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Energy loss of a heavy fermion in a hot QED plasma

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The energy loss dE/dx for a heavy lepton propagating through a high-temperature QED plasma is calculated to leading order in the QED coupling constant. The screening effects of the plasma are computed consistently using a resummation of perturbation theory in the small-momentum-transfer region. At large momentum transfer, recoil effects are properly taken into account. Our complete leading-order calculation differs significantly from previous calculations.

I. INTRODUCTION

One promising signature for the formation of a quark-gluon plasma in relativistic heavy-ion collisions is a change in the characteristics of the jets emitted in the collision. For example, the energy loss dE/dx for a jet propagating through a quark-gluon plasma could be rather different from one propagating through hadronic matter [1,2]. The asymptotic freedom of QCD suggests that for a quark-gluon plasma at sufficiently high temperature T, the energy loss dE/dx should be computable using perturbation theory in the running coupling constant $g_s(T)$. Unfortunately dE/dx cannot be computed by straightforwardly evaluating the lowest-order tree-level Feynman diagrams for scattering off of thermal quarks and gluons in the plasma because of infrared divergences due to the long-range interactions mediated by the gluon. These long-range interactions are screened in the plasma and in order to compute dE/dx to leading order in g_s, it is necessary to resum the thermal loop corrections which provide the screening.

It is an open question whether a perturbative calculation can provide a quantitative prediction for dE/dx at the temperatures achievable in heavy-ion collisions, which will not be much higher than the deconfinement transition temperature. There is reason for optimism, because perturbative predictions of static properties of the quark-gluon plasma are in reasonable accord with lattice-gauge-theory calculations, even at temperatures rather close to the phase transition [3]. Regardless of the answer, perturbation theory is valuable as a qualitative guide to the behavior of the plasma. While lattice gauge theory allows systematic calculations of the static properties of the plasma, it is completely impractical for measuring dynamical properties such as dE/dx. If supplemented by appropriate resummation techniques, perturbation theory can be used to systematically calculate dynamical as well as static properties at high temperature.

The first perturbative estimate of dE/dx in a quark-gluon plasma was made by Bjorken [1]. A high-energy quark or gluon loses energy by scattering off of thermal quarks and gluons in the plasma. In the tree-level calculation of dE/dx, gluon-exchange diagrams give rise to logarithmically infrared-divergent integrals over the momentum transfer q of the gluon. Bjorken estimated dE/dx by keeping only the logarithmically divergent integral over q and imposing physically reasonable upper and lower limits $q_{\max}$ and $q_{\min}$. Although Bjorken only discussed dE/dx for light quarks and gluons, the corresponding estimate for a heavy quark with velocity v in the high-temperature limit is

$$-\frac{dE}{dx} \simeq \frac{8\pi}{3}\left[1+\frac{n_f}{6}\right]\alpha_s^2 T^2 \left[\frac{1}{v}-\frac{1-v^2}{2v^2}\ln\frac{1+v}{1-v}\right]\ln\frac{q_{\max}}{q_{\min}}, \tag{1}$$

where $\alpha_s = g_s^2/4\pi$ and n_f is the number of light-quark flavors. In the case of a light quark with $v=1$, Bjorken's estimates for the upper and lower limits were $q_{\max}=\sqrt{4TE}$, where E is the energy of the quark, and $q_{\min}=\Lambda$, where Λ is some constant energy scale on the order of $0.5-1.0$ GeV.

A complete calculation of dE/dx to leading order in g_s should eliminate the ambiguities due to the choices of $q_{\min}$ and $q_{\max}$ in the approximation (1), with the upper and lower cutoffs on the logarithmic integral being provided automatically by the physics. A complete calculation is essential to make (1) into a quantitative estimate of dE/dx. While the dependence of $q_{\max}$ and $q_{\min}$ on the variables g_s, T, E, and v, could be determined by simple physical reasoning, it is certainly not trivial and indeed previous estimates have been in error. Furthermore the neglect in (1) of leading-order contributions other than those multiplied by the $\ln(q_{\max}/q_{\min})$ is only valid to the extent that they are made small by suitable choices of the multiplicative constants in $q_{\min}$ and $q_{\max}$. These constants can only be determined by a complete calculation of dE/dx to leading order in g_s.

A significant step in this direction was recently made by Thoma and Gyulassy [4], who calculated dE/dx for a heavy quark using plasma-physics techniques. They correctly included the plasma effects that screen the infrared divergences due to the long-range Coulomb and magnetic interactions, and thus obtained a result that was free of the ambiguity associated with the choice of the lower limit $q_{\min}$ in (1). However their result still suffered from an ambiguity associated with the choice of upper limit $q_{\max}$ in (1). Furthermore their calculation did not allow for recoil of the scattered thermal quark or gluon, which becomes important when the momentum transfer q becomes large. Thus while it was a complete and correct calculation of the soft q contribution to dE/dx, the calculation of Thoma and Gyulassy did not treat the hard momentum-transfer contribution correctly.

