Determination of QCD condensates from low-energy hadronic data

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The possibility of extracting the quark and gluon condensates from low-energy hadronic data is investigated. We propose a new method, which generalizes the usual finite-energy sum rules, taking into account explicitly the truncation error of the high-energy QCD expansion of the correlation functions. The method is applied to the e^+e^- annihilation into isospin $I=1$ hadrons. Our conclusion is that a precise determination of the four quark and gluon condensates is not possible unless a better theoretical knowledge of the high-energy QCD expansion and its truncation error is achieved.

I. INTRODUCTION

The quark and gluon condensates are fundamental quantities for the description of the long-distance regime of QCD. They reflect the nonperturbative nature of the QCD vacuum and are believed to be responsible for quark confinement.

A pure theoretical calculation of the condensates starting from the QCD Lagrangian is very difficult and only qualitative estimates were obtained up to now. Also, in spite of the large number of semiphenomenological analyses based on QCD sum rules, the values of the condensates obtained from experimental data are still ambiguous [1–10]. Actually, the large dispersion of the existing determinations is related to the instabilities inherent in the extrapolation of an analytic function affected by errors [11]. Indeed, the QCD sum rules correlate, via analyticity, the high-energy expansion of a Green's function to its low-energy values measurable in hadronic processes. In the absence of errors this correlation would be very strong, but it becomes much looser when errors are present. In practice, the truncated QCD expansion and the low-energy values of the physical amplitudes are affected by uncertainties. Therefore, various types of QCD sum rules, i.e., various types of analytic extrapolation, which are equivalent in the absence of errors, become in practice strongly inequivalent and give different results for the same input data. Of course, in these conditions one may ask whether a precise determination of the condensates from low-energy hadronic data is possible. In order to answer this question the uncertainties affecting the input of the QCD sum rules must be taken into account explicitly. This problem can be treated with techniques of analytic extrapolation and functional analysis [12–14]. In the approach proposed in Ref. [13] the imaginary part of the polarization amplitude along the whole timelike axis and its QCD expansion along a part of the spacelike axis are assumed to be known with *a priori* specified error channels, and the consistency of these input conditions is investigated with functional analysis methods. In a physical application [14] it is shown that the allowed range of the QCD condensates obtained in this way is very sensitive to the error channels.

In the present paper we approach the study of QCD sum rules from a similar viewpoint. We started from the idea that it is of interest to separate the two sources of uncertainty mentioned above, i.e., the experimental errors affecting the low-energy values of the amplitude and the theoretical error of the truncated QCD expansion. Actually, the large next-to-next-to-leading perturbative contributions calculated recently in the modified minimal subtraction ($\overline{\rm MS}$) scheme [15], the renormalization scheme dependence of the coefficients [16] and the effect of the higher terms in the operator-product expansion (OPE) (Ref. [17]) show that the truncation error of the QCD parametrization might be considerable. Therefore, in the present work we concentrate upon the effect of this theoretical uncertainty on the determination of the quark and gluon condensates. We propose a new method, which generalizes in a natural way the usual QCD finite-energy sum rules (FESR). Namely, we take as input the imaginary part of the polarization amplitude, supposed to be known exactly along a finite interval of the timelike axis, and also the QCD expansion, given together a specified error channel along a contour situated at high energies. The consistency of these input conditions, which can be investigated with methods of functional analysis, imposes restrictions on the values of QCD parameters, in particular the quark and gluon condensates.

The paper is organized as follows. In the next section we present the method and in Sec. III we apply it to the correlation function of the ρ meson, whose absorbtive part is known at low energies from the e^+e^- annihilation into $I=1$ hadrons. The conclusions of our analysis are presented in Sec. IV.

II. THE METHOD

We consider a correlation function $\Pi(s)$ defined through

$$\Pi_{\mu\nu}(q)=(q_\mu q_\nu-g_{\mu\nu}q^2)\Pi(s)$$
$$=i\int d^4x\, e^{iqx}\langle 0|Tj_\mu^\dagger(x)j_\nu(0)|0\rangle\ , \qquad (2.1)$$

where $s=q^2$ and j_μ is a hadronic current with specified quantum numbers. The function Π is real analytic in the complex s plane, with a physical cut from a positive threshold s_0 to infinity. The spectral function $\mathrm{Im}\Pi(s+i\varepsilon)$ is related, by unitarity, to observable quantities. Denoting by $s_{\max}$ the highest energy up to which experimental data are available we assume that

$$\mathrm{Im}\Pi(s+i\varepsilon)=\sigma(s), \quad s_0 \le s \le s_{\max} , \tag{2.2}$$

where $\sigma(s)$ is a known real function.

On the other hand, perturbative QCD and OPE [Refs. [1] and [18]) predict an expansion of the form

$$\Pi^{\mathrm{QCD}}(s)=\Pi^{\mathrm{pert}}(s)+\sum_{j\ge 2}\frac{C_{2j}}{s^j} \tag{2.3}$$

which is assumed to represent a realistic parametrization of the true function Π at large $|s|$. In Eq. (2.3), Π^{pert} denotes the pure perturbative terms and the coefficients C_{2j} are proportional to the quark and gluon condensates which reflect the nonperturbative character of the QCD vacuum. The explicit form of the expansion (2.3) in the case of the ρ correlation function will be specified in the next section.

In the usual derivation of the QCD sum rules one assumes that the function Π (or some of its derivatives) coincide with the parametrization Π^{QCD} (or the corresponding derivatives) along a certain contour Γ in the complex plane, starting at $s=s_{\max}$. However, at finite $|s|$ the expression (2.3) represents only an approximation of Π, obtained by truncating the perturbative series and OPE. Moreover, in general Π^{QCD} does not satisfy the analyticity properties of Π in the s plane (for instance, it has poles at $s=0$, which are absent from the true function Π). Therefore, instead of assuming that Π coincides with Π^{QCD} along Γ we shall impose the more realistic condition

$$|\Pi(s)-\Pi^{\mathrm{QCD}}(s)|\le \varepsilon^{\mathrm{QCD}}(s), \quad s\in\Gamma , \tag{2.4}$$

where $\varepsilon^{\mathrm{QCD}}$ is a prescribed theoretical error. We choose a reasonable contour Γ, consisting from two arcs (Fig. 1): Γ_1 is a contour in the complex plane, starting at $s=s_{\max}$ and going to infinity, and Γ_2 is the infinite interval $s\le s_2<0$ of the spacelike axis. For simplicity, we take in what follows Γ_1 to be the interval, $s>s_{\max}$ of the timelike axis. A more general contour lying in the complex plane can be treated similarly, by using a suitable conformal mapping.

