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COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review**. *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on "Time in quantum gravity: An hypothesis"

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We examine the solution of the issue of time in quantum gravity proposed recently by Rovelli. We find that the procedure does not work, at least in the presented form.

In recent papers, [1,2], Rovelli introduces very interesting new quantities, the so-called evolving constants. The existence of the evolving constants surely resolves some old problems (such as that of Bergmann's "frozen dynamics"). He proposes to use these variables to resolve the "time problem" of quantum gravity as well (see Refs. [1] and [2]).

The idea looks very promising, and one is seduced to play with these new quantities. However, one is quickly involved in difficulties, which can be traced to the following two points.

(1) To construct the evolving constants, one first has to solve the full system of classical equations of motion prior to quantization. Thus, the method is even more unrealistic than the usual reduction method, where one is to solve just the constraints prior to quantization. This problem is, however, at least a well-defined mathematical problem.

(2) Second, one has to order the factors in the classical evolving constants to finish the construction of the quantum operators. This is not even a well-defined mathematical problem. In fact, the factor ordering, in a broader sense, in the existing quantum theories is always based on some choice of time (such as normal ordering, see, e.g., [3]).

An error in Ref. [1] shows how difficult it is to find a reasonable factor ordering of the evolving constants. In Ref. [1] a model presymplectic system with the super Hamiltonian

$$K=\tfrac{1}{2}(p_1^2+p_2^2+q_1^2+q_2^2)-M \qquad \text{(I13)}$$

is used as an example of a non-Hamiltonian system (see also [4]). (We will refer to the equations by their reference numbers in Refs. [1] or [2] by adding just I or II, respectively, in front of the number). One easily finds the general solution of the classical equations of motion

$$q_1=(2A)^{1/2}\sin(\tau)\ ,\quad q_2=(2M-2A)^{1/2}\sin(\tau+\varphi)\ . \qquad \text{(I16)}$$

Here, τ is a physically irrelevant parameter, A and φ are constants of motion. Giving the values of A and φ fixes the classical trajectory uniquely. If one eliminates the parameter τ from (I16), one obtains

$$q_2=(M/A-1)^{1/2}[q_1\cos\varphi+(2A-q_1^2)^{1/2}\sin\varphi]\ . \qquad \text{(I18)}$$

The one-parameter set of quantities

$$Q(T)=(M/A-1)^{1/2}[T\cos\varphi+(2A-T^2)^{1/2}\sin\varphi]$$

have the following properties: (i) it describes the dependence of the variable q_2 on q_1 along the trajectory given by A and φ, and (ii) it consists of constants of motion (for each T one constant). This is an example of an "evolving constant." The following factor ordering is proposed for the quantity $\hat{Q}(T)$ in the quantum theory

$$\hat{Q}(T)=f(\hat{L}_z)[\hat{L}_yT+g(\hat{L}_z,T)\hat{L}_xg(\hat{L}_z,T)]f(\hat{L}_z)\ , \qquad \text{(I31)}$$

where f and g are functions defined by

$$f(x)=(\tfrac{1}{2}M-x)^{1/2}\ ,$$

$$g(x)=(M+2x-x^2)^{1/2}\ ,$$

the operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ are the well-known generators of the group SU(2) in the representation defined by its total angular momentum number j and

$$M^2=4j(j+1)$$

(see Ref. [1]). For the analysis of Ref. [1], the diagonalizability of the matrix $\hat{Q}(T)$ is important. $\hat{Q}(T)$ is, however, not a normal matrix, and so it is not diagonalizable. The calculation of the corresponding commutator yields

$$\begin{aligned}\langle m|[\hat{Q}(T),\hat{Q}(T)^\dagger]|m+2\rangle &= -4(1-\mathrm{i})L(m)L(m+1)g^{-1}(m,0)g^{-2}(m+1,0)\\ &\quad\times g^{-1}(m+2)G(m+1,T)\\ &\quad\times g^{1/2}(m,T)g^{1/2}(m+1,T)\neq 0\ ,\end{aligned}$$

where m is such that $g(m+1,T)$ is real and $g(m,T)$ is purely imaginary,

$$G(m+1,T)=\mathrm{i}T+g^{1/2}(m,T)g^{1/2}(m+1,T)$$

and

$$L(m)=\frac{1}{2}[j(j+1)-m(m+1)]^{1/2} .$$

Thus, the quite naturally looking factor order (I31) does not work; it is difficult now to decide what the proper ordering is to be.

