Errata

Erratum: Self-energy of a thin charged shell in general relativity [Phys. Rev. D 42, 4254 (1990)]

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In reference to our recent paper on the self-energy of a thin charged shell in general relativity, we want to note that, subsequent to publication, related papers came to our attention which we had overlooked and where some of our results had already been derived. Our Eq. (26) for the total energy of the shell was given by Kuchař [1] and, in a more general form, by Chase [2]. Kuchař also found the lower bound (29) for the radius of the shell. We are grateful to Charles Curry for bringing these papers to our attention.

Furthermore, we would like to mention the following point. In our paper we restricted ourselves to the case where the shell is outside the outer Nordström horizon, i.e., where the charge satisfies $|Q| \leq \sqrt{G}\, M$. In the meantime, we received a letter from Don Page in which he demonstrates how one can derive without this restriction the following generalized form of our energy bound (31) directly from the energy equation (26):

$$M \geq \frac{M_0}{2} + \frac{Q^2}{2R} .$$

We are grateful to Don Page for pointing this out to us.

[1] K. Kuchař, Czech, J. Phys. B **18**, 435 (1968).

[2] J. E. Chase, Nuovo Cimento **B67**, 136 (1970).

Erratum: Trace anomaly in $\lambda\phi^4$ theory near a fixed point [Phys. Rev. D 40, 444 (1989)]

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(i) Equation (3.2) should read

$$K(\lambda) = \exp\left[\int_{\lambda^*+\delta}^{\lambda} \frac{2\gamma_m(\lambda')}{\beta(\lambda')} d\lambda'\right] \left[K(\lambda^*+\delta) + \int_{\lambda^*+\delta}^{\lambda} \frac{2\gamma_m(\lambda') Z_{14}^1(\lambda')}{\beta(\lambda')} \exp\left[-\int_{\lambda^*+\delta}^{\lambda'} \frac{2\gamma_m(\lambda'')}{\beta(\lambda'')} d\lambda''\right]\right] .$$

(ii) Equation (3.4) should read

$$K(\lambda) \simeq K(\lambda^*+\delta)\left[\frac{\lambda-\lambda^*}{\delta}\right]^{\alpha} + Z_{14}^1(\lambda^*)\left[\frac{\lambda-\lambda^*}{\delta}\right]^{\alpha} - Z_{14}^1(\lambda^*) .$$

A similar correction is needed in Eq. (3.11).

(iii) Equation (3.5) should read

$$K(\lambda) + Z_{14}^1(\lambda^*) = (\lambda-\lambda^*)^{\alpha}[K(\lambda^*+\delta)\delta^{-\alpha} + Z_{14}^1(\lambda^*)\delta^{-\alpha}] .$$

Erratum: Two-dimensional Euclidean anomalous effective actions in exactly solvable Abelian models [Phys. Rev. D 43, 4088 (1991)]

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A page was missing from the original printed version of this paper. The entire corrected version of the article is printed on the following pages.

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Two-dimensional Euclidean anomalous effective actions in exactly solvable Abelian models

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We study two-dimensional Euclidean models involving massless Dirac spinors coupled to topologically trivial vector and axial-vector potentials in the compactifiable case as well as in the case of constant field strengths. In both cases we derive the different forms of the anomalies and, in so doing, clarify several delicate and controversial aspects involving regularization procedures, often debated in the literature.

I. INTRODUCTION

In recent years a considerable amount of work has been devoted to the study of fermionic effective actions and, in particular, to calculations of their anomalous transformation properties with respect to gauge symmetry groups.

In order to compare different methods of regularization, obviously leading to different results, it is worth investigating two-dimensional models involving fermion fields coupled to Abelian vector and axial-vector (VAV) gauge potentials, defined on a topologically trivial Euclidean manifold.

In two dimensions for compactifiable and topologically trivial potentials, the quantities $\det \not{D}$ and $\det \not{D}^\dagger \not{D}$, where $\not{D}$ is the Dirac operator, can be explicitly computed by means, for instance, of the decoupling technique[1,2] together with Seeley's method[3] extending the ζ-function regularization.[4] On the other hand in the case of constant fields, thereby leading to a particular example of noncompatifiable potentials, the eigenvalues and the eigenfunctions of the Dirac operator can be explicitly obtained using recently developed procedures.[5,6]

As a consequence one can compare $\det \not{D}$ and $\det \not{D}^\dagger \not{D}$ both in a compactifiable and in a noncompactifiable example and discuss their variations with respect to localized gauge transformations.

In Sec. II we carefully examine the compactifiable case. We first repeat the calculation of $\det \not{D}$ and afterwards we use the same technique to evaluate $\det \not{D}^\dagger \not{D}$. This calculation shows that, in general, $|\det \not{D}|^2 \neq \det \not{D}^\dagger \not{D}$, which is at variance with claims appearing in the literature. Moreover, it is shown that the gauge variation of $\det \not{D}$ leads to the consistent anomaly; $\det \not{D}^\dagger \not{D}$ is obviously invariant under usual gauge transformations but a polar decomposition of $\not{D}$ gives rise under gauge variations to anomalies in their "covariant" form.

Section III is devoted to obtaining the effective action for the constant field case. Here the difference between $|\det \not{D}|^2$ and $\det \not{D}^\dagger \not{D}$ is even more dramatic. Section IV is concerned with the derivation of the anomalies as variations of the effective actions under infinitesimal localized gauge transformations using standard first-order perturbation theory. The general features, which were present in the compactifiable case, are recovered here in spite of the fact that known theorems on elliptic operators cannot be trivially extended to this case.

As a result of this investigation, the relations among different recipes for obtaining the anomalies are clarified, at least in this context. Moreover, the complete solution of the constant field model may represent a first step towards the comprehension of models involving open boundaries.[7,8] In particular, in the constant field case, the infinite Landau degeneracy of the spectrum plays a remarkable role; one might wonder whether an analogous phenomenon would also take place within a more general context. These and other aspects are discussed further in Sec. V, while some technical details are deferred to the Appendixes.

II. THE EFFECTIVE ACTION FOR COMPACTIFIABLE POTENTIALS AND ITS CHIRAL ANOMALY

In this section we first present, for the sake of completeness, some known results[2] concerning the general form of the effective action, defined by means of the ζ-function regularization applied to the first-order elliptic Dirac-Weyl operator $\not{D}$,[3] in the case where the fermions interact with the general vector V_μ and axial-vector A_μ potentials. Then we apply the same rigorous technique to the evaluation of the determinant of second-order nonnegative operators $(\not{D}^\dagger \not{D})^{1/2}$ and $(\not{D}\not{D}^\dagger)^{1/2}$ in order to discuss the close relation between the two different effective actions. In this comparison some new and interesting features will emerge.

In order to reach those results, it is essential to consider external potentials suitably behaving at infinity in such a way that the model can be set on a compact two-dimensional sphere. The starting point is the classical action

$$S[V_\mu, A_\mu] = \int d^2x\, \psi^\dagger(\not\partial - ie\not V - ie\not A\gamma_5)\psi\ , \tag{2.1}$$

where $\gamma_0 = i\sigma_1$, $\gamma_1 = -i\sigma_2$, $\gamma_5 = -\sigma_3$, $\epsilon_{\mu\nu}\gamma_\nu = i\gamma_\mu\gamma_5$.

Here we choose to work with two real VAV potentials. Obviously, thanks to the properties of two-dimensional Dirac algebra, this choice is equivalent to having only one complex vector potential. Nevertheless, our choice is more natural if one wants to subsequently perform an extension to complex-valued VAV potentials; we shall do so when discussing $\det(\not D^\dagger \not D)^{1/2}$.

The quantum effective action $W[V_\mu, A_\mu]$ is defined as

$$W[V_\mu, A_\mu] = -\ln \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp(-S[V_\mu, A_\mu])\ . \tag{2.2}$$

(In this section we are concerned with compactifiable and topologically trivial potentials. In this case the Dirac-Weyl operator does not possess a nontrivial kernel.[9])

The above path integral can be evaluated, for instance, following the "decoupling" method, namely by the computation of suitable Jacobians. These Jacobians obviously require regularization procedures to be mathematically well defined. We shall use the ζ-regularization technique.

The crucial point is that, in two dimensions, the potentials can be expressed as the sum of a "gradient"- and a "curl"-type term, namely,

$$V_\mu = (\partial_\mu \alpha + \epsilon_{\mu\nu}\partial_\nu \beta)\frac{1}{e} \tag{2.3}$$

and

$$A_\mu = (\partial_\mu \rho + \epsilon_{\mu\nu}\partial_\nu \sigma)\frac{1}{e}\ , \tag{2.4}$$

the antisymmetric tensor $\epsilon_{\mu\nu}$ being normalized as $\epsilon_{01} = 1$. We introduce the decomposition (2.3) in Eq. (2.1) and notice that the dependence on α is irrelevant because it can be reabsorbed by the change of variables in Eq. (2.2):

$$\begin{aligned} \psi'(x) &= e^{i\alpha(x)}\psi(x)\ , \\ \psi'^\dagger(x) &= e^{-i\alpha(x)}\psi^\dagger(x)\ , \end{aligned} \tag{2.5}$$

which does not affect the integration measure.