In general, the constraints (2.2)–(2.4) do not define uniquely the unknown amplitude $\Pi(s)$. There is, in principle, a whole class of "admissible" functions which satisfy these constraints. If the input quantities $\sigma, \Pi^{\mathrm{QCD}}$, and $\varepsilon^{\mathrm{QCD}}$ are consistent, the admissible class contains at least one analytic function, otherwise this class will be empty. Thus, in order to check the consistency of the QCD expansion with the low-energy hadronic data and find, in particular, the allowed range of the QCD condensates we must investigate the content of the admissible class.

First, we perform the conformal mapping

$$w=\frac{\sqrt{s_{\max}}-\sqrt{s_{\max}-s}}{\sqrt{s_{\max}}+\sqrt{s_{\max}-s}} , \tag{2.5}$$

which applies the cut s plane onto the unit disk $|w|\le 1$, cut along the segment $(w_0,1)$, where $w_0=w(s_0)$. The arcs Γ_1 and Γ_2 become, respectively, the boundary of the unit disk, $|w|=1$, and the segment $(-1,w_2)$ (Fig. 2), while the conditions (2.2) and (2.4) can be written as

$$\mathrm{Im}\Pi(w+i\varepsilon)=\sigma(w), \quad w\in(w_0,1) , \tag{2.6}$$

$$|\rho(w)[\Pi(w)-\Pi^{\mathrm{QCD}}(w)]|\le 1, \quad w\in\Gamma_1 U\Gamma_2 , \tag{2.7}$$

where $\rho(w)=1/\varepsilon^{\mathrm{QCD}}(w)$.

Let us consider the extremal problem

$$\varepsilon_0=\min_{\{\Pi\}}\|\rho(\Pi-\Pi^{\mathrm{QCD}})\|_{L^\infty_\Gamma} , \tag{2.8}$$

where the minimization is performed with respect to all the functions Π analytic in the unit disk $|w|<1$, except for a cut along $(w_0,1)$, where the discontinuity is given by (2.6), and $\|\ \|_{L^\infty}$ denotes the L^∞_Γ norm, i.e., the essential supremum along Γ (on Γ_2, which is inside the analyticity domain, this reduces to the usual supremum).

Since Π does not coincide with Π^{QCD} along Γ, the real number ε_0 defined in (2.8) will be in general strictly posi-

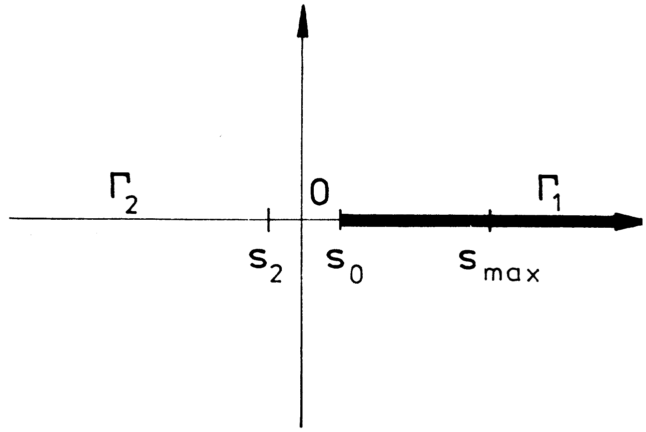

FIG. 1. The contours Γ_1 and Γ_2 in the complex s plane.

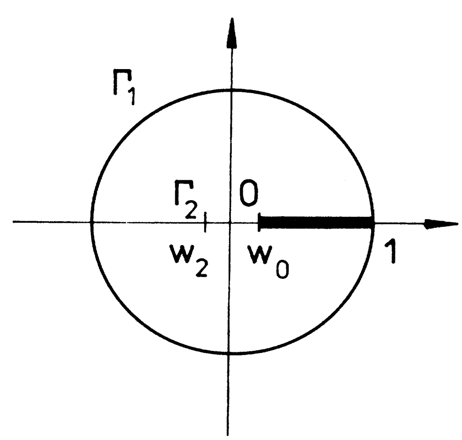

FIG. 2. The complex w plane, obtained by the conformal mapping (2.5).

tive. Its actual value depends on the input functions σ, Π^{QCD}, and ε^{QCD}. From (2.7) it follows that the consistency of these quantities requires that

$$\varepsilon_0 \leq 1 \ . \qquad (2.9)$$

If this inequality is satisfied, there exists at least one analytic function belonging to the admissible class defined above. On the contrary, if ε_0 is strictly greater than 1 the admissible class is empty and, in particular, the corresponding values of the QCD condensates are not acceptable. Thus, by means of the inequality (2.9) we can describe the allowed range in the space of the QCD parameters.

The extremal problem (2.8) with the constraint (2.6) can be treated by applying standard techniques of the optimization theory for analytic functions [19–21]. A similar problem was investigated in Ref. [22]. In what follows we shall first derive a lower bound for the quantity ε_0, by solving an extremal problem in L^∞ norm. Then, we shall present an approximate method, based on L^2 norm, which gives both upper and lower bounds for ε_0.

A. Lower bound for ε_0

A lower bound for ε_0 is obtained if we perform the minimization (2.8) upon a larger class of functions. Let us apply the disk $|w|<1$ with a slit along the real segment $(-1,w_2)$ onto the unit disk $|z|<1$. This can be achieved by two successive mappings;

$$u=\frac{4w}{(1-w)^2}\ ,\qquad z=\frac{\sqrt{u-u_2}-\sqrt{-u_2}}{\sqrt{u-u_2}+\sqrt{-u_2}}\ , \qquad (2.10)$$

where $u_2=u(w_2)$. In the variable z the contour Γ becomes the unit circle $|z|=1$. The slit $(-1,w_2)$ of the w plane, i.e., the interval $s<s_2$ of the spacelike axis, is applied onto a part of the unit circle $z=\exp(i\theta)$, while the interval $s>s_{max}$ of the timelike axis is applied on the remaining part. The segment (s_0,s_{max}) of the physical cut becomes the slit $(x_0,1)$, where $x_0=z(s_0)$ (Fig. 3). For convenience, the origin $s=0$ of the s plane was applied onto the origin $z=0$ of the z plane.