The resummation methods required to calculate dE/dx to leading order in g_s have only recently been developed. Their development was stimulated by the "plasmon problem," that one-loop calculations of the gluon damping rate give gauge-dependent answers. Braaten and Pisarski [5] resolved the problem by showing that thermal corrections from higher-loop diagrams contribute to the damping rate at leading order in g_s and must be resummed. They developed a method for carrying out the resummation, proved that the result was gauge invariant [5], and computed the damping rate explicitly to leading order in g_s (Ref. [6]). The resummation method was based on a distinction between "hard" momenta of order T and "soft" momenta of order $g_s T$. If a tree amplitude has soft external momenta of order $g_s T$, the one-loop thermal corrections proportional to T^2 contribute at the same order in g_s as the tree amplitude. It is these corrections that must be resummed. They are called "hard thermal loops," because they arise from integration regions where the loop momentum is hard. The resummation of the geometric series of hard thermal loop corrections to a propagator results in an effective propagator for soft particles. Braaten and Pisarski [5] developed a resummed perturbation expansion in which effective propagators are used for soft particles and tree-level propagators are used for hard particles. If all lines entering a vertex are soft, then it is replaced by an effective vertex which includes a hard thermal loop correction.

The effective propagator for soft gluons was calculated long ago by Klimov [7] and Weldon [8]. It screens the static Coulomb interaction, thus eliminating infrared divergences due to the long-range Coulomb force. The inverse of the electric screening length is $\sqrt{1+n_f/6}\, g_s T$. The purely static magnetic interaction is not screened, but there is screening at nonzero frequencies [8]. The inverse of the magnetic screening length behaves like $\omega^{1/3}$ as the frequency ω goes to zero. The approach to zero is sufficiently slow that the dynamical screening cuts off the infrared divergence due to the long-range magnetic interaction if the divergence is only logarithmic, as is the case for transport coefficients [9]. Thoma and Gyulassy showed that it is also true for dE/dx (Ref. [4]).

A new method for computing the effects of screening in a hot gauge theory has recently been developed by Braaten and Yuan [10]. It is applicable to any quantity for which the dynamical screening of the magnetic interaction cuts off the static magnetic divergence at the scale $g_s T$, and therefore can be applied to dE/dx. An arbitrary momentum scale q^* is introduced to separate the hard and soft regions of the momentum transfer q. The contribution from hard momentum transfer $q > q^*$ is computed using a tree-level propagator for the exchanged gluon, while the contribution from the soft region $q < q^*$ is computed using an effective gluon propagator. The dependence on the arbitrary scale q^* cancels upon adding the hard and soft contributions to get the complete result to leading order in g_s.

In this paper we apply the method of Braaten and Yuan [10] to compute to leading order in the QED coupling constant e the energy loss dE/dx for a heavy lepton propagating through a hot QED plasma of electrons, positrons, and photons. We focus on hot QED instead of hot QCD because it is simpler and the main purpose of the paper is to expose the calculational methods required to compute dE/dx. The calculation of the energy loss for a heavy-quark propagating through a hot quark-gluon plasma is a straightforward extension of the QED calculation. It will be presented elsewhere together with the phenomenological implications for heavy-ion collisions [11].

The outline of the paper is as follows. In Sec. II, the energy loss dE/dx is defined in terms of field-theoretical quantities which can be expressed perturbatively as sums of Feynman diagrams. In Sec. III, the contribution to dE/dx from the exchange of photons with hard momentum transfer q is calculated. A lower limit q^* on the momentum transfer is used to cut off the infrared divergences. In Sec. IV, the soft q contribution to dE/dx is computed using the imaginary-time formalism of thermal field theory. An effective photon propagator provides the screening that cuts off the infrared divergences at the scale eT. In Sec. V, an alternative method for calculating the soft contribution without using the imaginary-time formalism is presented. In Sec. VI, the hard and soft contributions are added to give the complete result for dE/dx to leading order in e. The additional calculations required to extend this result to the QCD plasma are described.

II. FIELD-THEORETIC DEFINITION OF dE/dx

Our first task is to express the energy loss dE/dx of a heavy fermion in terms of field-theoretic quantities. The calculation of dE/dx then reduces to analyzing and evaluating Feynman diagrams. We will present two formulas for dE/dx. The first formula expresses dE/dx in terms of a weighted integral of the differential interaction rate. It is used in Sec. III to compute the hard contribution to dE/dx. The second formula expresses dE/dx in terms of the imaginary part of the self-energy of the heavy fermion, which is a quantity that can be computed using the imaginary-time formalism. This formula is used in Sec. IV to compute the soft contribution to dE/dx.

We consider a high-energy muon (or any other heavy lepton) of mass M and momentum $\mathbf{p}$ propagating through a plasma of electrons, positrons, and photons in thermal equilibrium at a temperature T. The muon has energy E

and velocity $\mathbf{v}=\mathbf{p}/E$. We assume that $m_e \ll eT$ and set the electron mass m_e to zero. We also assume that $T \ll M,p$ and work to leading order in T/M and T/p. The muon loses energy by scattering off of thermal electrons, positrons, and photons. The interaction rate $\Gamma(E)$ for the muon can be expressed in terms of Feynman diagrams in the standard way. For example, the contribution to $\Gamma(E)$ from the process $e^-\mu \rightarrow e^-\mu$ is

$$\Gamma(E)=\frac{1}{2E}\int\frac{d^3p'}{(2\pi)^3 2E'}\int\frac{d^3k}{(2\pi)^3 2k}n_F(k)\int\frac{d^3k'}{(2\pi)^3 2k'}[1-n_F(k')](2\pi)^4\delta^4(P+K-P'-K')\tfrac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2\ , \quad (2)$$

where $P=(E,\mathbf{p})$ and $K=(k,\mathbf{k})$ are the four-momenta of the incoming muon and electron, respectively, while P' and K' are the momenta of the outgoing muon and electron. The phase space is weighted by a Fermi distribution $n_F(k)=(e^{k/T}+1)^{-1}$ for the incoming electron and a Pauli-blocking factor $1-n_F(k')$ for the outgoing electron. The matrix element $\mathcal{M}$ is given by a sum of Feynman diagrams, beginning with the tree-level diagram in Fig. 1. The square of $\mathcal{M}$ is summed over the spins of the electrons and the outgoing muon and averaged over the spin of the initial muon.