The term concerning $\beta(x)$ can also be reabsorbed by the change of variables

$$\begin{aligned} \psi'(x) &= e^{-\gamma_5\beta(x)}\psi(x)\ , \\ \psi'^\dagger(x) &= \psi^\dagger(x)e^{-\gamma_5\beta(x)}\ , \end{aligned} \tag{2.6}$$

which, however, gives rise to a nontrivial Jacobian. We indeed have[2]

$$J = \exp\left[-2\int d^2x \int_0^1 dr\, \mathrm{Tr}[\not D_r^{-s}\gamma_5\beta(x)]_{s=0}\right]\ , \tag{2.7}$$

where

$$\not D_r = e^{r\gamma_5\beta(x)}\not D e^{r\gamma_5\beta(x)}\ , \tag{2.8}$$

$$\not D = \not\partial - ie\not V - ie\not A\gamma_5\ . \tag{2.9}$$

It is worth noticing that the operator $\not D$ is not Hermitian; however, it is of an elliptic type and therefore the complex power $(\not D)^{-s}$ can be defined.[3]

The Jacobian J can be explicitly obtained by evaluating Seeley's coefficients and the result turns out to be

$$J[V_\mu, A_\mu] = \exp\left[-\frac{e}{2\pi}\int d^2x\, \beta(x)(\tfrac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu} + 2i\partial_\mu A_\mu)\right]\ , \tag{2.10}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$.

One can easily rewrite Eq. (2.10), taking Eq. (2.3) into account, in the form

$$J[V_\mu, A_\mu] = \exp\left\{-\frac{e^2}{2\pi}\int d^2x \left[V_\mu\left[\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right]V_\nu - 2i\partial_\mu A_\mu \epsilon_{\rho\lambda}\frac{\partial_\rho}{\partial^2}V_\lambda\right]\right\}\ . \tag{2.11}$$

Equation (2.2) becomes

$$W[V_\mu, A_\mu] = -\ln J[V_\mu, A_\mu] - \ln\int \mathcal{D}\psi^\dagger\mathcal{D}\psi \exp\left[-\int d^2x\, \psi^\dagger(\not\partial - ie\not A\gamma_5)\psi\right]\ . \tag{2.12}$$

It is worthwhile noting that the potential V_μ couples only in the Jacobian term and only with the divergence of A_μ.

The same procedure can obviously be repeated to perform the remaining functional integration, leading to the well-known final result

$$W[V_\mu, A_\mu] = -\ln J[V_\mu, A_\mu] - \ln \hat{J}[A_\mu]\ , \tag{2.13}$$

where

$$\hat{J}[A_\mu] = \exp\left\{\frac{e^2}{2\pi}\int d^2x\left[A_\mu\left[\delta_{\mu\nu} - \epsilon_{\mu\lambda}\epsilon_{\nu\rho}\frac{\partial_\lambda\partial_\rho}{\partial^2}\right]A_\nu\right]\right\}\ . \tag{2.14}$$

In Eq. (2.13) we have disregarded trivial constant terms. Again, in Eq. (2.14), the relevant part of A_μ is of the "gradient" type, namely, if $\partial_\mu A_\mu = 0$, the potential A_μ does not contribute to the effective action. To sum up we can say that, in Eqs. (2.3) and (2.4), the interacting terms are given by β and ρ, as was expected.

Since the effective action is obtained after a regularization procedure, in general, it is determined up to polynomial terms in the external fields of the correct canonical dimension, preserving the invariance under the vector transformation $V_\mu \to V_\mu - (1/e)\partial_\mu\theta$. We have chosen to work with dimensionless potentials and the coupling constant e with a canonical mass dimension equal to 1; however, it is apparent from Eq. (2.1) that one is free to shift the mass dimension from the coupling constant to the potentials without any change in the theory. Thanks to this freedom, it is clear that the only quantity which can be added to the effective action is $ae^2\int d^2x\, A_\mu A_\mu$, a being a subtraction dimensionless parameter. The same result can be obtained by starting from a more general definition[10] of the Jacobians in Eqs. (2.12) and (2.14).

The chiral cases correspond to the particular choices

$$A_\mu = \pm V_\mu = \tfrac{1}{2} C_\mu \,, \tag{2.15}$$

so that Eq. (2.13) becomes[11]

$$W[C_\mu] = -\frac{e^2}{8\pi}\left\{ \int d^2x \left[C_\mu \left[(\delta_{\mu\rho} \mp i\epsilon_{\mu\rho})\frac{\partial_\rho\partial_\lambda}{\partial^2} \times (\delta_{\nu\lambda} \pm i\epsilon_{\lambda\nu}) \right] C_\nu \right] - a\int d^2x\, C_\mu C_\mu \right\} . \tag{2.16}$$

It is clear that, in this case, no symmetry under the transformations of the "gradient" type is present.

From Eq. (2.13) the derivation of the expression for the chiral anomaly in its "consistent" form is straightforward. As a matter of fact, Eq. (2.13) is insensitive to the transformation $V_\mu \to V_\mu - (1/e)\partial_\mu\theta$, whereas the infinitesimal localized transformation $A_\mu \to A_\mu - (1/e)\partial_\mu\theta$ induces the variation

$$\delta W = \frac{ie}{2\pi}\int d^2x\, \theta(x)[\epsilon_{\rho\lambda}(\partial_\rho V_\lambda - \partial_\lambda V_\rho) - 2i\partial_\mu A_\mu] \,. \tag{2.17}$$

The last term can be reabsorbed by a redefinition of the Lagrangian subtracting a local counterterm, as we have already explained.

At this point we would like to compare the above results with the ones following from possible alternative definitions of the quantum effective action, still in the topologically trivial compactifiable case.[12] To this purpose let us introduce a unitary operator $\mathcal{U}$ acting in the space of the spinors and define

$$\det{}_{\mathcal{U}} \not{D} = \det\mathcal{U} \det(\mathcal{U}^\dagger \not{D}) \,. \tag{2.18}$$

Were we considering operators acting in a finite-dimensional space (matrices), we would obviously have

$$\det{}_{\mathcal{U}} \not{D} = \det\not{D} \,. \tag{2.19}$$

For infinite-dimensional operators, this is not the case.

We now perform on $\not{D}$ an infinitesimal gauge transformation of a vector type $[V_\mu \to V_\mu - (1/e)\partial_\mu\theta]$ and get

$$\not{D}' = \not{D} + i[\not{D},\theta] \tag{2.20}$$

and from Eq. (2.18) we obtain

$$\det{}_{\mathcal{U}} \not{D}' = \det\mathcal{U} \det(\mathcal{U}^\dagger \not{D}') \,. \tag{2.21}$$

One can easily convince oneself that, using the ζ-regularization technique,

$$\frac{\det_{\mathcal{U}}\not{D}'}{\det_{\mathcal{U}}\not{D}} = \frac{\det(\mathcal{U}^\dagger\not{D}')}{\det(\mathcal{U}^\dagger\not{D})} = \exp\{-i\,\mathrm{Tr}[(\mathcal{U}^\dagger\not{D})^{-s}\theta - (\not{D}\mathcal{U}^\dagger)^{-s}\theta]|_{s=0}\} \,. \tag{2.22}$$

We stress that the quantity in the exponent does not vanish for a general unitary operator $\mathcal{U}$.

By performing an infinitesimal gauge transformation of an axial-vector type $[A_\mu \to A_\mu - (1/e)\partial_\mu\varphi]$, we analogously get

$$\frac{\det_{\mathcal{U}}\not{D}'}{\det_{\mathcal{U}}\not{D}} = \frac{\det(\mathcal{U}^\dagger(\not{D}')}{\det(\mathcal{U}^\dagger\not{D})} = \exp\{-i\,\mathrm{Tr}[(\mathcal{U}^\dagger\not{D})^{-s}\gamma_5\varphi + (\not{D}\mathcal{U}^\dagger)^{-s}\gamma_5\varphi]|_{s=0}\} . \tag{2.23}$$

If we choose $\mathcal{U} = \mathbb{1}$, then obviously Eq. (2.22) becomes unity, expressing the vanishing of the (consistent) vector anomaly, whereas Eq. (2.23) obviously gives the (consistent) axial anomaly.