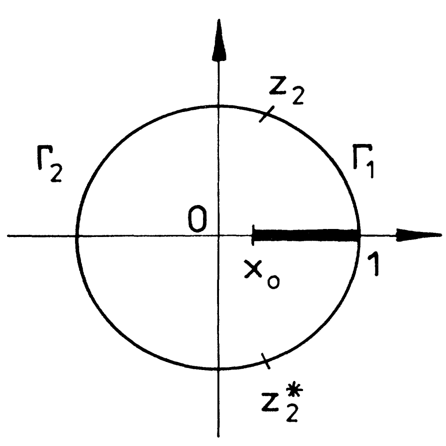

FIG. 3. The complex z plane, obtained by the conformal mapping (2.10).

Let us consider now the modified extremal problem

$$\varepsilon_0'=\min_{\{\Pi\}}\ \operatorname*{ess\,sup}_{|z|=1}|\rho(z)[\Pi(z)-\Pi^{QCD}(z)]| \qquad (2.11)$$

the minimization being performed with respect to the functions Π, real analytic in the disk $|z|<1$, except for a cut along the segment $(x_0,1)$ where, according to (2.6),

$$\mathrm{Im}\Pi(x+i\varepsilon)=\sigma(x),\quad x\in(x_0,1)\ . \qquad (2.12)$$

It is clear that this class of functions is larger than the class appearing in the initial minimization (2.8), since it contains, in addition to the functions analytic along the segment $(-1,w_2)$ of the w plane, functions having a discontinuity across this segment, introduced artificially by means of the conformal mapping. Therefore, the solution ε_0' of the new problem (2.11) will satisfy the inequality

$$\varepsilon_0'\leq\varepsilon_0\ . \qquad (2.13)$$

The number ε_0', which represents a lower bound for ε_0, can be calculated by an explicit algorithm.

We first take into account the constraint (2.12), by separating Π into two terms, one of them being analytic inside the unit disk $|z|<1$ and the other having in $|z|<1$ a prescribed nonanalytic part. Of course, this separation is not unique, but as we shall see below, the final result will not depend on this arbitrariness. Also, we take into account the asymptotic behavior of the function Π, which actually coincides with that of the model function Π^{QCD} (in the case treated below, Π grows logarithmically at $|s|\to\infty$). We write then

$$\Pi(z)=\varphi(z)f(z)+\frac{1}{\pi}\int_{x_0}^{1+\eta}\frac{\sigma(x)dx}{x-z}\ , \qquad (2.14)$$

where $\varphi(z)$ is a real analytic outer function (without zeros in $|z|<1$), having the same asymptotic behavior in the s variable as Π, η is an arbitrary positive number, and $\sigma(x)$ for $x>1$ is an arbitrary continuous extension of the input function $\sigma(x)$ given for $x\leq1$. By construction, the function $f(z)$ defined in (2.14) is real analytic and bounded in $|z|<1$, with no cut along $(x_0,1)$ and Eq. (2.14) establishes a one-to-one correspondence between the functions Π and the Hardy class H^∞ (Ref. [20]). By introducing (2.14) in (2.11) we write the extremal problem in the form

$$\varepsilon_0'=\min_{f\in H^\infty}\|\rho\varphi(f-h)\|_{L^\infty}\ , \qquad (2.15)$$

where

$$h(z)=\frac{\Pi^{QCD}(z)}{\varphi(z)}-\frac{1}{\pi\varphi(z)}\int_{x_0}^{1}\frac{\sigma(x)dx}{x-z}\ . \qquad (2.16)$$

The problem (2.15) can be solved by applying a well-known duality theorem, which relates a minimization problem in a Banach space to a maximization problem in the dual space [19,20]. According to this theorem we have

$$\varepsilon_0' = \sup_{F\in H_\rho^1} \left| \frac{1}{2\pi}\int_{|z|=1} F(z)h(z)dz \right| , \qquad (2.17)$$

where H_ρ^1 is the class of functions which are analytic in the unit disk and satisfy the condition

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{|F(\theta)|}{\rho(\theta)|\varphi(\theta)|} d\theta \le 1 \qquad (2.18)$$

on the boundary $z=\exp(i\theta)$.

The maximization problem (2.17) can be solved by an explicit algorithm. First, we write $F(z)$ as

$$F(z) = C(z)\varphi(z)\bar{F}(z) , \qquad (2.19)$$

where

$$C(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln\rho(\theta) d\theta \right] \qquad (2.20)$$

is an outer function having the modulus equal to $\rho(\theta)$ on the boundary (we assume that $\ln\rho$ is a function of class L^1) and

$$\bar{F}(z) = \sum_{n=0}^{\infty} F_n z^n \qquad (2.21)$$

belongs to the unit sphere of the Hardy class H^1; i.e., it satisfies the condition

$$\frac{1}{2\pi}\int_{|z|=1} |\bar{F}(z)||dz| \le 1 . \qquad (2.22)$$

By introducing Eqs. (2.19) and (2.21) in (2.17) and by applying the residues theorem we obtain

$$\varepsilon_0' = \max_{\{F_n\}} \sum_{n=0}^{\infty} F_n h_{-(n+1)} , \qquad (2.23)$$

where the maximization is performed with respect to the numbers $\{F_n\}$ subject to the condition (2.22) and h_{-n} are Fourier coefficients:

$$h_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(\theta)h(\theta)\exp(in\theta)d\theta$$
$$= \frac{1}{2\pi i}\int_{|z|=1} C(z)h(z)z^{n-1}dz, \quad n\ge 1 . \qquad (2.24)$$

The maximization (2.23) can be performed by applying a factorization theorem [20], according to which we can write

$$\bar{F}(z) = \sum_{n=0}^{\infty} F_n z^n = g(z)G(z) , \qquad (2.25)$$

where the functions g and G belong to the unit sphere of the Hilbert space H^2; i.e., they admit the developments

$$g(z) = \sum_{n=0}^{\infty} g_n z^n, \quad G(z) = \sum_{n=0}^{\infty} G_n z^n \qquad (2.26)$$

with the conditions

$$\sum_{n=0}^{\infty} g_n^2 \le 1, \quad \sum_{n=0}^{\infty} G_n^2 \le 1 . \qquad (2.27)$$

