phases discussed in Ref. [6]. The soliton bubbles will continue to contract at $T=T_c$ until they are stabilized by the small surface term in Eq. (27) which becomes

$$F=\frac{9}{2}\beta_c^2\frac{N^2}{4\pi R^3}+4\pi sR^2. \tag{29}$$

The soliton will stabilize at the minimum free energy

$$F_{\min}=5\left[\frac{3\pi}{4}\right]^{1/5}\beta_c^{4/5}s^{3/5}N^{4/5} \tag{30}$$

with a radius

$$R=\left[\frac{3}{4}\right]^{3/5}\left[\frac{1}{\pi}\right]^{2/5}\left[\frac{\beta_c^2}{s}\right]^{1/5}N^{2/5}. \tag{31}$$

The fermion number density is given by

$$\beta_c^3\rho=\frac{3\beta_c^3N}{4\pi R^3}=\left[\frac{s^4}{B^3}\right]^{3/20}N^{-1/5}. \tag{32}$$

This parameter $\beta^3\rho$ is the expansion parameter in Eqs. (B11) and (26). With $B=(100\text{ MeV})^4$ and $s=\frac{1}{6}(30\text{ GeV})^3$ this expansion parameter is less than $\frac{1}{10}$ provided that $N>10^{25}$. This shows that for the values of N of interest ($N\sim10^{50}$) the low-density, high-temperature expansion in Eq. (26) is quite adequate.

The soliton now possesses characteristics similar to those considered in Ref. [4] [see Eqs. (11)–(13)] but at a finite temperature T_c; in particular for large N the soliton is stable against break up [Eq. (30)] and the radius increases faster than $N^{1/3}$ [Eq. (31)] and fermion density decreases as N increases. Such solitons could have been formed in the early universe at a time $\sim10^{-4}$ s after the big bang, corresponding to the critical temperature $T_c\sim100$ MeV.

The total energy (the mass) is however quite different; at the minimum this is $M=(\partial/\partial\beta)(\beta F)$ with F given by Eq. (27). This yields a mass

$$M=\frac{16\pi}{3}BR^3+\frac{27}{8\pi}\beta_c^2\frac{N^2}{R^3} \tag{33}$$

which with Eq. (31) is

$$M=\frac{16\pi}{3}BR^3+8\pi sR^2 . \tag{34}$$

At the formation time the effects of gravity can be estimated with the Schwarzschild limit $R>R_c\sim 2GM$ and Eqs. (30) and (33) yield a critical fermion $N_c\sim 10^{52}$. This bound is considerably smaller than that found in Ref. [4] at zero temperature. On the other hand, the fermion number N_H inside the soliton horizon at $T_c=100$ MeV is $\sim 10^{49}$, smaller than N_c, so that black holes cannot be formed in the phase transition. Taking $N\sim 10^{49}$ yields, at the formation time, a maximum soliton radius of 2 km and a mass $1.5\times 10^{-3}M_{\odot}$. The radius of the horizon volume at the time of the phase transition ($\sim 10^{-4}$ s) can be estimated to be of the order 30 km so that the distance between solitons would be about ten times the soliton radius. Thus, the soliton could be considered as giving rise to high-density fluctuations in the mass distribution in the universe. It is an interesting consequence of this model that inhomogeneities of the order of the soliton size in the energy density of the universe can appear at this relatively early stage in its evolution.

Evolution of soliton stars. When the phase transition is completed the temperature will fall, the term proportional to volume in Eq. (27) will no longer be zero, and it will contribute to compress the soliton. By arguments similar to those leading to Eq. (9) it can be shown that the volume term will play the dominant role in compressing the soliton if

$$\frac{T_c-T}{T_c}\gg\frac{90}{7\pi}\left[\frac{27\beta_c^3 s}{\pi^3N^2}\right]^{1/5}\beta_c^3 s \tag{35}$$

and for $N=10^{49}$ this will be so when

$$\left[\frac{T_c-T}{T_c}\right]>10^{-12}$$

$$(s=\tfrac{1}{6}(30\ \text{GeV})^3,\beta_c=10^{-2}\ \text{MeV}^{-1}) . \tag{36}$$