A different choice of $\mathcal{U}$ is provided by the polar decomposition of operator $\not{D}$:[13]

$$\not{D} = U|\not{D}| \,, \tag{2.24}$$

$|\not{D}|$ being defined as $[\not{D}^\dagger\not{D}]^{1/2}$. In general, the operator U is only partially isometric, but in this case it turns out to be unitary [see parenthetical remark following Eq. (2.2) and Ref. 9]. Setting $\mathcal{U} = U$ in Eq. (2.18) we get

$$\det{}_U \not{D} = \det U \det|\not{D}| \,, \tag{2.25}$$

and, consequently, from Eqs. (2.22) and (2.23),

$$\frac{\det_U\not{D}'}{\det_U\not{D}} = \exp\{-i\,\mathrm{Tr}[|\not{D}|^{-s}\theta - (U|\not{D}|U^\dagger)^{-s}\theta]|_{s=0}\} \tag{2.26}$$

and

$$\frac{\det_U\not{D}'}{\det_U\not{D}} = \exp\{-i\,\mathrm{Tr}[|\not{D}|^{-s}\gamma_5\varphi + (U|\not{D}|U^\dagger)^{-s}\gamma_5\varphi]|_{s=0}\} \,, \tag{2.27}$$

respectively.

The definition (2.24) together with the possibility of inverting $|\not{D}|$, owing to the absence of null eigenvalues, allows us to set Eqs. (2.26) and (2.27) in the form

$$\frac{\det_U\not{D}'}{\det_U\not{D}} = \exp\{-i\,\mathrm{Tr}[(\not{D}^\dagger\not{D})^{-s}\theta - (\not{D}\not{D}^\dagger)^{-s}\theta]|_{s=0}\} \tag{2.28}$$

and

$$\frac{\det_U\not{D}'}{\det_U\not{D}} = \exp\{-i\,\mathrm{Tr}[(\not{D}^\dagger\not{D})^{-s}\gamma_5\varphi + (\not{D}\not{D}^\dagger)^{-s}\gamma_5\varphi]|_{s=0}\} \,, \tag{2.29}$$

respectively. Evaluation of relevant Seeley's coefficients eventually gives

$$\frac{\det_U \not{D}'}{\det_U \not{D}} = \exp\left[i\frac{e}{2\pi}\int \epsilon_{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)\theta(x) d^2x \right] \tag{2.30}$$

and

$$\frac{\det_U \not{D}'}{\det_U \not{D}} = \exp\left[i\frac{e}{2\pi}\int \epsilon_{\mu\nu}(\partial_\mu V_\nu - \partial_\nu V_\mu)\varphi(x) d^2x \right], \tag{2.31}$$

namely, the vector and axial-vector anomalies, respectively, in their "covariant" form.[12] [The distinction between "consistent" and "covariant" forms of gauge anomalies is standard in the non-Abelian case. The former follows from an infinitesimal variation of $\det\not{D}$, the latter from a variation of $\det(\not{D}^\dagger\not{D})$ with respect to complex extensions of the gauge group as we shall see below. In this context we adopt the same terminology also for the Abelian case.]

The expressions (2.28) and (2.29) are quite remarkable because they provide the link with the Fujikawa prescription for computing anomalies[12] as well as with the anomalies obtained by performing "extended," i.e., nonunitary, gauge transformations on $|\not{D}|$.[14] The first point is realized by formally writing

$$\det\not{D} = \int \mathcal{D}\psi^\dagger\mathcal{D}\psi \exp\left[\int (-\psi^\dagger\not{D}\psi) d^2x\right], \tag{2.32}$$

and by inserting, in Eq. (2.32), the polar decomposition (2.24), followed by a change of integration variables

$$\det\not{D} = \int \mathcal{D}(U\chi)\mathcal{D}\psi \exp\left[\int [-\chi|\not{D}|\psi] d^2x\right]. \tag{2.33}$$

This equation naturally leads us to the definition (2.25).

By transferring the whole variation on the measure

$$\not{D}' = (U + \hat{\delta}U)|\not{D}| \tag{2.34}$$

(this equation can be interpreted as a definition of $\hat{\delta}U$; keeping in mind that $U+\hat{\delta}U$ is no longer unitary), we recover Fujikawa's recipe and results. Alternatively, by transferring the whole variation on $|\not{D}|$, we can write

$$\not{D}' = U(|\not{D}| + \hat{\delta}|\not{D}|). \tag{2.35}$$

Here we remark that $\hat{\delta}|\not{D}|$ does not coincide with the variations of $|\not{D}|$ under the vector and axial-vector gauge transformations, as the latter would obviously only entail phase transformations and thereby would not affect $\det|\not{D}|$. On the other hand, one can show that $\hat{\delta}|\not{D}|$ corresponds to the variations of $\not{D}^\dagger\not{D}$ under the "extended" transformations

$$A_\mu \to A_\mu - \frac{i}{e}\epsilon_{\mu\nu}\partial_\nu\theta \tag{2.36}$$

and

$$V_\mu \to V_\mu - \frac{i}{e}\epsilon_{\mu\nu}\partial_\nu\varphi, \tag{2.37}$$

respectively, which alter the Hermiticity properties of the potentials V_μ and A_μ. As a consequence, one realizes that, in order to fully exploit the general properties of the effective action, one should consider complex VAV potentials. In this way the procedure adopted in Ref. 14 for obtaining the anomalies in their "covariant" form is justified. This can also be checked directly in the expression one obtains for

$$\det(\not{D}^\dagger\not{D}) = \int \mathcal{D}\psi^\dagger\mathcal{D}\psi \exp(-\hat{S}[V_\mu, A_\mu]) = e^{-2\hat{W}[V_\mu, A_\mu]}, \tag{2.38}$$

where we have defined

$$\hat{S}[V_\mu, A_\mu] = \int d^2x\, \psi^\dagger\not{D}^\dagger\not{D}\psi. \tag{2.39}$$

By using the decoupling technique, one immediately shows that $\hat{W}[V_\mu, A_\mu]$ depends neither on α nor on ρ [see Eqs. (2.3) and (2.4)]. As a consequence, we can take them as equal to zero.

The next step is to perform the change of variables

$$\psi \to \exp[\gamma_5\beta(x) + \sigma(x)]\psi, \quad \psi^\dagger \to \psi^\dagger\exp[-\gamma_5\beta(x) - \sigma(x)]. \tag{2.40}$$

Under this change of variables, which entails a trivial Jacobian, Eq. (2.39) becomes

$$\hat{S} = \int d^2x\, \psi^\dagger(\not{\partial} - 2ie\not{V} - 2ie\not{A}\gamma_5)\not{\partial}\psi. \tag{2.41}$$

In order to go on, one has to be careful because nonlocal transformations are to be considered. As a matter of fact, under the change of variables

$$\psi \to \frac{1}{\not{\partial}}(e^{2\beta\gamma_5}\not{\partial}\psi), \quad \psi^\dagger \to \psi^\dagger e^{2\beta\gamma_5}, \tag{2.42}$$

$\hat{S}$ can be written as

$$\hat{S} = \int d^2x\, \psi^\dagger(\not{\partial} - 2ie\not{A}\gamma_5)\not{\partial}\psi. \tag{2.43}$$

However, the corresponding Jacobian J is given by

$$\ln J = -\int_0^1 dr\, \omega'(r) \tag{2.44}$$

with

$$\omega'(r) = \lim_{s\to 0}\mathrm{Tr}[(\not{D}_r^\dagger\not{\partial})^{-s-1}\delta(\not{D}_r^\dagger\not{\partial})] \tag{2.45}$$

and

$$\not{D}_r^\dagger = \not{\partial} - 2ie\not{V}(1-r) - 2ie\not{A}\gamma_5, \quad 0 \le r \le 1,$$

δ representing the variation under transformation (2.42) with an infinitesimal β.

Quantity (2.45) can be computed using Seeley's methods (see Appendix A); the final result is

$$J = \exp\left[-\frac{e^2}{\pi}\int d^2x\, V_\mu\left[\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right]V_\nu\right] \tag{2.46}$$

and therefore $\hat{W}$ becomes

$$\hat{W}=\frac{e^2}{2\pi}\int d^2x\left[V_\mu\left[\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\partial^2}\right]V_\nu\right]$$
$$-\tfrac{1}{2}\ln\int\mathcal{D}\psi^\dagger\mathcal{D}\psi\exp\left[-\int d^2x\,\psi^\dagger(\not\partial-2ieA\gamma_5)\not\partial\psi\right]. \tag{2.47}$$

We can finally decouple field A_μ through the last change of variable

$$\psi\to\frac{1}{\not\partial}(e^{-2\sigma(x)}\not\partial\psi),$$
$$\psi^\dagger\to\psi^\dagger e^{2\sigma(x)}, \tag{2.48}$$

which transforms $\not\partial-2ie\not A\gamma_5$ into $\not\partial$, but involves the nontrivial Jacobian

$$\hat{J}=\exp\left[-\frac{e^2}{\pi}\int d^2x\,A_\mu\left[\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\partial^2}\right]A_\nu\right]. \tag{2.49}$$

To sum up, the final result is, up to irrelevant terms,

$$\hat{W}=\frac{e^2}{2\pi}\int d^2x\left[V_\mu\left[\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\partial^2}\right]V_\nu\right.$$
$$\left.+A_\mu\left[\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\partial^2}\right]A_\nu\right]. \tag{2.50}$$

One can easily realize that, starting from operator $\not D\not D^\dagger$, the same result will follow.