The average time between muon interactions is $1/\Gamma$, so the average distance travelled by the muon between interactions is $\Delta x = v/\Gamma$, where v is the velocity of the muon. The average energy lost by the muon per interaction is

$$\Delta E=\frac{1}{\Gamma}\int_M^\infty dE'(E-E')\frac{d\Gamma}{dE'}(E,E')\ , \quad (3)$$

where $d\Gamma/dE'$ is the differential interaction rate with respect to the final-state muon energy E'. The integral extends over all energies E', because there is a small probability that the muon will gain energy in the collision. The rate of energy loss dE/dx per distance traveled is the ratio of ΔE to Δx:

$$-\frac{dE}{dx}=\frac{1}{v}\int_M^\infty dE'(E-E')\frac{d\Gamma}{dE'}(E,E')\ . \quad (4)$$

For example, the contribution to $-dE/dx$ from the process $e^-\mu\rightarrow e^-\mu$ is given by (2) with $(E-E')/v$ inserted in the integrand.

The factor of $E-E'$ in (4) is essential in making dE/dx calculable with presently available methods for resumming perturbation theory. If the interaction rate Γ is calculated using the tree-level scattering diagram in Fig. 1, it has a quadratic infrared divergence. The extra factor of $E-E'$ makes the infrared divergence in dE/dx only logarithmic. The use of an effective propagator for the exchanged photon softens the divergences, so that Γ is only logarithmically divergent [12] while dE/dx becomes infrared finite [4]. Thus for dE/dx, infrared divergences are screened by plasma effects at the scale eT, and the resummation of hard thermal loops is sufficient to calculate it to leading order in e. For Γ, screening also involves the smaller energy scale e^2T, and its calculation to leading order in e requires more elaborate resummation methods [13].

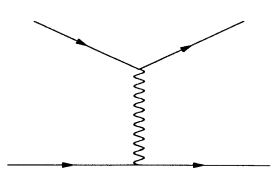

FIG. 1. Tree-level Feynman diagram for the scattering process $e^-\mu\rightarrow e^-\mu$.

The energy loss can also be expressed in terms of the muon self-energy $\Sigma(P)$. As shown by Weldon [14], the interaction rate averaged over the two spin states of the muon is

$$\Gamma(E)=-\frac{1}{2E}\frac{1}{1+e^{-E/T}}\sum_s \bar{u}(P,s)\mathrm{Im}\Sigma(E+i\epsilon,\mathbf{p})u(P,s)\ , \quad (5)$$

where $u(P,s)$ is the spinor for a muon with four-momentum $P=(E,\mathbf{p})$ and spin s. Representing the sum over the spin s by a Dirac trace, the interaction rate becomes

$$\Gamma(E)=-\frac{1}{2E}[1-n_F(E)]\mathrm{tr}[(P\cdot\gamma+M)\mathrm{Im}\Sigma(E+i\epsilon,\mathbf{p})]\ . \quad (6)$$

The imaginary part of $\Sigma(P)$ can be expressed as a sum of integrals over phase space weighted by statistical distributions [14]. The integrands are squares of amplitudes for processes of the form $X\mu\rightarrow X'\mu$, where X and X' are initial and final states containing one or more electrons, positrons, or photons. For each such term, the energy of the final-state muon E' can be identified, and $-dE/dx$ is obtained as in (4) by inserting $(E-E')/v$ inside the integrand in $\mathrm{Im}\Sigma$.

The advantage of the formula (6) for $\Gamma(E)$ is that $\Sigma(P)$ can be calculated using the imaginary-time formalism for thermal field theory. This is important, because the resummation methods required to compute the effects of screening have been developed using the imaginary-time formalism and have not yet been extended to the real-time formalism.

The formula (4) gives the energy loss of a heavy fermion. At relativistic temperatures, the energy loss of a light particle, such as a high-energy electron or photon, is not as easy to define. The problem is that there is a significant probability for a high-energy electron or photon to lose a large fraction of its energy without that energy being dispersed in the plasma. For example, a high-energy electron with four-momentum P can scatter off of a thermal electron, positron, or photon via the processes $e^- e^\pm \to e^- e^\pm$ and $e^-\gamma \to e^-\gamma$ and produce a pair of almost collinear particles with momenta $(1-x)P$ and xP. While the electron loses a fraction x of its energy and momentum, this energy and momentum is not dispersed into the plasma. Instead it is merely transferred to a collinear particle. In this ultrarelativistic situation, the concept of dE/dx for an individual light particle loses its usefulness. It is more appropriate to consider dE/dx for a jet, which is defined to be a collection of particles with collinear momenta. A scattering event in which a high-energy electron splits into a collinear pair of particles corresponds to evolution of the jet and not to a large loss of energy. The problem of defining dE/dx for the jet associated with a high-energy light particle and calculating it to leading order in the coupling constant is under investigation [11]. In this paper, we consider only the energy loss of a heavy particle where kinematics prevents splitting into collinear particles.