It is interesting to observe that the volume term dominates when the temperature has fallen from the critical temperature by only 1 °C. After this small temperature change the surface term can be neglected and, minimizing expression (27) for F with respect to R yields

$$R=\frac{3}{\pi^{2/3}}\left[\frac{5}{56}\right]^{1/6}\beta_c N^{1/3}\frac{\beta/\beta_c}{[(\beta/\beta_c)^4-1]^{1/6}} \tag{37}$$

and a mass

$$M=\frac{\partial}{\partial\beta}\beta F=\frac{4}{3}\pi R^3\left[B+\frac{7\pi^2}{60\beta^4}\right]+\frac{27}{8\pi}\frac{\beta^2N^2}{R^3}$$

$$=\frac{16\pi}{3}BR^3 . \tag{38}$$

These formulas hold so long as our expansion parameter $\beta^3\rho$ is less than 1. As the temperature cools this condition is eventually violated.

A soliton star formed at $T=T_c$ could undergo several processes. They could evaporate into hadrons either from its surface [7] or bubbles can be nucleated inside [8]. Estimates of the evaporation rate from the surface can be made [7]. For nucleons the number of evaporated particles would be $\sim 10^{46}$. This estimate does not include reabsorption of hadrons into the soliton so that 10^{46} can be considered as an upper limit. The rate of bubble nucleation inside the soliton can also be estimated [8]. It is proportional to $e^{-W_c/T}$ where W_c is the maximum of the free energy needed to form a bubble. One can show that

$$W_c=\frac{16\pi}{3}\frac{s^3}{B-7\pi^2/180\beta^4} \tag{39}$$

and for the parameter values considered here, bubble nucleation is completely negligible. This qualitative analysis suggests that soliton stars formed at $T=T_c$ with $N>10^{46}$ could survive to the present epoch.

Neglecting any evaporation or nucleation which would change N, and if the star were to cool uniformly then the thermodynamics can be analyzed numerically. Figure 1 shows the evolution in radius as a function of temperature until the cold configuration of Eq. (21) is reached. It can be seen that most of the soliton shrinking takes place for T very close to T_c. At $T=T_c$, $R(T_c)=174R(0)$, and at its cold configuration, the soliton would have a very small mass $\sim 10^{-8}M_{\odot}$.

CONCLUSIONS

The Lee-Wick model at zero temperature can provide soliton stars with very different properties depending on the choice of the parameter values. We have examined the possibility of a transition between these different types of solutions by introducing temperature dependence into the model. It is indeed found that at some critical temperature a phase transition takes place and soliton stars with characteristics considered by Lee and Pang could be formed, although the values for their mass and size are drastically modified. These solitons induce significant inhomogeneities in the mass density of the

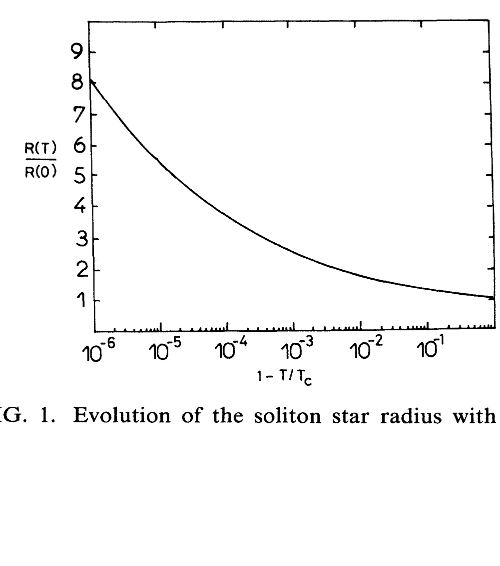

FIG. 1. Evolution of the soliton star radius with temperature.

universe at an early stage. One speculation could be that these small local mass fluctuations will be amplified in the course of time to form much larger condensations of matter. We follow through the evolution of those of the soliton stars which would have survived until the present time without modification and find that they would be condensed objects with a mass $\sim 10^{-8} M_{\odot}$.