From (2.25) and (2.26) we obtain

$$F_n = \sum_{k=0}^{n} g_k G_{n-k}, \quad n=0,1,\ldots , \qquad (2.28)$$

so that (2.23) becomes

$$\varepsilon_0' = \max_{\{g_n\}\{G_k\}} \sum_{n,k=1}^{\infty} H_{nk} g_{n-1} G_{k-1} , \qquad (2.29)$$

where the maximization is performed with the constraints (2.27) and H is an infinite Hankel matrix [20] defined as

$$H_{nk} = h_{-(n+k-1)}, \quad n,k=1,2,\ldots . \qquad (2.30)$$

In (2.29) one may recognize the norm of the matrix H in the Hilbert space l^2 of numerical sequences [20]. The initial functional extremal problem was thus reduced to a numerical problem. The norm of an infinite matrix is evaluated in practice by truncating it at a finite range. Numerical programs which perform this calculation are available [23] and in our case a good accuracy was obtained with 100–150 coefficients.

The input data of the problem, particularly the QCD condensates, enters in the expression of the coefficients h_{-n} defined in (2.24). Using Eq. (2.16) and the general expression (2.3) of $\Pi^{\rm QCD}$ and applying the residues theorem we can write (2.24) in the convenient form

$$h_{-n} = \frac{1}{\pi}\int_0^1 [\sigma(x) - \mathrm{Im}\Pi^{\rm QCD}(x)]C(x)x^{n-1}dx$$
$$+ \sum_{j\ge 2} C_{2j}\xi_j^{(n)} , \qquad (2.31)$$

where $\mathrm{Im}\Pi^{\rm QCD}$ is the absorbtive part of the theoretical expansion of Π and $\xi_j^{(n)}$ are real numbers defined as

$$\xi_j^{(n)} = \frac{1}{2\pi i}\int_{|z|=1} \frac{C(z)z^{n-1}}{s^j} dz . \qquad (2.32)$$

We notice that the outer function φ, the number η and the arbitrary extension of $\sigma(s)$ above $x=1$ do not appear in the above equations, so that the result contains only known quantities.

In order to calculate explicitly the numbers $\xi_j^{(n)}$ we recall that the origin $s=0$ of the energy plane was applied by the conformal mappings (2.5) and (2.10) into the origin $z=0$ of the z plane. It follows that $\xi_j^{(n)}=0$ for $n>j$, since in this case the integrand of (2.32) is analytic in $|z|<1$. The residues theorem gives

$$\xi_j^{(n)} = \frac{1}{(j-1)!}\frac{d^{j-1}}{ds^{j-1}}[C(z)a(s)z^{n-1}]_{s=z=0}, \quad n\le j , \qquad (2.33)$$

where

$$a(s) = \frac{dz}{ds} \qquad (2.34)$$

is the derivative of the conformal mapping. In the physical application of the method presented below, we keep only terms with $j\le 3$ in the expansion of $\Pi^{\rm QCD}$. By a straightforward calculation we obtain the expressions

$$\xi_2^{(1)} = C_z'(0)a^2(0) + C(0)a_s'(0) ,$$

$$\xi_2^{(2)}=C(0)a^2(0)\ ,$$
$$\xi_3^{(1)}=0.5[C_z''(0)a^3(0)+3C_z'(0)a(0)a_s'(0)+C(0)a_s''(0)]\ ,$$
$$\xi_3^{(2)}=\xi_2^{(1)}a(0)+0.5C(0)a(0)a_s'(0)\ , \tag{2.35}$$
$$\xi_3^{(3)}=C(0)a^3(0)\ ,$$

all the other coefficients $\xi_j^{(n)}$ being zero.

In the above expressions appear the derivatives of the outer function $C(z)$ at $z=0$, which can be calculated from Eq. (2.20):

$$C(0)=\exp\left[\frac{1}{\pi}\int_0^\pi \ln\rho(\theta)d\theta\right]\ ,$$
$$C_z'=\frac{2C(0)}{\pi}\int_0^\pi \cos\theta\ln\rho(\theta)d\theta\ ,$$
$$C_z''(0)=\frac{2}{\pi}C_z'(0)\int_0^\pi\cos\theta\ln\rho\,d\theta \tag{2.36}$$
$$+C(0)\left[-\frac{4}{\pi}\int_0^\pi\ln\rho\,d\theta+\frac{8}{\pi}\int_0^\pi\cos^2\theta\,d\theta\right]$$

and the derivatives of the function $a(s)$ defined in (2.34). Using the expressions of the conformal mappings (2.5) and (2.10), we obtain, after a straightforward calculation,

$$a(s)=z_s'=z_w'w_s'\ ,$$
$$a'(s)=z_w''(w_s')^2+z_w'w_s''\ ,$$
$$a''(s)=z_w'''(w_s')^3+3z_w''w_s'w_s''+z_w'w_s'''\ ,$$

the expressions of the derivatives at the origin being

$$z_w'=-u_2^{-1},\quad z_w''=-4u_2^{-1}(1+u_2^{-1})\ ,$$
$$z_w'''=-30u_2^{-3}-48u_2^{-2}-18u_2^{-1}\ ,$$
$$w_s'=0.25s_{\max}^{-1},\quad w_s''=0.25s_{\max}^{-2}\ ,$$
$$w_s'''=15s_{\max}^{-3}/32$$

[u_2 was defined below Eq. (2.10)].

The above equations provide an explicit algorithm for finding the solution of the extremal problem (2.11) which, according to (2.13), yields a lower bound for the solution of the initial extremal problem (2.8).