III. HARD CONTRIBUTION TO dE/dx

The method of calculating the effects of screening developed by Braaten and Yuan [10] involves introducing an arbitrary momentum scale q^* to separate the region of hard momentum transfer $q \sim T$ from the soft region $q \sim eT$. It should be chosen so that $eT \ll q^* \ll T$, which is possible in the weak-coupling limit $e \to 0$. The contribution from the hard region $q > q^*$ is calculated using tree-level scattering diagrams. The lower limit q^* acts as an infrared cutoff, so the result of integrating over hard q has the form $A_{\rm hard} + B \ln(T/q^*)$. The contribution from soft momentum transfer $q < q^*$ is computed using an effective photon propagator, which cuts off the logarithmic divergence at the scale eT. In order to match onto the hard contribution, the result of integrating over soft q must have the form $B \ln(q^*/eT) + A_{\rm soft}$. The dependence on the arbitrary scale q^* cancels between the soft and hard contributions, and the complete result to leading order in e is $A_{\rm hard} + B\ln(1/e) + A_{\rm soft}$. The logarithm of $1/e$ is simply a reflection of the fact that this quantity receives contributions from all momentum scales from T down to eT.

In this section, we calculate the hard contribution to dE/dx. At temperatures T much smaller than the muon mass M, the only processes that contribute to dE/dx at leading order in e are scattering from electrons and positrons: $e^\pm\mu \to e^\pm\mu$. The only other process that is not obviously of higher order in e is Compton scattering: $\gamma\mu \to \gamma\mu$. The tree-level Feynman diagrams for $\gamma\mu \to \gamma\mu$ are shown in Fig. 2, and the matrix element is

$$i\mathcal{M} = -ie^2 \epsilon_\mu(K',\lambda')^* \epsilon_\nu(K,\lambda) \bar{u}(P',s') \times \left[\frac{1}{2P\cdot K}\gamma^\mu[(P+K)\cdot\gamma + M]\gamma^\nu - \frac{1}{2P'\cdot K}\gamma^\nu[(P'-K)\cdot\gamma + M]\gamma^\mu \right] u(P,s) , \tag{7}$$

where P, s, K, and λ are the four-momenta and spins of the incoming muon and photon, and the corresponding primed variables refer to the outgoing muon and electron. Since the incoming photon is thermal, factors of K in the numerator give rise to terms that are suppressed by T/M and can be dropped. Using the Dirac equation $(P\cdot\gamma - M)u(P,s) = 0$, the matrix element in (7) reduces to

$$i\mathcal{M} = -ie^2 \epsilon_\mu(K',\lambda')^* \epsilon_\nu(K,\lambda) \times \left[\frac{P^\mu}{P\cdot K} - \frac{P'^\mu}{P'\cdot K} \right] \bar{u}(P',s')\gamma^\nu u(P,s) . \tag{8}$$

Since $P' \simeq P$ to leading order in T/M, there is a cancellation between the two diagrams in Fig. 2. Thus the contribution to dE/dx from Compton scattering is suppressed by $(T/M)^2$.

The calculation of the hard contribution to dE/dx from $e^\pm\mu \to e^\pm\mu$ is straightforward. It is given by inserting $(E-E')/v$ inside the integral in (2) and multiplying by 2 to take into account scattering of e^+ as well as e^-:

$$\left[-\frac{dE}{dx}\right]_{\rm hard} = \frac{1}{E}\int \frac{d^3p'}{(2\pi)^3 2E'} \int \frac{d^3k}{(2\pi)^3 2k} n_F(k) \int \frac{d^3k'}{(2\pi)^3 2k'} [1 - n_F(k')] \times (2\pi)^4 \delta^4(P+K-P'-K') \tfrac{1}{2} \sum_{\rm spins} |\mathcal{M}|^2 \frac{\omega}{v} \theta(q - q^*) , \tag{9}$$

where $\omega = E - E'$ and $\mathbf{q} = \mathbf{p} - \mathbf{p}'$ are the energy and momentum of the virtual photon. The θ function imposes the restriction to the region of hard momentum transfer $q > q^*$. The differential interaction rate $d\Gamma/d^3q$ is an observable and hence gauge invariant. As a consequence, the separation of Γ or dE/dx into contributions from soft and hard momentum transfer q is also gauge invariant. Thus each contribution can be calculated in any convenient gauge. In this section, we use the covariant Feynman gauge to calculate the hard contribution to dE/dx. In Sec. IV, the Coulomb gauge will be used to calculate the soft contribution.