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APPENDIX A

In this appendix we introduce finite temperatures into the Lee-Wick model and consider its thermodynamics. Most of the formulas derived in this appendix are to be found in other literature [9] but we believe our method to be more simple and can be applied to other field-theoretical models.

Boson fields

We first consider the σ field in isolation, and as with all scalar fields the theory can be considered to be that of an infinite set of coupled harmonic oscillators.

For a single oscillator the partition function $Z=\sum_{n=0}^{\infty} e^{-\beta(n+1/2)\epsilon}$ where $\epsilon=\hbar\omega$ is the level spacing. The free energy F defined by $Z=e^{-\beta F}$ is

$$F=\frac{\epsilon}{2}+\frac{1}{\beta}\ln(1-e^{-\beta\epsilon}) \tag{A1}$$

and the internal energy

$$E=\frac{\partial}{\partial\beta}(\beta F)=\frac{\epsilon}{2}+\frac{\epsilon}{e^{\beta\epsilon}-1} . \tag{A2}$$

For a free scalar field theory with Lagrangian density

$$\mathcal{L}=\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma-\frac{m^2}{2}\sigma^2 \tag{A3}$$

we have the field equation $\partial^2\sigma/\partial t^2=\nabla^2\sigma-m^2\sigma$ and the Fourier components q_k obey the harmonic-oscillator equation $d^2q_k/dt^2=-E_k^2 q_k$ with $E_k=\sqrt{k^2+m^2}$.

The total free energy is the sum of the harmonic-oscillator free energies, normalized in a volume V,

$$F=V\int\frac{d^3k}{(2\pi)^3}\left[\frac{E_k}{2}+\frac{1}{\beta}\ln(1-e^{-\beta E_k})\right] , \tag{A4}$$

and the total internal energy

$$E=V\int\frac{d^3k}{(2\pi)^3}\left[\frac{E_k}{2}+\frac{E_k}{e^{\beta E_k}-1}\right] . \tag{A5}$$

For an interacting field theory there are no simple exact expressions for the thermodynamic functions. However, if the field can be considered to be executing only small fluctuations about a fixed mean value, then the free field method yields useful approximations. For example, in the $\lambda\sigma^4$ theory

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma-\tfrac{1}{2}m^2\sigma^2-\frac{\lambda}{4}\sigma^4-\gamma\sigma . \tag{A6}$$

The field configuration of minimum energy has

$$\frac{\partial}{\partial\sigma}\left[\frac{m^2}{2}\sigma^2+\frac{\lambda}{4}\sigma^4+\gamma\sigma\right]=0 . \tag{A7}$$

We have included in $\mathcal{L}$ the interaction of the σ field with an external variable force γ which is chosen to constrain this minimum to be at $\sigma=\bar{\sigma}$, a static uniform field. Now write

$$\sigma=\bar{\sigma}+\sigma' . \tag{A8}$$

The lowest order in σ' the Lagrangian density is that of a free field

$$\mathcal{L}=-\frac{m^2}{2}\bar{\sigma}^2-\frac{\lambda}{4}\bar{\sigma}^4-\gamma\bar{\sigma}+\frac{1}{2}\partial_\mu\sigma'\partial^\mu\sigma'-\frac{M^2}{2}\sigma'^2 \tag{A9}$$

with $M^2=m^2+3\lambda\bar{\sigma}^2$. If the fluctuations σ' are small this is a good approximation to the full Lagrangian and yields a free energy

$$F=V\left[\frac{m^2}{2}\bar{\sigma}^2+\frac{\lambda}{4}\bar{\sigma}^4+\gamma\bar{\sigma}+\int\frac{d^3k}{(2\pi)^3}\left[\frac{E_k}{2}+\frac{1}{\beta}\ln(1-e^{-\beta E_k})\right]\right] \tag{A10}$$

with $E_k=\sqrt{M^2+k^2}=\sqrt{m^2+3\lambda\bar{\sigma}^2+k^2}$ and, with the removal of the external force ($\gamma=0$)

$$F=V\left[\frac{m^2}{2}\bar{\sigma}^2+\frac{\lambda}{4}\bar{\sigma}^4+\int\frac{d^3k}{(2\pi)^3}\left[\frac{E_k}{2}+\frac{1}{\beta}\ln(1-e^{-\beta E_k})\right]\right] . \tag{A11}$$