Some statements and equations in Ref. [2] could be understood, at first glance at least, as an attempt to avoid the construction described in points (1) and (2), and to define the classical or quantum evolving constants in a direct way. In particular, the classical equations

$$\{Q(T;q_n,p_n),K(q_n,p_n)\}=0 , \tag{II10}$$

$$Q(T;T,q_2,\ldots,q_N,p_1,\ldots,p_N)=q_2 \tag{II11}$$

are proposed as definition equations for the evolving constant $Q(T)$. These equations imply that

$$\frac{\partial Q}{\partial T}\{q_1,K\}=\{q_2,K\} . \tag{II17}$$

(II17) is proposed as the propagation equation for Q in T. However, (II10) and (II11) *alone* and in the form they stand do not define the quantity $Q(T)$ properly. We observe that then the Poisson brackets $\{q_1,K\}$ and $\{q_2,K\}$ do not contain T so that (II17) can be integrated at once:

$$Q(T;q_n,p_n)=\frac{\{q_2,K\}}{\{q_1,K\}}T+Q^0(q_n,p_n) . \tag{1}$$

However, (1) is true only if the trajectories are straight lines [as in the example, Eq. (II12)].

In fact, Eq. (II17) is a transcription of the true relation

$$dq_2/dq_1=\dot{q}_2/\dot{q}_1 ,$$

where

$$\dot{q}_n=\alpha\{q_n,K\}$$

is the derivative of the solution $q_n(\tau)$ with respect to the parameter τ (α is the lapse function). Equation (II17) can, therefore, be useful for the definition of the quantity $Q(T)$ only if one prescribes the full system:

$$\frac{\partial Q_n}{\partial q_1}=\left[\frac{\{q_n,K\}}{\{q_1,K\}}\right]_{q=Q,p=P} , \quad n=2,\ldots,N , \tag{2}$$

$$\frac{\partial P_n}{\partial q_1}=\left[\frac{\{p_n,K\}}{\{q_1,K\}}\right]_{q=Q,p=P} , \quad n=1,\ldots,N , \tag{3}$$

of coupled classical equations of motion, with q_1 chosen as parameter. Thus, the point (1) cannot be circumvented.

The following equations are proposed in Ref. [2] as the quantum version of Eqs. (II10) and (II11):

$$[\hat{Q}(T),\hat{K}]=0 , \tag{II29}$$

$$\hat{Q}(\hat{q}_1)=\hat{q}_2 . \tag{II30}$$

It is very difficult to give any sense to Eq. (II30). First, within the Rovelli method of quantization as described in Ref. [1], only those classical quantities can become operators that have weakly vanishing Poisson brackets with the super-Hamiltonian. Then (II28) is trivial and $\hat{q}_1$ and $\hat{q}_2$ are not well defined. Rovelli's method requires, therefore, the two steps (1) and (2). Second, within the usual Dirac method, $\hat{q}_1$ and $\hat{q}_2$ are operators, but not self-adjoint ones (the space on which they are defined is not a Hilbert space). Still, what can (II30) mean? We can try to give it a sense as follows.

Let Ψ_q be an eigenstate of the operator $\hat{q}_1$ (we disregard the fact that $\hat{q}_1$ is not self-adjoint):

$$\hat{q}_1\Psi_q=q\Psi_q .$$

Then, instead of (II30), we write

$$\hat{Q}(q)\Psi_q=\hat{q}_2\Psi_q . \tag{4}$$

It is not clear if Eqs. (II29) and (4) can be used to define $\hat{Q}(T)$. As we have seen, the classical analogs of them could not. In any case, the quantum analogy of Eq. (II17),

$$\frac{\partial\hat{Q}}{\partial T}[\hat{q}_1,\hat{K}]=[\hat{q}_2,\hat{K}] , \tag{II32}$$

does not follow from (II29) and (4). One can just postulate Eq. (II32) as it stands. Suppose that a mathematical prescription based on (II32) could be found which would define quantities such as $\hat{Q}(T)$ uniquely. Then, in general, such $\hat{Q}(T)$'s were unlikely to satisfy (II29). Rovelli apparently considers the Eqs. (II29), (II30), and (II32) as tentative and only shows how the scheme works for Hamiltonian systems with

$$K=p_1+H(q_1,\ldots,q_N,p_2,\ldots,p_N) .$$

References [1] and [2] do not show how one can avoid steps (1) and (2) in constructing the evolving constant operators. Thus, the *time* problem is not solved, but only transformed into the *factor-ordering* problem. The equivalence of the two problems is more or less well known [3].

[1] C. Rovelli, Phys. Rev. D **42**, 2638 (1990).
[2] C. Rovelli, Phys. Rev. D **43**, 442 (1991).
[3] K. V. Kuchar, in *Highlights of Gravitation and Cosmology*, Proceedings of the Conference, Goa, India, 1987, edited by B. R. Iyer *et al.* (Cambridge University Press, New York, 1989).
[4] P. Hajicek, Class. Quantum Grav. **7**, 871 (1990).