The matrix element $\mathcal{M}$ in (9) is given by the tree diagram in Fig. 1. In the Feynman gauge, the matrix ele-

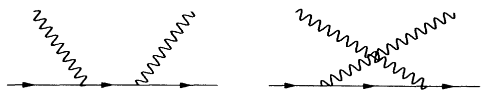

FIG. 2. Tree-level Feynman diagrams for the scattering process $\gamma\mu \to \gamma\mu$.

ment is

$$i\mathcal{M} = -\frac{e^2}{Q^2}\bar{u}(P',s')\gamma^\mu u(P,s)\bar{u}(K',\lambda')\gamma_\mu u(K,\lambda) , \tag{10}$$

where $Q=(\omega,\mathbf{q})$ and $Q^2=\omega^2-q^2$. Squaring the matrix element, averaging over the spin s of the incoming muon, and summing over the spins s', λ, and λ' of the other three particles, we get

$$\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 = 16\frac{e^4}{Q^4}(P\cdot KP'\cdot K' + P\cdot K'P'\cdot K - M^2 K\cdot K') . \tag{11}$$

The muon is assumed to have large momentum $p \gg T$ as well as large mass $M \gg T$. The Fermi distribution $n_F(k)$ in (9) restricts the energy k to be at most of order T. Conservation of energy and momentum, together with the condition $p, M \gg T$ then constrains the energy k' to also be on the order of T. The difference between the initial and final quark velocities $\mathbf{v}$ and $\mathbf{v}'$ is therefore of order T/E. Assuming $v \gg T/E$ and setting $\mathbf{v}'=\mathbf{v}$, the expression in (11) reduces to

$$\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 = 16\frac{e^4}{Q^4}EE'\left[2(k-\mathbf{v}\cdot\mathbf{k})(k'-\mathbf{v}\cdot\mathbf{k}') + \frac{1-v^2}{2}(\omega^2-q^2)\right] , \tag{12}$$

where $\mathbf{v}=\mathbf{p}/E$ is the velocity of the muon. The momentum δ function can be used to compute the integral over $\mathbf{p}'$. Inserting the approximation $E' \simeq E - \mathbf{v}\cdot\mathbf{q}$, the energy δ function reduces to $\delta(\omega-\mathbf{v}\cdot\mathbf{q})$. The expression (9) for dE/dx has now been reduced to

$$\left[-\frac{dE}{dx}\right]_{\text{hard}} = \frac{4\pi e^4}{v}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k}n_F(k)\int\frac{d^3k'}{(2\pi)^3}\frac{1}{k'}[1-n_F(k')]\delta(\omega-\mathbf{v}\cdot\mathbf{q})\frac{\omega}{(\omega^2-q^2)^2} \times\left[2(k-\mathbf{v}\cdot\mathbf{k})(k'-\mathbf{v}\cdot\mathbf{k}') + \frac{1-v^2}{2}(\omega^2-q^2)\right]\theta(q-q^*) , \tag{13}$$

where $\omega=k'-k$ and $\mathbf{q}=\mathbf{k}'-\mathbf{k}$. In the Pauli-blocking factor $1-n_F(k')$, the term $n_F(k')$ can be dropped because the corresponding term in the integrand is odd under the interchange of $\mathbf{k}$ and $\mathbf{k}'$ and integrates to zero. This simplification makes it possible to evaluate the integrals in (13) analytically.

In order to simplify the expression (13) further, it is convenient to make use of the fact that dE/dx is independent of the direction of $\mathbf{v}$. We can therefore average the integrand over the directions of $\mathbf{v}$. The integrals that are required are

$$\int\frac{d\Omega}{4\pi}\delta(\omega-\mathbf{v}\cdot\mathbf{q}) = \frac{1}{2vq}\theta(v^2q^2-\omega^2) , \tag{14}$$

$$\int\frac{d\Omega}{4\pi}\delta(\omega-\mathbf{v}\cdot\mathbf{q})v^i = \frac{1}{2vq}\theta(v^2q^2-\omega^2)\frac{\omega}{q}\hat{q}^i , \tag{15}$$

$$\int\frac{d\Omega}{4\pi}\delta(\omega-\mathbf{v}\cdot\mathbf{q})v^iv^j = \frac{1}{2vq}\theta(v^2q^2-\omega^2)\left[\frac{v^2q^2-\omega^2}{2q^2}\delta^{ij} + \frac{3\omega^2-v^2q^2}{2q^2}\hat{q}^i\hat{q}^j\right] , \tag{16}$$

where $\int d\Omega$ represents integration over the angles of $\mathbf{v}$. At this point, it is convenient to change the remaining integration variables from k, k', and $\cos\theta=\hat{k}\cdot\hat{k}'$ to k, q, and ω. The formula (13) for dE/dx reduces to

$$\left[-\frac{dE}{dx}\right]_{\text{hard}} = \frac{e^4}{4\pi^3v^2}\int_0^\infty dk\, n_F(k)\left[\int_{q^*}^{2k/(1+v)}\frac{dq}{q^2}\int_{-vq}^{+vq}d\omega\,\omega + \int_{2k/(1+v)}^{2k/(1-v)}\frac{dq}{q^2}\int_{q-2k}^{+vq}d\omega\,\omega\right] \times\left[\frac{3\omega^2}{4q^2} - \frac{v^2}{4} - \frac{1-v^2}{2}\frac{q^2}{q^2-\omega^2} + 3\frac{k(k+\omega)}{q^2} - (1-v^2)\frac{k(k+\omega)}{q^2-\omega^2}\right] . \tag{17}$$