In thermal equilibrium the mean field $\bar{\sigma}$ would adopt a configuration to make F a minimum. This result is the same as that found in Ref. [9] in the one-loop approximation. Also the contribution of the zero-point energy to the free energy density (the first term in the integrand) is temperature independent. It was shown in Ref. [9] that this term corresponds to a renormalization of the parameters of the Lagrangian (A6) and can be discarded.

For the Lee-Wick model with a scalar field only

$$\mathcal{L}_\sigma=\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma-\frac{a}{2}\sigma^2-\frac{b}{3}\sigma^3-\frac{c}{4}\sigma^4 \tag{A12}$$

the same method leads to the free energy

$$F=V\left[\frac{a}{2}\bar{\sigma}^2+\frac{b}{3}\bar{\sigma}^3+\frac{c}{4}\bar{\sigma}^4+\frac{1}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln(1-e^{-\beta E_k})\right] \tag{A13}$$

with

$$E_k=\sqrt{k^2+M^2},\quad M^2=\frac{\partial^2}{\partial\bar\sigma^2}U(\bar\sigma)=a+2b\bar\sigma+3c\bar\sigma^2\ ,$$

and the total internal energy

$$E=\frac{\partial}{\partial\beta}\beta F$$
$$=V\left[\frac{a}{2}\bar\sigma^2+\frac{b}{3}\bar\sigma^3+\frac{c}{4}\bar\sigma^4+\int\frac{d^3k}{(2\pi)^3}\left[\frac{E_k}{e^{\beta E_k}+1}\right]\right]\ . \tag{A14}$$

In Eqs. (A13) and (A14), we have discarded the zero-point energy contribution for the same reason as that used in the case of $\lambda\sigma^4$ theory.

Fermion fields

We now turn to the fermion field and consider first the free field case. For fermions of mass m we have the Dirac Lagrange density

$$\mathcal{L}=\bar\Psi(\gamma^\mu\partial_\mu-m)\Psi \tag{A15}$$

and the field equations $(\gamma^\mu\partial_\mu-m)\Psi=0$. Now the Fourier components $q_{ks}(p_{ks})$ corresponding to fermions (antifermions) in their two spin states obey the Schrödinger equations

$$i\frac{dq_{ks}}{dt}=E_kq_{ks},\quad i\frac{dp_{ks}}{dt}=E_kp_{ks}\ ,$$

corresponding to fermion oscillators with energy $E_k=\sqrt{k^2+m^2}$. For a single oscillator the partition function is

$$Z=\sum_{n=0}^{1}e^{-\beta nE_k}=1+e^{-\beta E_k}=e^{-\beta F}\ . \tag{A16}$$

The total free energy is the sum of the free energies for the individual oscillators and so is

$$F=-\frac{4}{\beta}V\int\frac{d^3k}{(2\pi)^3}\ln(1+e^{-\beta E_k})\ . \tag{A17}$$

The factor 4 comes from the fermions and antifermions in their two spin states.

The above simple form is obtained by treating the fermions and antifermions in exactly the same footing but is inappropriate for situations in which one has a net conserved fermion number N. In this case one can proceed with the introduction of a chemical potential μ by adding a term μN into the Hamiltonian, or equivalently by modifying the Lagrange density into

$$\mathcal{L}=\bar\Psi(\gamma^\mu\partial_\mu-m)\Psi+\mu\bar\Psi\gamma^0\Psi\ . \tag{A18}$$

$\bar\Psi\gamma^0\Psi$ is the fermion number density and the chemical potential μ appears as a Lagrange multiplier.