Equation (2.50) is apparently gauge invariant, as was expected from the starting definition. If we compare Eq. (2.50) with Eq. (2.13), we notice that, in spite of the fact that the same regularization technique is adopted, the usually claimed relation $|W|=\hat{W}$ does not apply; i.e., $|W|$ is not, in general, gauge invariant. Nevertheless, as we have already explained, we are free to add the local counterterm $e^2a\int d^2x\,A_\mu A_\mu$ to W. If we choose $a=1$, gauge invariance is restored for $|W|$ and the equality

$$|W(a=1)|=\hat{W} \tag{2.51}$$

follows. By performing the infinitesimal transformations (2.36) and (2.37) in Eq. (2.50), one immediately recovers Eqs. (2.30) and (2.31).

We would like to conclude this section by remarking that only the choice $\mathcal{U}=\mathbb{1}$ leads, under infinitesimal gauge variations, to the anomalies in their "consistent" form. As a matter of fact, only in this case do successive variations obey the usual group composition law. If, instead, we first operate a polar decomposition, neither the variation $\hat{\delta}U$, nor the variation $\hat{\delta}|\not D|$ are genuine gauge transformations. This is the ultimate reason why anomalies in the "covariant" form unavoidably ensue.

III. THE EFFECTIVE ACTION FOR CONSTANT FIELDS

In this section we shall consider the same effective action, but for vector and axial-vector potentials which do not allow a compactification of the model on a two-dimensional sphere. To be more specific, we shall choose uniform and constant fields leading to potentials which increase at infinity. The previous technique does not apply to this case; in turn, the problem of determining the spectrum and the eigenfunctions for the relevant fermionic determinants can be explicitly solved.

The interest in this study is twofold: it provides us with detailed expressions whose variations can be compared with the corresponding quantities obtained by means of procedures proposed by Andrianov and Bonora[15] and by Fujikawa;[12] in turn, this comparison allows us to point out the differences related to the unbounded nature of the potentials, a problem which is now starting to be investigated on a more general basis.[7,8]

Once again, we consider Eq. (2.1) and choose for the potentials the quantities

$$V_\mu=\left[-\frac{\mathcal{E}x}{2},\frac{\mathcal{E}\tau}{2}\right],$$
$$A_\mu=\left[\mu\frac{\tau}{2}+\frac{Bx}{2},\mu\frac{x}{2}-\frac{B\tau}{2}\right], \tag{3.1}$$

with constant $\mathcal{E}$ and B. We remark that $\partial_\mu V_\mu=0$, namely, we have not introduced in V_μ a term of the "gradient" type, which would be irrelevant at this stage.

The operator $\not D=\not\partial-ie\not V-ie\not A\gamma_5$ is not Hermitian, as is well known. The first method in order to evaluate its determinant is by replacing the term $ie\not A\gamma_5$ with $e\alpha\not A\gamma_5$. Then $\not D(\alpha)$ is Hermitian as long as α is real. It is easy to show that choice (3.1) for real α is equivalent to the following:

$$V_\mu=\left[-\frac{(\mathcal{E}-\alpha\mu)x}{2},\frac{(\mathcal{E}-\alpha\mu)\tau}{2}\right],$$
$$A_\mu=\left[\frac{Bx}{2},-\frac{B\tau}{2}\right], \tag{3.2}$$

and then we can set $\mu=0$, by redefining $\mathcal{E}$. We remark, however, that, should α take its "physical" value i, the resulting field would no longer be Hermitian. The problem of evaluating the spectrum of $\not D(\alpha)$ will be solved algebraically. We shall, of course, recover the well-known Landau levels; in addition, we shall obtain an orthonormal Segal-Bargmann basis, which is necessary to neatly discuss degeneracy and to perform first-order perturbation theory (see Sec. IV).

First we introduce the complex variable $z=\tau+ix=\rho e^{i\theta}$. Then,

$$\partial_z=\tfrac{1}{2}(\partial_\tau-i\partial_x). \tag{3.3}$$

Field B does not enter the spectrum of $\not D(\alpha)$ because it can be canceled by a phase transformation on the wave function, in agreement with the result in the compactifiable case.

Next we introduce the operators

$$\delta_+=\frac{i}{\sqrt{k}}\left[\partial_z-\frac{k\bar{z}}{2}\right]=\delta_-^\dagger, \tag{3.4}$$

$\bar{z}$ being the complex conjugate of z and $k=e\mathcal{E}/2$, which we have chosen positive for the sake of definiteness, and the operators

$$\omega_+ = \frac{i}{\sqrt{k}}\left[\partial_{\bar z} - \frac{kz}{2}\right] = \omega_-^\dagger \ . \tag{3.5}$$

They satisfy the algebra

$$[\delta_-,\delta_+] = [\omega_-,\omega_+] = 1 \ , \tag{3.6}$$

all the other commutators vanishing.

In terms of those operators we have

$$\not{D} = 2\sqrt{k}\begin{bmatrix} 0 & \delta_- \\ \delta_+ & 0 \end{bmatrix} . \tag{3.7}$$

Henceforth, an infinite degeneracy is present related to the action of $\omega_\pm$. The spectrum is easily obtained, realizing that

$$\not{D}^2 = 4k\begin{bmatrix} \delta_-\delta_+ & 0 \\ 0 & \delta_+\delta_- \end{bmatrix} , \tag{3.8}$$

describing harmonic oscillators. As a consequence, the eigenvalues are

$$\lambda_n^2 = 4kn \ , \tag{3.9}$$

namely,

$$\lambda_n = \pm 2\sqrt{kn} \ , \quad n = 0,1,\ldots \ . \tag{3.10}$$

In order to discuss their degeneracy, it is useful to realize a representation of the algebra (3.6) in the Segal-Bargmann space of analytic functions, which becomes a Hilbert space when endowed with the measure

$$d\mu = \frac{k}{2\pi}\exp(-k|z|^2)dz\, d\bar z$$

in the usual definition of the scalar product. The kernel of $\not{D}$ is described by the functions $\binom{0}{\varphi_0}$, where

$$\delta_-\varphi_0 = 0 \ . \tag{3.11}$$

The solutions are

$$\varphi_0 = \exp\left[-\frac{k}{2}|z|^2\right] f_0(z) \ , \tag{3.12}$$

$f_0(z)$ being an analytic function of z. The degeneracy can be resolved by noticing that

$$\omega_+\varphi_0 = -i\sqrt{k}\, z\varphi_0 \ . \tag{3.13}$$

Then the polynomials

$$f_0^{(q)} = \frac{(\sqrt{k}\,)^q}{\sqrt{q!}} z^q, \quad q = 0,1,\ldots, \tag{3.14}$$

provide us with an orthonormal basis with respect to the mentioned measure.

Now the "excited" states can be obtained acting with δ_+ on φ_0; each one of them is degenerate according to Eq. (3.14). If we define

$$\begin{aligned}\varphi_n^{(q)} &= \frac{\sqrt{k}}{\sqrt{2\pi}\sqrt{n!q!}}\delta_+^n\omega_+^q \exp\left[-\frac{k|z|^2}{2}\right] \\ &= \sqrt{k/2\pi}\exp\left[-\frac{k|z|^2}{2}\right](-iz\sqrt{k}\,)^q(-i\bar z\sqrt{k}\,)^n \\ &\quad\times \sum_{h=0}^{\infty}\frac{(-k|z|^2)^{-h}}{h!}\frac{\sqrt{n!q!}}{\Gamma(q-h+1)\Gamma(n-h+1)}\end{aligned} \tag{3.15}$$

[the series is actually a finite sum as $h \le \min(n,q)$], it is easy to check that the eigenfunctions of $\not{D}$,

$$\phi_{n(\pm)}^{(q)} = \frac{1}{\sqrt{2}}\begin{bmatrix} \pm\varphi_{n-1}^{(q)} \\ \varphi_n^{(q)} \end{bmatrix}, \quad n = 0,1,\ldots,\varphi_{-1} \equiv 0 \ , \tag{3.16}$$

correspond to the eigenvalues $\pm 2\sqrt{nk}$.