The limits of integration arise from taking into account the restriction $-vq<\omega<+vq$ from the θ functions in the integrals (14)–(16), the restriction to hard momentum transfer $q>q^*$, and $\omega>q-2k$ which follows from $\omega=k'-k$ and $q=|\mathbf{k}'-\mathbf{k}|$. The integrals over ω and q can be evaluated analytically. The only integral which is not elementary is

$$\int_{2k/(1+v)}^{2k/(1-v)} dq\frac{1}{q}\ln\frac{k}{q-k} = \mathrm{Sp}\left[\frac{1+v}{2}\right] - \mathrm{Sp}\left[\frac{1-v}{2}\right] + \frac{1}{2}\ln\frac{1+v}{1-v}\ln\frac{1-v^2}{4} , \tag{18}$$

where $\mathrm{Sp}(x)$ is the dilogarithm or Spence function:

$$\mathrm{Sp}(x) = -\int_0^x dt \frac{1}{t} \ln(1-t) \; . \tag{19}$$

The remaining integrals over k can also be evaluated analytically:

$$\int_0^\infty dk\, n_F(k) k = \frac{\pi^2 T^2}{12} \; , \tag{20}$$

$$\int_0^\infty dk\, n_F(k) k \ln \frac{k}{q^*} = \frac{\pi^2 T^2}{12} \left[\ln \frac{2T}{q^*} + 1 - \gamma + \frac{\zeta'(2)}{\zeta(2)} \right] , \tag{21}$$

where $\gamma \simeq 0.577\,22$ is Euler's constant and $\zeta(z)$ is Riemann's zeta function with $\zeta'(2)/\zeta(2) \simeq -0.569\,96$. The final result for the contribution to dE/dx from hard momentum transfer is

$$\left[-\frac{dE}{dx} \right]_{\rm hard} = \frac{e^4 T^2}{24\pi} \left\{ \left[\frac{1}{v} - \frac{1-v^2}{2v^2} \ln \frac{1+v}{1-v} \right] \left[\ln \frac{4T}{q^*} - \frac{1}{2} \ln(1-v^2) + \frac{3}{2} - \gamma + \frac{\zeta'(2)}{\zeta(2)} \right] \right.$$
$$\left. - \frac{1-v^2}{4v^2} \left[\mathrm{Sp}\left[\frac{1+v}{2} \right] - \mathrm{Sp}\left[\frac{1-v}{2} \right] + \frac{1}{2} \ln \frac{1+v}{1-v} \ln \frac{1-v^2}{4} \right] - \frac{2}{3} v \right\} . \tag{22}$$

A logarithmic infrared divergence appears as $q^* \to 0$. A calculation in the Coulomb gauge would reveal that the divergence receives contributions both from Coulomb scattering and from magnetic scattering. The Coulomb contribution to the expression in square brackets multiplying $\ln(4T/q^*)$ is $2v/3$, and the remainder comes from the magnetic interaction.

The hard contribution to dE/dx can be written more compactly in the form

$$\left[-\frac{dE}{dx} \right]_{\rm hard} = \frac{e^4 T^2}{24\pi} \left[\frac{1}{v} - \frac{1-v^2}{2v^2} \ln \frac{1+v}{1-v} \right] \left[\ln \frac{T}{q^*} + \ln \frac{E}{M} + A_{\rm hard}(v) \right] , \tag{23}$$

where we have used $E/M = 1/\sqrt{1-v^2}$. The function $A_{\rm hard}(v)$ decreases monotonically as a function of the velocity from 1.239 at $v=0$ to 1.072 at $v=1$. If we wish to approximate (23) by replacing the expression in parentheses by a simple logarithm $\ln(q_{\max}/q^*)$ as in (1), the value of $q_{\max}$ that minimizes the maximum error over the entire range $0<v<1$ is $q_{\max} = 3.2T/\sqrt{1-v^2}$. This value of $q_{\max}$ has a dependence on v that differs significantly from the estimate $q_{\max} = 4Tp/(E-p+4T) \simeq 4Tv/(1-v)$ of Thoma and Gyulassy [4], which was based on a consideration of the maximum energy transferred in the scattering process.

IV. SOFT CONTRIBUTION TO dE/dx

The soft contribution to dE/dx has already been calculated correctly by Thoma and Gyulassy, using a method borrowed from plasma physics [4]. To use their result in our calculation, it is necessary only to impose an upper limit q^* on the integral over the momentum transfer in their expression for dE/dx. In this section, we verify their result by repeating the calculation using the imaginary-time formalism for thermal field theory. The advantage of this approach is that the resummation methods required to calculate the soft contribution [5] have only been developed systematically within the imaginary-time formalism. For example, the power-counting methods of Ref. [5] can be used to verify that our calculation does indeed include all terms of leading order in e. The disadvantage is that the imaginary-time formalism involves calculation in Euclidean space followed by analytic continuation to Minkowski space, which is not very conducive to intuitive interpretation. This will be remedied in Sec. V, where an alternative field-theoretic calculation, which is more intuitive but less rigorous, will be presented.

To calculate dE/dx using the imaginary-time formalism, we begin with Weldon's formula (6), which expresses the interaction rate $\Gamma(E)$ in terms of the muon self-energy $\Sigma(P)$. The hard contribution calculated in Sec. III comes from the two-loop self-energy diagram in Fig. 3. The imaginary part of Σ comes from cutting the diagram through the virual muon line and the two electron lines, thus producing the square of the diagram in Fig. 1. In the integration region where the momentum $\mathbf{q}$ flowing through the photon line is soft, hard thermal loop corrections to the photon propagator contribute at leading order in e and must be resummed. The resulting diagram is shown in Fig. 4, where the blob on the soft photon line represents an effective photon propagator $\Delta^{\mu\nu}(Q)$. The effective propagator is obtained by summing the

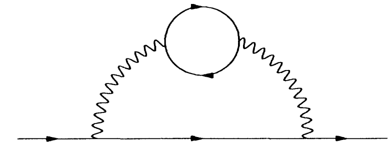

FIG. 3. Two-loop Feynman diagram for the muon self-energy $\Sigma(P)$.