B. Approximate solution based on L^2 norm

In this subsection we shall describe an approximate procedure which allows us to obtain both upper and lower bounds for the quantity ε_0. To this end we shall use the fact that the function Π is real analytic [i.e., $\Pi(s^*)=\Pi^*(s)$] and the contour Γ_2 is a part of the real axis of the s plane. We recall that in the modified problem (2.11) we perform the minimization upon a larger class of functions, including in principle functions which have a discontinuity along the artificial cut introduced along the contour Γ_2 (i.e., for $s<s_2$). In order to restrict the minimization upon the initial class of functions, with no cut along the spacelike axis, we must solve the extremal problem (2.11) with the additional constraint

$$\mathrm{Im}\Pi(\theta)=0,\quad 0\in\Gamma_2\ , \tag{2.37}$$

Γ_2 being now the part of the unit circle $|z|=1$ corresponding to the region $s<s_2$ of the s plane (Fig. 3). In this way, the functions Π allowed in the minimization are real along the spacelike axis, having no discontinuity along the artificial cut introduced by the conformal mapping.

The extremal problem (2.11) with the constraint (2.37) is not easy to solve. However, its solution can be approached by solving a sequence of problems in L^2 norm. The same procedure was applied for similar problems in Refs. [24] and [25]. Using the method discussed in these papers, one can prove the equality

$$\varepsilon_0=\sup_{\substack{g\in H^2\\ \|g\|_{L^2}\le 1}}\ \min_{\{\Pi\}}\|\rho g(\Pi-\Pi^{\mathrm{QCD}})\|_{L^2}\ , \tag{2.38}$$

where the minimization must be done upon functions Π real analytic in $|z|<1$ except for a cut along $(x_0,1)$, and which satisfy the constraints

$$\mathrm{Im}\Pi(x+i\varepsilon)=\sigma(x),\quad x\in(x_0,1)\ ,$$
$$\mathrm{Im}\Pi(\theta)=0,\quad \theta\in\Gamma_2 \tag{2.39}$$

and the final supremum is taken upon arbitrary outer functions g in the unit sphere of the Hilbert space H^2. In practice, we shall first perform the minimization in L^2 norm with the conditions (2.39) for a fixed g, which yields the quantity

$$\varepsilon_2(g)=\min_{\{\Pi\}}\|\rho g(\Pi-\Pi^{\mathrm{QCD}})\|_{L^2}\ . \tag{2.40}$$

It is clear that for every fixed g with the required properties, $\varepsilon_2(g)$ represents a lower bound for ε_0. In order to approach ε_0 we must perform a final maximization upon g. We shall treat this problem approximately, using to this end a particular but very suitable class of functions g. As shown in the previous works [24,25], such a choice proves to be

$$g(z)=\frac{\sqrt{1-\alpha^2}}{(1-\alpha z)}\ , \tag{2.41}$$

where $|\alpha|<1$ is an arbitrary parameter, such that $|g(\theta)|^2$ is the Jacobian of a conformal mapping of the unit disk $|z|<1$ onto itself. Taking g of the form (2.41), the maximization in (2.38) amounts to varying the parameter α and taking the largest value of $\varepsilon_2(g)$ thus obtained, which is expected to approach ε_0 quite closely from below. On the other hand, as we shall see, the optimal function Π_0 (depending on g), which achieves the minimum in (2.40) can be explicitly computed. We can use this function to evaluate the L^∞ norm

$$\varepsilon_0''=\|\rho(\Pi_0-\Pi^{\mathrm{QCD}})\|_{L^\infty}\ . \tag{2.42}$$

From the very definition of ε_0 it follows that this number represents an upper bound for it. In this way, we obtain both upper and lower bounds for ε_0, allowing us to approach closely this quantity.

In the remaining part of this subsection we shall treat the extremal problem (2.40) with the constraints (2.39) for fixed g. First, as in the preceding subsection we shall use the first condition (2.39) by introducing the function f,

$$f(z)=\Pi(z)-\frac{1}{\pi}\int_{x_0}^{1}\frac{\sigma(x)dx}{x-z}\ , \qquad (2.43)$$

which is analytic in $|z|<1$ and satisfies the condition

$$\mathrm{Im}f(\theta)=-\frac{1}{\pi}\mathrm{Im}\int_{x_0}^{1}\frac{\sigma(x)dx}{x-e^{i\theta}}\equiv\psi(\theta),\quad \theta\in\Gamma_2\ . \qquad (2.44)$$

In (2.43) we renounced to the outer function φ and the extension of σ above $x=1$ appearing in the corresponding equation (2.14) since, as above, they can be shown to disappear from the final result. Moreover, when Π has the asymptotic properties mentioned above, the function f defined in (2.43) belongs to the class H^2 in $|z|<1$, which we will actually require below.

In terms of f the extremal problem (2.40) writes as

$$\varepsilon_2(g)=\min_{\{f\}}\|Cg(f-h)\|_{L^2}\ , \qquad (2.45)$$

where $C(z)$ is the outer function defined in (2.20),

$$h(\theta)=\Pi^{\mathrm{QCD}}(\theta)-\frac{1}{\pi}\int_{x_0}^{1}\frac{\sigma(x)dx}{x-e^{i\theta}} \qquad (2.46)$$

is a complex function given on the boundary of the unit disk, and the minimization is performed upon functions f real analytic in $|z|<1$, subjected to the condition (2.44) on a part of the boundary.

We shall solve this constrained extremal problem by applying the general method of Lagrange multipliers [21]. It is convenient to write the Lagrangian of the problem in the form

$$L(f,\lambda)=\|Cg(f-h)\|_{L^2}^2+\frac{1}{\pi}\int_{\Gamma_2}(\mathrm{Im}f-\psi)|Cg|\lambda(\varphi)d\varphi\ , \qquad (2.47)$$

where the real function λ is the Lagrange multiplier and the real factor $|Cg|$ was introduced in the last integral for the simplicity of the subsequent calculations. Since we deal with real analytic functions [i.e., $\mathrm{Im}f(-\varphi)=-\mathrm{Im}f(\varphi)$], we can assume without loss of generality that λ is an odd function, the contribution of an even part vanishing in the integral along the symmetric interval Γ_2 (Fig. 3). Further, we can write

$$\mathrm{Im}f=if^*-i\,\mathrm{Re}f$$

and notice that only the first term contributes in (2.47), due to the same argument.