The determinant of $\not{D}^2$ can be defined by omitting the null eigenvalue and by means of the ζ-function-regularization technique:

$$\zeta_{\not{D}^2} = \Omega\sum_{n=1}(\lambda_n^2/\Lambda^2)^{-s} \ , \tag{3.17}$$

Λ being a dimensionful regularization parameter and Ω being the degeneracy (which does not depend on n) associated to any eigenvalue. We get

$$\zeta_{\not{D}^2} = \Omega\left[\frac{4k}{\Lambda^2}\right]^{-s}\zeta(s) \ , \tag{3.18}$$

$\zeta(s)$ being the Riemann ζ function.

To evaluate Ω, we count the number of eigenfunctions (3.16) for $n = 0$, whose maximum lies inside a circle of (large) radius R in the complex plane

$$\begin{aligned} &q - k\rho_{\max}^2 = 0 \ , \\ &\rho_{\max} \le R \ , \\ &q \le kR^2 \ . \end{aligned} \tag{3.19}$$

Then, recalling that there is a one-to-one correspondence between the degenerate basis for any other value of n, due to the fact that ω_+ and δ_+ commute, we get

$$\Omega = 2\sum_q^{q_{\max}} \sim 2k\int d\rho_{\max}^2 = 2kR^2 = \frac{2kS}{\pi} \ , \tag{3.20}$$

the factor 2 being due to the extra ($\pm$) degeneracy and S being a (large) two-dimensional area. The same result would follow by considering the behavior of the heat kernel for small values of the "temperature." The effective action takes the expression

$$\tfrac{1}{2}\zeta'_{\not{D}^2}(0) = \frac{eS(\mathcal{E}-\alpha\mu)}{4\pi}\ln\left[\frac{e(\mathcal{E}-\alpha\mu)}{\pi\Lambda^2}\right] . \tag{3.21}$$

It exhibits the logarithmic behavior which is characteristic of unbounded potentials.[16]

The case with $e\mathcal{E} < 0$ does not entail new features, apart from the different shift due to the μ quantity, according to Eq. (3.2). We remark that Eq. (3.21), although quite different from Eq. (2.13), depends neither on B nor on a possible "gradient" term in V_μ, in agreement with the result obtained in the compactifiable case. One might define the determinant of $\not{D}$, by continuing Eq. (3.21) to $\alpha=i$. In this way a complex result will obviously ensue.

A different procedure is started by considering the two positive operators $\not{D}^\dagger\not{D}$ and $\not{D}\not{D}^\dagger$. As their spectra are invariant under "gradient"-type transformations, we can choose the potentials as in Eq. (3.1) with $\mu=0$. Thereby, we get

$$\not{D}=\begin{bmatrix} 0 & 2i\partial_{\bar z}+iz\dfrac{e(\mathcal{E}-B)}{2} \\ 2i\partial_z-i\bar z\dfrac{e(\mathcal{E}+B)}{2} & 0\end{bmatrix} =2\begin{bmatrix} 0 & \delta_1\sqrt{k_1} \\ \delta_2^\dagger\sqrt{k_2} & 0\end{bmatrix}, \tag{3.22}$$

where we have set $k_1=e(\mathcal{E}-B)/2$, $k_2=e(\mathcal{E}+B)/2$ and we have chosen them positive for the time being. We have the commutation relations

$$[\delta_1,\delta_1^\dagger]=[\delta_2,\delta_2^\dagger]=1 . \tag{3.23}$$

Correspondingly, we introduce the operators

$$\omega_{1,2}=\frac{i}{\sqrt{k_{1,2}}}\left[\partial_z+\bar z\frac{k_{1,2}}{2}\right], \tag{3.24}$$

$$\omega_{1,2}^\dagger=\frac{i}{\sqrt{k_{1,2}}}\left[\partial_{\bar z}-\frac{k_{1,2}}{2}z\right], \tag{3.25}$$

such that

$$[\omega_1,\delta_1]=[\omega_1,\delta_1^\dagger]=[\omega_2,\delta_2]=[\omega_2,\delta_2^\dagger]=0 , \tag{3.26}$$

whereas

$$[\omega_1,\omega_1^\dagger]=[\omega_2,\omega_2^\dagger]=1 . \tag{3.27}$$

Now $\not{D}\not{D}^\dagger$ and $\not{D}^\dagger\not{D}$ become

$$\not{D}\not{D}^\dagger=4\begin{bmatrix} k_1\delta_1\delta_1^\dagger & 0 \\ 0 & k_2\delta_2^\dagger\delta_2\end{bmatrix}, \tag{3.28}$$

$$\not{D}^\dagger\not{D}=4\begin{bmatrix} k_2\delta_2\delta_2^\dagger & 0 \\ 0 & k_1\delta_1^\dagger\delta_1\end{bmatrix}. \tag{3.29}$$

It is clear from the above expressions that the eigenvalue equations for the up and down components decouple.

Therefore, the sets of the eigenvalues are, respectively,

$$\Sigma_{\not{D}\not{D}^\dagger}=U\{4k_1(n+1),4k_2m\} , \tag{3.30a}$$

$$\Sigma_{\not{D}^\dagger\not{D}}=U\{4k_2(p+1),4k_1q\} , \tag{3.30b}$$

with $n,m,p,q=0,1,\ldots$. We notice that the null eigenvalue is present in both the spectral sets. Regarding this we remark that if in Eq. (3.22) we choose $k_1\langle 0k_2\rangle 0$ it follows that

$$\not{D}=2\begin{bmatrix} 0 & \delta_1^\dagger\sqrt{-k_1} \\ \delta_2^\dagger\sqrt{k_2} & 0\end{bmatrix} \tag{3.31}$$

and thereby

$$\begin{aligned}\Sigma_{\not{D}\not{D}^\dagger}&=U\{4|k_1|n,4k_2m\} ,\\ \Sigma_{\not{D}^\dagger\not{D}}&=U\{4|k_1|(p+1),4k_2(q+1)\} ,\end{aligned} \tag{3.32}$$

namely, the first spectral set possesses the null eigenvalue twice, whereas $\Sigma_{\not{D}^\dagger\not{D}}$ does not.

Thus, changes of sign of k_1,k_2 alter the kernel of $\not{D}$, without altering the index of $\not{D}$. This phenomenon has already been noticed in Ref. 6, in the four-dimensional case.

If we disregard the null eigenvalue, the ζ functions turn out to be

$$\zeta_{\not{D}\not{D}^\dagger}=\zeta_{\not{D}^\dagger\not{D}} =\zeta(s)\left[\Omega_1\left[\frac{4|k_1|}{\Lambda^2}\right]^{-s}+\Omega_2\left[\frac{4|k_2|}{\Lambda^2}\right]^{-s}\right], \tag{3.33}$$

where

$$\Omega_{1,2}=\frac{|k_{1,2}|S}{\pi}, \tag{3.34}$$

in analogy with Eq. (3.20). Then,

$$\tfrac{1}{2}\zeta'_{\not{D}\not{D}^\dagger}(0)=\tfrac{1}{2}\zeta'_{\not{D}^\dagger\not{D}}(0) =\frac{S}{4\pi}\left[|k_1|\ln\frac{2|k_1|}{\pi\Lambda^2}+|k_2|\ln\frac{2|k_2|}{\pi\Lambda^2}\right]. \tag{3.35}$$

Equation (3.35) bears no relation to Eq. (3.21) unless we have trivially $A_\mu=0$. As a consequence, should one define

$$|\det{}'\not{D}(\alpha=i)|=[\det{}'(\not{D}\not{D}^\dagger)]^{1/2} ,$$

where the prime means that the null eigenvalue is disregarded, the difference between the definitions of $\det{}'\not{D}$ according to Eqs. (3.21) and (3.35), respectively, would not simply be a phase factor. Moreover, at variance with the compactifiable case, this difference cannot be eliminated by adding a suitable polynomial counterterm to the effective action.

IV. THE CHIRAL ANOMALY IN THE CASE OF CONSTANT FIELDS

We are interested in obtaining the variation of the effective action we have studied in the preceding section, under an infinitesimal localized gauge transformation. The general method in Sec. II requires a compactification of the problem and therefore does not apply, whereas an explicit solution of the Dirac equation for arbitrary (noncompact) potentials is not easily available. Nevertheless, if we are only interested in infinitesimal variations, we can apply first-order perturbation theory, using the eigenvalues and the eigenfunctions of the preceding section as unperturbed solutions.

As a first step we shall compute the variation of the eigenvalues (3.10) of the Hermitian operator $\not{D}(\alpha)$. The calculation is complicated by the circumstance that each eigenvalue is infinitely degenerate. One can easily check that a transformation on the vector field $V_\mu\to V_\mu$

$+(1/e)\partial_\mu\theta$ does not influence the spectrum. Therefore, let us study the transformation $A_\mu \to A_\mu+(1/e)\partial_\mu\theta$, θ being any infinitesimal function of (τ,x) with compact support.