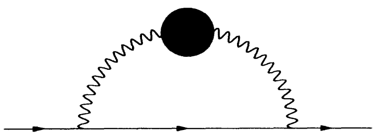

FIG. 4. Feynman diagram for the muon self-energy $\Sigma(P)$ with effective photon propagator.

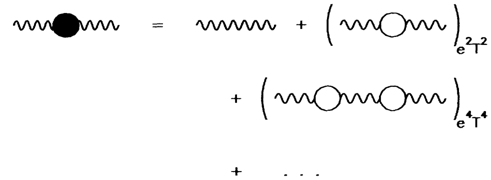

FIG. 5. Diagrammatic definition of the effective photon propagator.

geometric series of one-loop self-energy corrections proportional to e^2T^2, as shown diagrammatically in Fig. 5. The diagram in Fig. 4 includes all loop corrections that contribute to the imaginary part of $\Sigma(P)$ at leading order in e, as can be verified using the power-counting rules developed in Ref. [5].

In Minkowski space, the Feynman rules for the self-energy diagram in Fig. 4 give

$$\Sigma(P)=ie^2\int\frac{d^4Q}{(2\pi)^4}\Delta^{\mu\nu}(Q)\gamma_\mu\frac{1}{(P-Q)\cdot\gamma-M}\gamma_\nu\ . \tag{24}$$

To compute the thermal contributions in the imaginary-time formalism, we make the replacement $\int dq_0/2\pi \to iT\sum_{q_0}$, where the sum is over the discrete imaginary values $q_0=i2n\pi T$ for the photon energy. The sum over q_0 is evaluated for discrete imaginary values $p_0=i(2n+1)\pi T$ of the muon energy, and only then is p_0 analytically continued to the real Minkowski energy $p_0=E+i\epsilon$ required in (6).

The hard and soft contributions to dE/dx are separately gauge invariant. The most convenient gauge for evaluating the soft contribution is Coulomb gauge, where the only nonzero components of the effective propagator are

$$\Delta^{00}(Q)=\Delta_l(q_0,q)\ , \tag{25}$$

$$\Delta^{ij}(Q)=\Delta_t(q_0,q)(\delta^{ij}-\hat{q}^i\hat{q}^j)\ . \tag{26}$$

The four-momentum of the photon is $Q=(q_0,\mathbf{q})$ and $\hat{\mathbf{q}}=\mathbf{q}/q$. The longitudinal and transverse effective propagators are [7]

$$\Delta_l(\omega,q)^{-1}=q^2-\frac{3}{2}m_\gamma^2\left[\frac{\omega}{q}\ln\frac{\omega+q}{\omega-q}-2\right]\ , \tag{27}$$

$$\Delta_t(\omega,q)^{-1}=\omega^2-q^2+\frac{3}{2}m_\gamma^2\left[\frac{\omega(\omega^2-q^2)}{2q^3}\ln\frac{\omega+q}{\omega-q}-\frac{\omega^2}{q^2}\right]\ . \tag{28}$$

Inserting (25) and (26) into the self-energy (24), the trace required in (6) becomes

$$\mathrm{tr}[(P\cdot\gamma+M)\Sigma(P)]=-4e^2T\sum_{q_0}\int\frac{d^3q}{(2\pi)^3}\frac{1}{(P-Q)^2-M^2}\{\Delta_l(Q)(p_0^2+p^2-p_0q_0-\mathbf{p}\cdot\mathbf{q}+M^2)+2\Delta_t(q)[p_0^2-p_0q_0+\mathbf{p}\cdot\mathbf{q}-(\mathbf{p}\cdot\hat{\mathbf{q}})^2-M^2]\}\ . \tag{29}$$

The easiest way to compute the sum over q_0 is to introduce spectral representations for the propagators. For the muon propagator, the spectral representation is

$$\frac{1}{(P-Q)^2-M^2}=-\frac{1}{2E'}\int_0^{1/T}d\tau' e^{(p_0-q_0)\tau'}\{[1-n_F(E')]e^{-E'\tau'}-n_F(E')e^{+E'\tau'}\}\ , \tag{30}$$

where $E'=\sqrt{M^2+(\mathbf{p}-\mathbf{q})^2}$. For the effective propagators in (27) and (28), the spectral representations are

$$\Delta_l(q)=-\int_0^{1/T}d\tau\, e^{q_0\tau}\int_{-\infty}^{+\infty}d\omega\,\rho_l(\omega,q)[1+n_B(\omega)]e^{-\omega\tau}\ , \tag{31}$$

$$\Delta_t(Q)=-\int_0^{1/T}d\tau\, e^{q_0\tau}\int_{-\infty}^{+\infty}d\omega\,\rho_t(\omega,q)[1+n_B(\omega)]e^{-\omega\tau}\ , \tag{32}$$