We use now the expansions in power series,

$$\begin{aligned}
C(z)g(z)f(z)&=\sum_{n=0}^{\infty}f_nz^n\ ,\\
f_n&=\frac{1}{2\pi}\int_{-\pi}^{\pi}C(\varphi)g(\varphi)f(\varphi)e^{-in\varphi}d\varphi\ ,\\
C(z)g(z)h(z)&=\sum_{n=-\infty}^{\infty}h_nz^n\ ,\\
h_n&=\frac{1}{2\pi}\int_{-\pi}^{\pi}C(\varphi)g(\varphi)h(\varphi)e^{-in\varphi}d\varphi\ ,
\end{aligned} \qquad (2.48)$$

and notice that

$$\begin{aligned}
f^*(\varphi)|C(\varphi)g(\varphi)|&=f^*(\varphi)C^*(\varphi)g^*(\varphi)e^{i\Phi(\varphi)}\\
&=\sum_{n=0}^{\infty}f_ne^{-in\varphi}e^{i\Phi(\varphi)}\ ,
\end{aligned}$$

where $\Phi(\varphi)$ is the phase of the outer function Cg, which can be computed using the formula [20,24]

$$\Phi(\varphi)=\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\frac{\rho(\theta)|g(\theta)|}{\rho(\varphi)|g(\varphi)|}\cot\frac{\theta-\varphi}{2}d\theta\ . \qquad (2.49)$$

Using these expansions in (2.47) we write the Lagrangian in the form

$$\begin{aligned}
L(f,\lambda)=&\sum_{n=0}^{\infty}(f_n-h_n)^2+\sum_{n=1}^{\infty}h_{-n}^2\\
&+\frac{i}{\pi}\sum_{n=0}^{\infty}f_n\int_{\Gamma_2}\lambda(\varphi)e^{i\Phi(\varphi)-in\varphi}d\varphi\\
&-\frac{1}{\pi}\int_{\Gamma_2}\psi(\varphi)|C(\varphi)g(\varphi)|d\varphi\ .
\end{aligned} \qquad (2.50)$$

The unconstrained minimization upon the functions f can now be easily performed. We impose the conditions

$$\frac{\partial L}{\partial f_n}=0,\quad n=0,1,2,\ldots \qquad (2.51)$$

which gives

$$f_n=h_n-\frac{i}{2\pi}\int_{\Gamma_2}\lambda(\varphi)e^{i\Phi(\varphi)-in\varphi}d\varphi,\quad n\geq 0\ . \qquad (2.52)$$

We must now use the condition (2.44) in order to find the Lagrange multiplier λ. As we shall see, λ will be obtained by solving an integral equation. First, using Eqs. (2.48) and (2.52) we obtain the expression of the optimal function $f(z)$ for any $|z|<1$:

$$f(z)=C^{-1}(z)g^{-1}(z)\frac{1}{2\pi}\left[\int_{-\pi}^{\pi}\frac{h(\varphi)C(\varphi)g(\varphi)d\varphi}{1-ze^{i\varphi}}-i\int_{\Gamma_2}\frac{\lambda(\varphi)e^{i\Phi(\varphi)}d\varphi}{1-ze^{i\varphi}}\right]. \qquad (2.53)$$

We set now in this expression $z=re^{i\theta}$, take the limit $r\to 1$ and use the constraint (2.44), which can be written in the equivalent form

$$\mathrm{Im}f(\theta)|C(\theta)g(\theta)|=\psi(\theta)|C(\theta)g(\theta)|\ .$$

By applying the Plemelj relation [26]

$$\lim_{r\to 1}\frac{1}{2\pi}\int_{\Gamma}\frac{u(\varphi)d\varphi}{1-re^{i(\theta-\varphi)}}=\frac{1}{2}u(\theta)+\frac{1}{2\pi}\int_{\Gamma}\frac{u(\varphi)d\varphi}{1-e^{i(\theta-\varphi)}} \qquad (2.54)$$

where the Cauchy principal part appears in the last integral, we obtain after a straightforward calculation the following integral equation for the Lagrange multiplier λ:

$$\lambda(\theta)+\frac{1}{2\pi}\int_{\Gamma_2}\lambda(\varphi)\frac{\sin[\Phi(\theta)-\Phi(\varphi)+(\theta-\varphi)/2]}{\sin(\theta-\varphi)/2}d\varphi=\frac{1}{\pi}\mathrm{Im}\left[e^{-i\Phi(\theta)}\int_{-\pi}^{\pi}\frac{h(\varphi)C(\varphi)g(\varphi)d\varphi}{1-e^{i(\theta-\varphi)}}\right]-\psi(\theta)|C(\theta)g(\theta)| \ . \tag{2.55}$$

In deriving this equation we took into account the fact that

$$\mathrm{Im}h(\theta)=\psi(\theta), \quad \theta\in\Gamma_2$$

since the QCD expansion is real along the spacelike axis.

The properties of the integral equation (2.55) depend on the behavior of the function Φ and ultimately, as follows from (2.49), on the smoothness and the zeros of the product $\rho|g|$ along the contour Γ_2. If Φ is Hölder continuous, the equation is of Fredholm type [26], and can be solved easily with numerical methods. If, on the other hand, Φ is discontinuous, Eq. (2.55) is singular and can be solved by reducing it to a Hilbert-Riemann boundary value problem for piecewise analytic functions [26]. Once the Lagrange multiplier is found by solving this equation, we use it in order to evaluate the optimal function f written in (2.53) and the minimal L^2 norm (2.45). It is convenient to use the expression

$$\varepsilon_2^2(g)=\sum_{n=0}^{\infty}(f_n-h_n)^2r^{2n}+\sum_{n=1}^{\infty}h_{-n}^2r^{-2n} \quad \text{for } r\to 1 \ , \tag{2.56}$$

where we introduce f_n from (2.52) and h_n as defined in (2.48). Using the fact that the quantity

$$f_n-h_n=-\frac{i}{2\pi}\int_{\Gamma_2}\lambda(\varphi)e^{i\Phi(\varphi)-in\varphi}d\varphi, \quad n\geq 0$$

is real, we write the first term in (2.56) in the form

$$\frac{1}{2\pi}\int_{\Gamma_2}\lambda(\theta)d\theta\frac{1}{2\pi}\int_{\Gamma_2}\lambda(\varphi)\frac{e^{i\Phi(\varphi)-i\Phi(\theta)}}{1-re^{i(\theta-\varphi)}}d\varphi \quad \text{for } r\to 1 \ .$$