The perturbation matrix for the eigenvalue λ_n is

$$M_{q,r}^{(n)}=\pm 2\alpha\sqrt{nk}\int\theta(z,\bar z)dz\,d\bar z(\bar\varphi_n^{(q)}\varphi_n^{(r)}-\bar\varphi_{n-1}^{(q)}\varphi_{n-1}^{(r)})\ ,\qquad n\ge 1\ , \tag{4.1}$$

where the eigenfunctions are the ones defined in Eq. (3.15); the plus or minus sign coincides with the sign chosen for λ_n and the null eigenvalue remains unperturbed. The indices (q,r) refer to the degeneracy controlled by the operators $\omega_\pm$.

The matrix $M^{(n)}$ is Hermitian; as a consequence, it can be set in a diagonal form by a suitable unitary transformation on the basis. As the operators ω and δ commute, the unitary operator implementing this transformation in the whole Hilbert space is block diagonal.

The transformation obviously depends on θ. However, we will show that the anomaly does not depend on the basis. Let us therefore write, in the diagonal basis,

$$\begin{aligned}M_{q,r}^{(n)}&=\pm 2\alpha\sqrt{nk}\,\delta_{q,r}\int\theta(z,\bar z)dz\,d\bar z(\bar\xi_n^{(q)}\xi_n^{(q)}-\bar\xi_{n-1}^{(q)}\xi_{n-1}^{(q)})\\ &=\pm 2\alpha\sqrt{nk}\,\delta_{q,r}(\theta_n^{(q)}-\theta_{n-1}^{(q)})\ .\end{aligned} \tag{4.2}$$

Now, if we consider the variation of $\zeta_{D\!\!\!/^2}(s)$, we get

$$\zeta_{D\!\!\!/^2+\delta D\!\!\!/^2}(s)=\zeta_{D\!\!\!/^2}(s)-4\alpha s\sum_{n=1}\sum_q(\lambda_n^2/\Lambda^2)^{-s}(\theta_n^{(q)}-\theta_{n-1}^{(q)})\ . \tag{4.3}$$

As we shall discuss in the following, Eq. (4.3) involves delicate considerations related to the degeneracy labeled by the index q of the spectrum. We remark that the indices n and q appear symmetrically at the level of the eigenfunctions, whereas the ζ-function technique apparently regularizes only the n dependence. Because of this fact, a careful limit will be necessary in order to handle the dependence on q properly. The first evidence of the crucial role this degeneracy plays was encountered when evaluating dimension Ω. As a matter of fact, in Appendix B we show that

$$\sum_{q=0}^{\infty}\theta_n^{(q)}=\Theta=\frac{k}{2\pi}\int dz\,d\bar z\,\theta(z,\bar z)\ , \tag{4.4}$$

quantity Θ being independent of n. The sum over q, being a trace, can be computed in any basis, in particular, using the eigenfunctions $\varphi_n^{(q)}$ of Eq. (3.15). As a consequence, the variation in Eq. (4.3) should vanish, in spite of the fact that perturbation $\theta(z,\bar z)$ is localized.

One reaches a completely different conclusion if one first restricts the sum over q introducing a suitable cutoff, which is removed only at the very end of the calculation. The simplest way to achieve this goal is by introducing a damping factor in Eq. (4.3), namely,

$$\theta_n^{(q)}\to e^{-\epsilon q}\theta_n^{(q)}\ , \tag{4.5}$$

where ϵ is a positive parameter which will eventually be set equal to zero. Then one can prove (see the Appendix B) that, if θ is a real positive measure,

$$\sum_{q=0}^{\infty}\theta_n^{(q)}e^{-\epsilon q}=\Theta_n<\Theta\ . \tag{4.6}$$

From Eq. (4.3) we derive

$$\begin{aligned}\zeta'_{D\!\!\!/^2+\delta D\!\!\!/^2}(s)=\zeta'_{D\!\!\!/^2}(s)&-4\alpha\sum_{n=1}(\lambda_n^2/\Lambda^2)^{-s}(\Theta_n-\Theta_{n-1})\\ &+4\alpha s\sum_{n=1}(\lambda_n^2/\Lambda^2)^{-s}\\ &\qquad\times\ln(\lambda_n^2/\Lambda^2)(\Theta_n-\Theta_{n-1})\ .\end{aligned} \tag{4.7}$$

Usually, without the extra damping factor, Eq. (4.7) is considered for large enough Res and then analytically continued at $s=0$. In our case, the convergence in n (for any value of s) is actually guaranteed by parameter ϵ, so that the limit $s\to 0$ in the series can be naively performed, giving

$$\zeta'_{D\!\!\!/^2+\delta D\!\!\!/^2}(0)=\zeta'_{D\!\!\!/^2}(0)+4\alpha\Theta_0\ . \tag{4.8}$$

Nevertheless, the limit $\epsilon\to 0$ will be safe only in a suitable region of the complex s plane, Re$s>0$ in the present case.

Following this procedure we finally get

$$\delta\zeta'_{D\!\!\!/^2}(0)=\frac{e\alpha(\mathcal{E}-\alpha\mu)}{\pi}\int\theta(z,\bar z)dz\,d\bar z\ , \tag{4.9}$$

in full agreement with the result (2.17) of the compactifiable case, in spite of the different form the effective action takes in the two instances. In order to obtain the covariant form of the anomalies, we can start by considering variations of the operators $D\!\!\!/ D\!\!\!/^\dagger$ and $D\!\!\!/^\dagger D\!\!\!/$, as given in Eqs. (3.28) and (3.29).

From those expressions one immediately realizes that the functions

$$\begin{aligned}\phi_{n,q}^{(u)}&=\begin{bmatrix}\varphi_{n,q}^{(1)}\\ 0\end{bmatrix},\quad n,q\ge 0\ ,\\ \phi_{m,r}^{(d)}&=\begin{bmatrix}0\\ \varphi_{m,r}^{(2)}\end{bmatrix},\quad m,r\ge 0\ ,\end{aligned} \tag{4.10}$$

with

$$\varphi_{n,q}^{(1,2)}=\left[\frac{k_{1,2}}{2\pi}\right]^{1/2}\frac{1}{\sqrt{n!q!}}(\delta_{1,2}^\dagger)^n(\omega_{1,2}^\dagger)^q\exp\left[-\frac{k_{1,2}}{2}|z|^2\right] \tag{4.11}$$

are eigenfunctions of the operator $D\!\!\!/ D\!\!\!/^\dagger$ corresponding to the spectrum [(3.30a)], whereas

$$\begin{aligned}\psi_{n,q}^{(u)}&=\begin{bmatrix}\varphi_{n,q}^{(2)}\\ 0\end{bmatrix},\quad n,q\ge 0\ ,\\ \psi_{m,r}^{(d)}&=\begin{bmatrix}0\\ \varphi_{m,r}^{(1)}\end{bmatrix},\quad m,r\ge 0\ ,\end{aligned} \tag{4.12}$$

are eigenfunctions of $D\!\!\!/^\dagger D\!\!\!/$, corresponding to the spectrum [(3.30b)]. Now, from Eq. (3.22), we derive

$$\not{D}\psi^{(d)}_{m,r}=2\sqrt{mk_1}\phi^{(u)}_{m-1,r},\quad \phi^{(u)}_{-1,r}\equiv 0\ ,$$
$$\not{D}\psi^{(u)}_{n,q}=2\sqrt{(n+1)k_2}\phi^{(d)}_{n+1,q}\ , \tag{4.13}$$

namely, $\not{D}$ turns eigenfunctions of $\not{D}^\dagger\not{D}$ into eigenfunctions of $\not{D}\not{D}^\dagger$.