Including the chemical potential the partition function Z is related to the grand potential Ω by

$$Z=e^{-\beta\Omega} \tag{A19}$$

with

$$\Omega=-\frac{2V}{\beta}\int\frac{d^3k}{(2\pi)^3}[\ln(1+e^{-\beta(E_k-\mu)})+\ln(1+e^{-\beta(E_k+\mu)})]\ . \tag{A20}$$

The first term in the integrand is the fermion contribution, the second is from the antifermions; the factor of 2 takes account of spin. The fermion number N and number density ρ are determined through the relations

$$N=-\left[\frac{\partial\Omega}{\partial\mu}\right]_\beta=\rho V\ . \tag{A21}$$

The free energy

$$F=\Omega+\mu N \tag{A22}$$

and is generally considered to be a function of β and ρ [μ is eliminated in favor of ρ through Eq. (A21)]. The internal energy

$$E=\frac{\partial}{\partial\beta}(\beta F)_\rho\ . \tag{A23}$$

Turning to the complete Lee-Wick model we have

$$\mathcal{L}=\mathcal{L}_\sigma+\bar\Psi\left[\gamma^\mu\partial_\mu-m\left[1-\frac{\sigma}{\sigma_0}\right]\right]\Psi+\mu\bar\Psi\gamma^0\Psi\ . \tag{A24}$$

This Lagrange density corresponds to a set of coupled boson and fermion oscillators. For an approximate evaluation of the thermodynamic functions one can proceed as for the σ field alone by considering only small fluctuations about fixed mean values of the fields $\sigma=\bar\sigma+\sigma'$, $\Psi=\Psi_m+\Psi'$.

The contribution to the free energy of the σ' field alone is identical to Eq. (A13). The fermion field alone contributes

$$F_{\text{fermion}}=\mu N+V\bar\Psi_m m\left[1-\frac{\bar\sigma}{\sigma_0}\right]\Psi_m-\frac{2}{\beta}\int\frac{d^3k}{(2\pi)^3}[\ln(1+e^{-\beta(E_k-\mu)})+\ln(1+e^{-\beta(E_k+\mu)})]$$

with

$$E_k=\sqrt{k^2+m_f^2},\quad m_f=m\left[1-\frac{\bar\sigma}{\sigma_0}\right]\ . \tag{A25}$$

A special case of this formula corresponding to $\mu=0$ is given in Ref. [9]. The free energy is also modified by the terms $\bar\Psi_m\sigma'\Psi'$ and $\bar\Psi'\sigma'\Psi_m$ which couple Bose and fermion fluctuations. Including these modifications would require the solutions of the coupled, but still linear field equations for σ' and Ψ'. One can anticipate that they would give rise, for example, to a temperature-dependent renormalization of the fermion mass parameter m. We do not consider further these terms since, in the temperature range of interest here the fluctuations σ' of the σ field are highly suppressed and give rise to negligibly small effects.

APPENDIX B

In this appendix we derive the relevant formulas used in the body of the paper in the low-density, high-temperature approximation and for different limits of the fermion mass.

The low-density, high-temperature expansion

Inside the soliton we take the fermion mass to be zero and then, by a change of scale $\beta k = x$, Eq. (A20) becomes

$$\Omega = -\frac{V}{\pi^2\beta^4}\int_0^\infty x^2 dx[\ln(1+e^{-(x+\beta\mu)}) + \ln(1+e^{-(x-\beta\mu)})] . \quad (B1)$$

In this work we are particularly concerned with the soliton characteristics at the time of formation and at this time the fermion density $\rho = N/V$ is small or more particularly $\beta^3\rho$ is small. This corresponds to $\beta\mu$ being also small and then an expansion of Ω in powers of $\beta\mu$ is appropriate. To this end consider the function

$$G(a) = \int_0^\infty x^2 dx \ln(1+e^{-(x+a)}) = \sum \frac{g_n}{n!}a^n , \quad (B2)$$

$$g_0 = G(0) = \int_0^\infty x^2 dx \ln(1+e^{-x}) = \frac{7\pi^4}{360} , \quad (B3)$$

this is a standard result [9].