It is convenient to use now the integral equation (2.55) satisfied by the function λ, so finally this term writes as

$$\sum_{n=0}^{\infty}(f_n-h_n)^2=\frac{1}{2\pi}\int_{\Gamma_2}\lambda(\theta)d\theta\left[\frac{1}{2\pi}\mathrm{Im}e^{-i\Phi(\theta)}\int_{-\pi}^{\pi}\frac{h(\varphi)C(\varphi)g(\varphi)d\varphi}{1-e^{i(\theta-\varphi)}}-\frac{1}{2}\psi(\theta)|C(\theta)g(\theta)|\right] \ . \tag{2.57}$$

As concerns the second term in (2.56), we can use the expression of the negative-frequency Fourier coefficients h_{-n} from (2.31), with the only difference that everywhere the outer function $C(z)$ is now multiplied by the weight function $g(z)$. After a straightforward calculation we can write this term in the compact form

$$\sum_{n=1}^{\infty}h_{-n}^2=\frac{1}{\pi^2}\int_{x_0}^{1}\int_{x_0}^{1}\frac{\Delta(x)\Delta(y)}{1-xy}dx\,dy+\sum_{j=2}^{3}C_{2j}^2\left[\sum_{k=1}^{j}\xi_j^{(k)^2}\right]+2\sum_{j=1}^{3}C_{2j}\left[\sum_{k=1}^{j}M_k\xi_j^{(k)}\right] , \tag{2.58}$$

where C_{2j} are the QCD condensates, $\xi_j^{(k)}$ are the coefficients defined in (2.32) and calculated explicitly in (2.35), with $\rho(\theta)$ replaced by $\rho(\theta)|g(\theta)|$, the function Δ is defined as

$$\Delta(x)=[\sigma(x)-\mathrm{Im}\Pi^{\mathrm{QCD}}(x)]C(x)g(x) \ , \tag{2.59}$$

and M_k represent the moments of this function:

$$M_k=\frac{1}{\pi}\int_{x_0}^{1}\Delta(x)x^{k-1}dx, \quad k\geq 1 \ . \tag{2.60}$$

By summing Eqs. (2.57) and (2.58) and taking the square root of the result, we obtain the L^2 norm $\varepsilon_2(g)$, which represents, for a fixed g with the required properties, a lower bound for the quantity ε_0. Actually, the inequalities

$$\varepsilon_2^2(g)\leq 1 \tag{2.61}$$

which follow from (2.9), represent a family of rigorous sum rules which must be satisfied by the input quantities $\sigma, \Pi^{\mathrm{QCD}}$, and $\varepsilon^{\mathrm{QCD}}$. In particular, (2.61) describes explicitly an allowed domain in the space of the QCD condensates C_{2j}, when all the other quantities are fixed. Obviously, the final maximization upon the functions g amounts to taking the intersection of all these domains, which yields the optimal domain described by the inequality (2.9).

On the other hand, knowing the optimal function $f(z)$ from (2.53) we can reconstruct the optimal Π using (2.43) and then evaluate the L^∞ normal (2.42), which gives an upper bound for ε_0. Thus, the above procedure allows an efficient approximation of ε_0 both from below and from above.

III. APPLICATION

We applied the above method to the correlation function of the ρ meson. In this case the physical cut starts at $s_0=(2m_\pi)^2$ and the spectral function $\mathrm{Im}\Pi$ is related to the total cross section of the e^+e^- annihilation into isospin $I=1$ hadrons. Experimental data are available up to $s_{\max}=4.0\ \mathrm{GeV}^2$. We used a parametrization of these data by the $\rho(770)$ and ρ' resonances [27]. The experimental data, expressed in terms of the ratio

$$R=\sigma(e^+e^-\to I=1\ \text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)$$

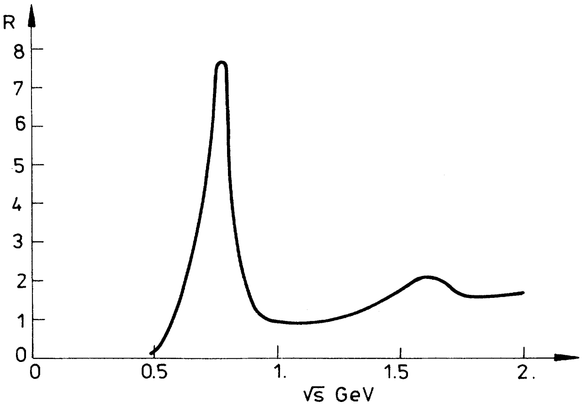

FIG. 4. The input experimental data below $s_{max}=4.0$ GeV^2, expressed in terms of the ratio R as a function of energy.

are shown in Fig. 4. Along the contour Γ we adopted the usual QCD perturbative prediction in the $\overline{\rm MS}$ scheme with $n_f=3$ flavors, plus the nonperturbative one, parametrized by the lowest dimension condensates [14]:

$$\Pi^{\rm QCD}(s)=-2\tau+V_2\left[-2\ln\tau+\frac{\pi^2}{12\tau^2}\right]+V_3\frac{\ln 2\tau+1}{\tau}+2\frac{V_4}{\tau}+V_5\ln 2\tau\frac{\ln 2\tau+1}{\tau^2}+V_6\frac{\ln 2\tau+0.5}{\tau^2}+\frac{C_4}{s^2}+\frac{C_6}{s^3} \quad (3.1)$$

with

$$\tau=0.5\ln\left[\frac{s}{\Lambda^2_{\overline{\rm MS}}}\right],$$
$$V_2=2/\beta_1,\quad \beta_1=11-2n_f/3=9,$$
$$V_3=2(-153+19n_f)/(3\beta_1^3),$$
$$V_4=R_2V_2^2,\quad R_2=1.9857-0.115n_f,\quad (3.2)$$
$$V_5=2(-153+19n_f)/(9\beta_1^5),$$
$$V_6=2R_2V_2V_3-V_5.$$