Let us now consider the infinitesimal variation $\not{D}\to(1-\gamma_5\alpha)\not{D}(1-\gamma_5\alpha)$, leading to

$$\delta(\not{D}^\dagger\not{D})=-\gamma_5\alpha\not{D}^\dagger\not{D}-\not{D}^\dagger\not{D}\gamma_5\alpha-2\not{D}^\dagger\gamma_5\alpha\not{D}\ . \tag{4.14}$$

Taking the matrix elements between eigenfunctions $\psi_{n,q}$, we get the first-order corrections to the eigenvalues. For instance,

$$M_n^{(q,r)}=(\psi^{(u)}_{n,q},\delta(\not{D}^\dagger\not{D})\psi^{(u)}_{n,r})$$
$$=8k_2(n+1)$$
$$\times\int dz\,d\bar z\,\alpha(z,\bar z)(\bar\varphi^{(2)}_{n,q}\varphi^{(2)}_{n,r}-\bar\varphi^{(2)}_{n+1,q}\varphi^{(2)}_{n+1,r})\ , \tag{4.15}$$

where we have used Eqs. (4.12). An analogous equation holds for the down components. Moreover, we can choose a basis in which M_n is diagonal. Then,

$$M_n^{(q)}=8k_2(n+1)[\alpha_n^{(q)}(k_2)-\alpha_{n+1}^{(q)}(k_2)],\quad n\ge 0 \tag{4.16}$$

and

$$M_m^{(q)}=-8k_1m[\alpha_m^{(q)}(k_1)-\alpha_{m-1}^{(q)}(k_1)],\quad m\ge 1\ . \tag{4.17}$$

The variation occurring in the ζ function is

$$\delta\zeta_{\not{D}^\dagger\not{D}}(s)=-2s\sum_{n,q}[\alpha_n^{(q)}(k_2)-\alpha_{n+1}^{(q)}(k_2)]\lambda_n^{-s}$$
$$+2s\sum_{m,q}\alpha_m^{(q)}(k_1)-\alpha_{m-1}^{(q)}(k_1)]\lambda_m^{-s}\ , \tag{4.18}$$

leading to the anomaly

$$\delta\zeta'_{\not{D}^\dagger\not{D}}(0)=-2\sum_q[\alpha_0^{(q)}(k_2)+\alpha_0^{(q)}(k_1)] \tag{4.19}$$

(again see Appendix B).

Now the trace can be computed in basis (4.12) and we get

$$\delta\zeta'_{\not{D}^\dagger\not{D}}(0)=-\frac{k_1+k_2}{\pi}\int\alpha(z,\bar z)dz\,d\bar z$$
$$=-\frac{e\mathcal{E}}{\pi}\int\alpha dz\,d\bar z\ , \tag{4.20}$$

in full agreement with Eq. (2.31). The same procedure applies to the variation $\not{D}\to(1-\theta)\not{D}(1+\theta)$ and to the different choices of sign for k_1 and k_2.

In conclusion we can say that, in spite of the quite different form the effective action takes in the compactifiable and the noncompatifiable cases, its variations under infinitesimal transformations of compact support are given by the same expressions.

It is now instructive to compare these results with the ones obtained directly by means of a path-integral calculation of Eq. (2.2), according to the suggestion proposed by Fujikawa. As we have noticed in Sec. II, the path integral has only a formal meaning and requires a specific regularization. In the present case we know the spectral sets of $\not{D}^\dagger\not{D}$ and $\not{D}\not{D}^\dagger$ [Eqs. (4.10) and (4.12)] explicitly and also the partially isometric operator U [Eqs. (4.13)]. We can now evaluate action S in Eq. (2.1) by expanding spinor ψ on the basis of the eigenstates of $\not{D}^\dagger\not{D}$ and $\psi^\dagger$ on the basis of $\not{D}\not{D}^\dagger$, namely,

$$\psi=\sum_{n,q=0}^{\infty}(a_{n,q}\psi^{(u)}_{n,q}+b_{n,q}\psi^{(d)}_{n,q})\ ,$$
$$\psi^\dagger=\sum_{m,r=0}^{\infty}(\bar\alpha_{m,r}\phi^{\dagger(u)}_{m,r}+\bar\beta_{m,r}\phi^{\dagger(d)}_{m,r})\ . \tag{4.21}$$

Taking Eq. (4.13) into account, we get

$$S\equiv\int d^2x\,\psi^\dagger\not{D}\psi=2\sum_{\substack{n=0\\q=0}}\sqrt{(n+1)k_2}\alpha_{n,q}\bar\beta_{n+1,q}$$
$$+2\sum_{\substack{n=1\\q=0}}\sqrt{nk_1}b_{n,q}\bar\alpha_{n-1,q}\ . \tag{4.22}$$

We notice that, in Eq. (4.22), the Grassmann coefficients $\bar\beta_{0,q}$ and $b_{0,q}$ do not appear. As a consequence, the generating functional

$$Z(V_\mu,A_\mu)=\exp(-W[V_\mu,A_\mu])\ ,$$

if naively computed via Berezin integration, would vanish. Let us therefore try to redefine the functional integral over ψ and $\psi^\dagger$ by omitting the integrations over $\bar\beta_{0,r}$ and $b_{0,q}$. This would correspond to a modified definition of the determinant. In doing so, one immediately gets

$$\int\prod_{\substack{n=0\\q=0}}da_{n,q}\prod_{\substack{m=1\\r=0}}db_{m,r}\prod_{\substack{s=0\\t=0}}d\alpha_{l,s}\prod_{\substack{k=1\\t=0}}d\bar\beta_{k,t}\exp\left[-2\sum_{\substack{j=0\\k=0}}\sqrt{(j+1)k_2}a_{j,k}\bar\beta_{j+1,k}-2\sum_{\substack{j=1\\k=0}}\sqrt{jk_1}b_{j,k}\alpha_{j-1,k}\right]$$
$$=|\det{}'\not{D}^\dagger\not{D}|^{1/2}=|\det{}'\not{D}\not{D}^\dagger|^{1/2}\ , \tag{4.23}$$

where the infinite products appearing on the left-hand side are supposed to be regularized, by means for instance of the ζ-function technique.

As we have already noticed, Eq. (4.23) is invariant under the usual gauge transformations on the vector and axial-vector potentials. The first lesson we learn is that the quantity in Eq. (4.23) has nothing to do with $|\det{}'\not{D}|$.

The second remark concerns the integrations over the variables $\bar\beta_{0,r}$ and $b_{0,q}$ we have omitted. The corresponding measure of integration $\prod_q db_{0,q}\prod d\bar\beta_{0,r}$ is not gauge invariant. We indeed have

$$b_{0,q}[\theta]=\int \psi_{0,q}^{\dagger(d)}(z,\bar z)e^{i\theta(z,\bar z)}\psi(z,\bar z)dz\,d\bar z$$
$$\sim b_{0,q}(0)+i\int \psi_{0,q}^{\dagger(d)}(z,\bar z)\theta(z,\bar z)\psi(z,\bar z)dz\,d\bar z$$
$$=b_{0,q}(0)+i\sum_{n,r} b_{n,r}(0)\Theta_{0,q;n,r}^{(1)}\ , \qquad (4.24)$$

where

$$\Theta_{0,q;n,r}^{(1)}=\int dz\,d\bar z\,\theta(z,\bar z)\bar\varphi_{0,q}^{(1)}(z,\bar z)\varphi_{n,r}^{(1)}(z,\bar z)\ . \qquad (4.25)$$

We are interested in the change of variables $b_{0,q}(0)\to b_{0,q}(\theta)$ for infinitesimal θ. The variation of the measure turns out to be

$$\delta\left[\prod_q db_{0,q}\right]=i\sum_q \Theta_{0,q;0,q}^{(1)}$$
$$=\frac{ik_1}{2\pi}\int dz\,d\bar z\,\theta(z,\bar z)\ , \qquad (4.26)$$

in full agreement with Eq. (4.20). The analogous calculation for $\bar\beta_{0,r}$ gives the term involving k_2.

We conclude that the covariant anomalies can be obtained either by varying the full fermionic integration measure or by performing infinitesimal local transformations belonging to a noncompact gauge group on quantity $[\det' D\!\!\!/^{\dagger} D\!\!\!/]^{1/2}$, as we have already discussed in the compact case. In this example involving constant fields, one recovers the remarkable fact that the covariant anomalies are related to the density of the degeneracy of the eigenvalues. Whether this feature generalizes to arbitrary noncompactifiable Dirac-type operators is an interesting open problem, at least to our knowledge.

V. CONCLUDING REMARKS

Although they are probably unphysical, two-dimensional models allow detailed explicit calculations to be performed which, in turn, lead to the clarifying of several delicate features concerning the origin and the form of the anomalies.

The study of compactifiable potentials in Sec. II has shown that, in general, there is no equality between $|\det D\!\!\!/|$ and $(\det D\!\!\!/^{\dagger} D\!\!\!/)^{1/2}$, $D\!\!\!/$ being the Dirac operator in the presence of vector and axial-vector gauge potentials. We have shown that, while gauge variations of $\det D\!\!\!/$ give rise to anomalies in their consistent form, by considering variations after a polar decomposition of the Dirac operator, one instead recovers the "covariant" form of the anomalies.

The same difference appears even more drastically in a model characterized by constant fields, namely, with potentials increasing at infinity. In this case, the decoupling techniques do not work; however, the model is exactly solvable in terms of eigenvalues and eigenfunctions, so that the above features can again be explicitly tested.

One can appreciate even more the difference of regularizing the theory by analytically continuing the operator $(D\!\!\!/)^{-s}$, a procedure which eventually leads under gauge transformations to the consistent form of the anomalies, or by first considering the polar decomposition $D\!\!\!/=U|D\!\!\!/|$ and then transferring the variation of $D\!\!\!/$ either on U or on $|D\!\!\!/|$ [see Eqs. (2.34) and (2.35)].