More generally,

$$g_n = \int_0^\infty x^2 dx \frac{d^n}{dx^n}\ln(1+e^{-x}) \quad (B4)$$

on integrating by parts

$$g_2 = 2\int_0^\infty \frac{x\, dx}{e^{x+1}} = \frac{\pi^2}{6} \quad (B5)$$

and, for $n \geq 4$,

$$g_n = 2\frac{d^{n-4}}{dx^{n-4}}\frac{1}{1+e^x} \text{ evaluated at } x=0 . \quad (B6)$$

In particular $g_4 = 1$ and $g_6 = 0$ and we have the expansion

$$G(a)+G(-a) = \frac{7\pi^4}{180} + \frac{\pi^2}{6}a^2 + \frac{1}{12}a^4 + O(a^8) , \quad (B7)$$

hence

$$\Omega = -\frac{V}{\beta^4}\left[\frac{7\pi^2}{180} + \frac{(\beta\mu)^2}{6} + \frac{(\beta\mu)^4}{12\pi^2} + O((\beta\mu)^8)\right] . \quad (B8)$$

From Eq. (A21) the fermion number density is given by

$$3\beta^3\rho = \beta\mu + \frac{(\beta\mu)^3}{\pi^2} + O((\beta\mu)^7) \quad (B9)$$

which on inversion gives

$$\beta\mu = 3\beta^3\rho - \frac{1}{\pi^2}(3\beta^3\rho)^3 + \cdots . \quad (B10)$$

From Eq. (A22) the free energy density as a function of temperature and density ρ is

$$f = \frac{F}{V} = \frac{1}{\beta^4}\left[-\frac{7\pi^2}{180} + \frac{3}{2}(\beta^3\rho)^2 - \frac{27}{4\pi^2}(\beta^3\rho)^4 + \cdots\right] . \quad (B11)$$

Finally from Eq. (A23) the internal energy density

$$E = \frac{1}{\beta^4}\left[+\frac{7\pi^2}{60} + \frac{9}{2}(\beta^3\rho)^2 - \frac{243}{4\pi^2}(\beta^3\rho)^4 + \cdots\right] . \quad (B12)$$

The fermion free-energy large-mass limit

In thermal equilibrium the fermions outside the soliton have the same chemical potential as those inside. Since μ is small both inside and outside and because $\sigma = 0$ the fermion mass m is large. For the temperatures relevant to this work, $\beta m \gg 1$, and the exponentials $e^{-\beta(E_k \pm \mu)}$ in Eq. (A20) are very small. Expanding the logarithm gives, to first order,

$$\Omega = -\frac{4V}{\beta}\cosh\beta\mu \int \frac{d^3k}{(2\pi)^3} e^{-\beta\sqrt{m_0^2+k^2}} . \quad (B13)$$

The angular integral is trivial and making a change of scale $k = mx$ gives

$$\Omega = -\frac{2Vm_0^3}{\beta\pi^2}\cosh\beta\mu \int_0^\infty x^2 dx\, e^{-\beta m\sqrt{1+x^2}} \quad (B14)$$

$$= -\frac{2Vm_0^3}{\beta\pi^2}\cosh(\beta\mu)e^{-\beta m}\int_0^\infty x^2 dx\, e^{-\beta m x^2/2} , \quad (B15)$$

$$\Omega = -V\left[\frac{2}{\pi}\right]^{1/2}\frac{\cosh(\beta\mu)(\beta m)^{3/2}e^{-\beta m}}{\beta^4\pi} . \quad (B16)$$

Outside the soliton the fermion number density [Eq. (A21)]

$$\rho_{\text{out}} = -\frac{1}{V}\frac{\partial\Omega}{\partial\mu} = \frac{\sqrt{6}}{\pi}(\beta m)^{3/2}e^{-\beta m}\rho_{\text{inside}} \quad (B17)$$

and the free energy density [see Eq. (A22)]

$$f_{\text{out}} = \frac{\Omega}{V} + \mu\rho = \frac{6}{\pi^2}\left[\frac{10}{7}\right]^{1/2}(\beta m)^{3/2}e^{-\beta m}f_{\text{inside}} . \quad (B18)$$

Both are exponentially small.

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