In (3.1) we neglected terms of order $1/s$ which are proportional to the squared quark masses. The nonperturbative terms are related to the quark and gluon condensates by [14]

$$C_4=\frac{\pi^2}{3}\langle\alpha_s/\pi GG\rangle+4\pi^2(m_u\langle\bar{u}u\rangle+m_d\langle d\bar{d}d\rangle),$$
$$C_6=896\pi^3\alpha_s\langle\bar{\Psi}\Psi\rangle^2/81, \quad (3.3)$$

where $\alpha_s(s)$ is the strong running coupling

$$\alpha_s(s)/\pi=2(1+\cdots)/(\beta_1\tau).$$

The perturbative error was chosen of the order $(\alpha_s/\pi)^2$, amplified at low energy by a factor $(1-\alpha_s/\pi)^{-1}$ in order to take into account the strong-coupling uncertainty when $\alpha_s=1$, and the nonperturbative one of the order $1/s^4$. More precisely, the error corridor was taken of the form [14]

$$\varepsilon^{\rm QCD}(s)=\frac{V_7}{|\tau|^2}\left|1-\frac{2}{\beta_1\tau}\right|^{-1}+\frac{C_8}{|s|^4} \quad (3.4)$$

with [14]

$$V_7=8R_2/\beta_1^3,$$
$$C_8=\max\{|C_4|m_{\rm sc}^4,|C_6|m_{\rm sc}^6\}. \quad (3.5)$$

The coefficient C_8 was assumed to be related to a typical instanton or hadron size $m_{\rm sc}^{-1}$, which will be specified below. Also, the specific choice (3.5) ensures a narrow error corridor when the input values of the lowest-order condensates C_4 and C_6 are small and a large error for large C_4 and C_6. We adopted the above parametrizations following Ref. [14].

The theoretical parameters used as input in our analysis are the QCD scale $\Lambda_{\overline{\rm MS}}$, the hadron mass $m_{\rm sc}$ and the QCD condensates C_4 and C_6. According to the above discussion, we have to calculate the minimal norm ε_0 for various sets of values of the input parameters. The values yielding an ε_0 less than or equal to 1 will be considered acceptable, while those for which ε_0 is larger than 1 must be rejected.

We first took $m_{\rm sc}=m_\rho$ and $\Lambda=0.1$ GeV and computed ε_0 for various sets of condensates C_4 and C_6 suggested by previous analyses [1–14]. It turned out that the values of ε_0 (actually, in many cases even of the lower bound ε_0' derived in Sec. II A) were systematically greater than 1. As an illustration, for $C_4=C_6=0$ we obtained ε_0 equal to 28.0, for the "standard values (Ref. [1])" $C_4=0.4$ GeV^4, $C_6=0.06$ GeV^6 ε_0 was 23.5, while for the pairs $C_4=0.5$ GeV^4, $C_6=2.0$ GeV^6 and $C_4=-0.5$ GeV^4, $C_6=-2.0$ GeV^6 ε_0 was, respectively, equal to 9.0 and 7.7. According to the above discussion, this indicates an inconsistency between the low-energy data and the QCD parametrization and its associated error, for these values of the parameters.

For $\Lambda=0.300$ GeV the values of ε_0 turned out to be considerably lower, remaining however still greater than 1. Thus, ε_0 was equal to 8.3 for $C_4=C_6=0$, 3.8 for the "standard values" mentioned above, 1.74 for $C_4=0.5$, $C_6=2.0$ and 2.2 for $C_4=-0.5$, $C_6=-2$.

When the hadronic mass scale $m_{\rm sc}$ was lowered ($m_{\rm sc}=0.300$ GeV) the values of ε_0 were larger than the results quoted above, by a factor of at least 2. This feature was noticed also in Ref. [14], and shows that for this mass scale the nonperturbative error in (3.4) is underestimated. Actually, the results mentioned above suggest that the perturbative part of the error channel (3.4) is underestimated too. We recall that, following Ref. [14], we took the scale V_7 of this error equal to R_2, i.e., to the coefficient of the $(\alpha_s/\pi)^2$ term in the perturbative expansion of Π. In fact, if the recent calculation of the α_s^3 coefficient [15] is confirmed, this choice is really an underestimation, since $R_3=67$, while $R_2=1.6$. Therefore, we repeated the calculations (with $m_{sc}=m_\rho$) for gradually increased values of V_7. It appeared that values of ε_0 below 1 could be obtained only if V_7 were multiplied by at least a factor of 5. In this case, for $\Lambda=0.300$ GeV (a value which seems to be favored with respect to 0.100 GeV), we obtained an allowed region in the C_4,C_6 plane, including near the boundary the "standard values," and excluding, for instance, the origin $C_4=C_6=0$. Our results are in agreement with those obtained in Ref. [14]. Thus, while values of ε_0 below and around 1 were obtained for pairs $C_4>0$, $C_6>0$ and $C_4<0$, $C_6<0$, much greater values were yielded by condensates of opposite signs. Moreover, the results indicate a correlation between the allowed values of the quark and gluon condensates, very similar to that noticed in Ref. [14].

We recall that in our analysis the absorbtive part of the polarization amplitude was assumed to be known exactly below s_{max}. The results discussed above were obtained for the experimental curve shown in Fig. 4. When the experimental data were increased or decreased by 5%, the values of ε_0 increased in a significant way. Thus, for $\Lambda=0.300$ GeV, $m_{sc}=m_\rho$ and the perturbative error channel amplified by 5, as discussed above, we obtained for the "standard values" $C_4=0.04$. $C_6=0.06$, ε_0 equal to 1.1 using the data of Fig. 4, 4.6 for data decreased by 5% and 4.3 for data increased by 5%. A similar increase was obtained for other values of the condensates.

IV. CONCLUSIONS

In the present paper we proposed a rigorous method for testing the consistency of the low-energy experimental data on the imaginary part of a correlation function with its high-energy QCD expansion given with an associated truncation error. The method is suitable for cases when accurate experimental data are available at low energies. In particular, in Sec. II B we derived a family of rigorous sum rules for the QCD condensates, which generalize the usual FESR, taking into account explicitly the truncation error associated with the QCD parametrization, while in the usual treatments this error is out of control.

In Sec. III we analyzed with this method the data on the e^+e^- annihilation into $I=1$ hadrons. The results are in qualitative agreement with those obtained in Ref. [14], where the same data were analyzed with a similar technique. However, our results indicate that the error channel (3.4) is underestimated, which is in agreement with a large coefficient of the α_s^3 term in the perturbative expansion. Our analysis shows that, even with accurate experimental data, a precise determination of the QCD condensates is not possible without a better knowledge of the QCD expansion and its truncation error.

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