Finally, in the case of constant fields, one verifies that the "covariant" form of the anomaly appears to be related to the degeneracy of the null eigenvalue and that the integral of the vector anomaly coincides with the index of the Dirac operator. It might be interesting to investigate whether these properties persist for more general noncompact potentials.

APPENDIX A

We compute the expression (2.45),

$$\lim_{s\to 0}\mathrm{Tr}[(D\!\!\!/_r^{\dagger}\partial\!\!\!/)^{-s-1}\delta(D\!\!\!/_r^{\dagger}\partial\!\!\!/)]\ , \qquad (A1)$$

using Seeley's coefficients for evaluating the trace in $s=0$, as an analytic continuation. The first step concerns the explicit writing of the infinitesimal variation of $(D\!\!\!/_r^{\dagger}\partial\!\!\!/)$ under (2.42):

$$\delta(D\!\!\!/_r^{\dagger}\partial\!\!\!/)=-2\gamma_5\beta(D\!\!\!/_r^{\dagger}\partial\!\!\!/)-(D\!\!\!/_r^{\dagger})2\beta\gamma_5\partial\!\!\!/\ . \qquad (A2)$$

The insertion of (A2) into the trace (A1) leads to the only nonvanishing contribution:

$$\lim_{s\to 0}\mathrm{Tr}[(D\!\!\!/_r\partial\!\!\!/)^{-s}2\beta\gamma_5]$$
$$=\int d^2x\,\mathrm{Tr}[K_0(x,x;D\!\!\!/_r^{\dagger}\partial\!\!\!/)2\gamma_5\beta(x)]\ , \qquad (A3)$$

where $K_0(x,x;D\!\!\!/_r^{\dagger}\partial\!\!\!/)$ is the analytic continuation in $s=0$ of the continuous kernel $K_{-s}(x,y;D\!\!\!/_r^{\dagger}\partial\!\!\!/)_{x=y}$, belonging to the operator $(D\!\!\!/_r^{\dagger}\partial\!\!\!/)^{-s}$ which are well defined for $\mathrm{Re}(s)>1$. The quantity

$$\mathrm{Tr}[K_0(x,x;D\!\!\!/_r^{\dagger}\partial\!\!\!/)2\gamma_5\beta]$$

is found with the help of Seeley's coefficients:

$$\mathrm{Tr}[K_0(x,x;D\!\!\!/_r^{\dagger}\partial\!\!\!/)2\gamma_5\beta]$$
$$=\frac{i}{(2\pi)^2}\frac{1}{2}\int_{|\xi|=1}\int_0^{\infty}dt\,\mathrm{Tr}[b_{-4}(x,\xi,it)2\beta\gamma_5]\ . \qquad (A4)$$

The coefficient $b_{-4}(x,\xi,\lambda)$ can be obtained by the equations

$$b_{-m-l}(a_m-\lambda)+\Sigma\left[\frac{\partial}{\partial\xi}\right]^{\alpha}b_{-m-j}\left[-i\frac{\partial}{\partial xj}\right]^{\alpha}\frac{a_{m-k}}{\alpha!}=0,\quad l>0;\ j<l,j+k+|\alpha|=l\ , \qquad (A5)$$

where α is the multiindex defined by

$$D\!\!\!/_r^{\dagger}\partial\!\!\!/=\sum_{\alpha\le r}A_{\alpha}(x)D_{\alpha}$$

with

$$D_{\alpha}=\prod_{1}^{2}{}_j\left[-i\frac{\partial}{\partial xj}\right]^{\alpha j},\quad |\alpha|=\sum_1^2\alpha_j$$

and

$$a_{m-j}(x,\xi)=\sum_{|\alpha|=m-j} A_\alpha(x)\xi^\alpha \ , \tag{A6}$$

so that

$$\begin{aligned} &a_2=\xi^2 \ , \\ &a_1=-2e\mathcal{A}(1-r)\xi\!\!/-2e\mathcal{B}\xi\!\!/\gamma_5 \ , \\ &a_0=0 \ , \end{aligned} \tag{A7}$$

and

$$\begin{aligned} b_{-4}(x,\xi,\lambda)=&\frac{2\xi^\mu}{(\xi^2-\lambda)^3}ie\partial_\mu[2\mathcal{A}(1-r)\xi\!\!/+2\mathcal{B}\xi\!\!/\gamma_5] \\ &+\frac{e^2}{(\xi^2-\lambda)^3}[2\mathcal{A}(1-r)\xi\!\!/-2\mathcal{B}\xi\!\!/\gamma_5] \ . \end{aligned} \tag{A8}$$

The explicit calculation of (A4) using (A8) gives

$$\lim_{s\to 0}\mathrm{Tr}[(\not{D}_r^\dagger\not{\partial})^{-s}2\beta\gamma_5]=\frac{e^2}{\pi}\int d^2x\ \epsilon_{\mu\nu}F^{\mu\nu}\beta(x)(1-r) \ . \tag{A9}$$

The Jacobian turns out to be [(2.44) and (2.45)]

$$\begin{aligned} J[A_\mu]&=\exp\left[-\int_0^1 dr\frac{e^2}{\pi}\int d^2x\ \epsilon_{\mu\nu}F^{\mu\nu}\beta(x)(1-r)\right] \\ &=\exp\left[-\frac{e}{\pi}\int d^2x\ A_\nu\epsilon_{\nu\mu}\partial_\mu\beta(x)\right] \ . \end{aligned} \tag{A10}$$

Remembering that

$$A_\nu(x)=\partial_\nu\alpha(x)+\epsilon_{\nu\rho}\partial_\rho\beta(x) \ ,$$

we immediately find

$$J[A_\mu]=\exp\left[-\frac{e^2}{\pi}\int d^2x\ A_\mu\left[\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\partial^2}\right]A_\nu\right] \ . \tag{A11}$$

APPENDIX B

In this appendix we study the behavior of the quantity

$$\Theta_n(\epsilon)=\sum_{q=0}^\infty \theta_n^{(q)}e^{-\epsilon q} \tag{B1}$$

of Eq. (4.6). To this purpose we introduce the generating function

$$\begin{aligned} G_\epsilon(w_1,w_2)&=\sum_{\substack{q=0\\n=0}}^\infty\frac{w_1^n w_2^q}{n!q!}\theta_n^{(q)}e^{-\epsilon q} \\ &=G_0(w_1,w_2e^{-\epsilon}) \ . \end{aligned} \tag{B2}$$

Obviously,

$$\int_0^\infty e^{-w_2}dw_2G_\epsilon(w_1,w_2)=\sum_{n=0}^\infty\frac{w_1^n}{n!}\Theta_n(\epsilon) \ . \tag{B3}$$

Now, from Eq. (3.15), we get

$$\begin{aligned} \theta_n^{(q)}=&\int dz\ d\bar z\ \theta(z,\bar z)\frac{k}{2\pi}e^{-k|z|^2}n!q!(k|z|^2)^{n+q} \\ &\times\sum_h\frac{(-k|z|^2)^{-h}}{h!(q-h)!(n-h)!}\sum_j\frac{(-k|z|^2)^{-j}}{j!(q-j)!(n-j)!} \ . \end{aligned} \tag{B4}$$

Then, one obtains[17]

$$\begin{aligned} G_\epsilon(w_1,w_2)=&\frac{k}{\pi}\int dz\ d\bar z\ e^{-k|z|^2}\theta(z,\bar z) \\ &\times\sum_{q=0}^\infty(-1)^qI_q(2(|k|z|^2w_1)^{1/2}) \\ &\times I_q(2(k|z|^2w_2e^{-\epsilon})^{1/2}) \\ &\times I_q(2(w_1w_2e^{-\epsilon})^{1/2}) \ , \end{aligned} \tag{B5}$$

I_q being the modified Bessel function of order q. Equation (B3) now gives[18,19]

$$\begin{aligned} \sum_{n=0}^\infty\frac{w_1^n}{n!}\Theta_n(\epsilon)=&\frac{k}{2\pi}e^{w_1e^{-\epsilon}} \\ &\times\int dz\ d\bar z\ \theta(z,\bar z)e^{-k|z|^2(1-e^{-\epsilon})} \\ &\times I_0(2(k|z|^2w_1)^{1/2}(1-e^{-\epsilon})) \ . \end{aligned} \tag{B6}$$

From Eq. (B6) one immediately sees that

$$\Theta_n(0)=\frac{k}{2\pi}\int dz\ d\bar z\ \theta(z,\bar z)=\Theta \ , \tag{B7}$$

independent of n and that, for a suitable $\chi<1$,

$$\Theta_n(\epsilon)<\chi^n(\epsilon)\Theta \ , \tag{B8}$$

if $\theta(z,\bar z)$ is any real positive measure, as we wanted to prove.

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