where $n_B(\omega)=(e^{\omega/T}-1)^{-1}$ is the Bose distribution. The spectral functions are defined by $\rho_l(\omega,q)=-\mathrm{Im}\Delta_l(\omega+i\epsilon,\mathbf{q})/\pi$ and $\rho_t(\omega,q)=-\mathrm{Im}\Delta_t(\omega+i\epsilon,\mathbf{q})/\pi$, and are given explicitly in Ref. [15]. They are odd functions of ω and we define them to be positive for positive ω. Their support consists of the spacelike interval $-q<\omega<+q$, together with δ-function contributions at the timelike points $\omega=\pm\omega_l(q)$ for ρ_l and $\omega=\pm\omega_t(q)$ for ρ_t. The functions $\omega_l(q)$ and $\omega_t(q)$ are the dispersion relations for longitudinal photons, or plasmons, and transverse photons, respectively. For spacelike frequencies $|\omega|<q$, the spectral functions are proportional to the squares of the effective propagators in (27) and (28):

$$\rho_l(\omega,q)=\frac{3m_\gamma^2\omega}{2q}|\Delta_l(\omega+i\epsilon,q)|^2\ , \tag{33}$$

$$\rho_t(\omega,q)=\frac{3m_\gamma^2\omega(q^2-\omega^2)}{4q^3}|\Delta_t(\omega+i\epsilon,q)|^2\ . \tag{34}$$

After introducing the spectral representations, the sum over q_0 reduces to the δ function $\delta(\tau-\tau')$ or its derivative. After using the δ function to integrate over τ', the integral over τ yields an energy denominator

$$\int_0^{1/T} d\tau\, e^{(p_0\mp E'-\omega)\tau}=(e^{(p_0\mp E'-\omega)/T}-1)\frac{1}{p_0\mp E'-\omega}\ . \tag{35}$$

At the discrete imaginary energies $p_0=i(2n+1)\pi T$, we have $e^{p_0/T}=-1$. Only after using this identity to eliminate the exponential of p_0 can the muon energy be analytically continued to the value $p_0=E+i\epsilon$ required in (6). The imaginary part of the self-energy $\Sigma(P)$ in (6) comes from the imaginary part of the energy denominator in (35):

$$\mathrm{Im}\frac{1}{(E+i\epsilon)\mp E'-\omega}=-i\pi\delta(E\mp E'-\omega)\ . \tag{36}$$

The imaginary part of the trace in (29) now reduces to

$$\begin{aligned}\mathrm{tr}[(P\cdot\gamma+M)\mathrm{Im}\Sigma(P)]\\ =-4\pi e^2(1+e^{-E/T})\int\frac{d^3q}{(2\pi)^3}\int_{-\infty}^{+\infty}d\omega[1+n_B(\omega)]\frac{1}{2E'}\{[1-n_F(E')]\delta(E-E'-\omega)-n_F(E')\delta(E+E'-\omega)\}\\ \times\{\rho_l(\omega,q)(2E^2-E\omega-\mathbf{p}\cdot\mathbf{q})+2\rho_t(\omega,q)[p^2-E\omega+\mathbf{p}\cdot\mathbf{q}-(\mathbf{p}\cdot\hat{\mathbf{q}})^2]\}\ .\end{aligned} \tag{37}$$

The expression (37) is the result of evaluating the diagram in Fig. 4 without any approximations. We now simplify it using the assumption that $M,p\gg T$. The second δ function cannot contribute for ω less than or on the order of T, so it can be dropped. The Fermi distribution $n_F(E')$ is exponentially suppressed, so it can also be deleted. Using $E'\simeq E-\mathbf{v}\cdot\mathbf{q}$, the first δ function reduces to $\delta(\omega-\mathbf{v}\cdot\mathbf{q})$, which constrains the frequency ω to the spacelike interval $-vq<\omega<+vq$. Using this δ function to evaluate the angular integral for $\mathbf{q}$ and inserting (37) into Weldon's formula (6), the interaction rate reduces to

$$\Gamma(E)=\frac{e^2}{2\pi v}\int_0^\infty dq\, q\int_{-vq}^{+vq}d\omega[1+n_B(\omega)]\left[\rho_l(\omega,q)+\left[v^2-\frac{\omega^2}{q^2}\right]\rho_t(\omega,q)\right]\ . \tag{38}$$

We first determine the order of magnitude of the contribution to Γ in (38) from the integration region where q and ω are hard. With q and ω of order T, the effective propagators appearing in the spectral functions (33) and (34) reduce to the tree-level propagators $\Delta_l\simeq 1/q^2$ and $\Delta_t\simeq 1/(\omega^2-q^2)$. Aside from a multiplicative factor of m_γ^2 in the spectral functions, the only scale in the integral is T. By dimensional analysis since Γ has dimensions of energy, the hard contribution to Γ must be of order $e^2m_\gamma^2/T\sim e^4T$. This estimate is identical to that which would be obtained by considering the contribution to the interaction rate from the tree-level scattering process in Fig. 1. We next consider the contribution to Γ from the integration region where q and ω are soft. Since ω is of order eT, we can expand the Bose factor in (38) around $\omega=0$,

$$1+n_B(\omega)=\frac{T}{\omega}+\frac{1}{2}+\cdots\ , \tag{39}$$

and keep only the leading term T/ω. Aside from this explicit factor of T, the only scale in the integral is eT. By dimensional analysis, the soft contribution to Γ must be of order e^2T. Thus the hard contribution to Γ is suppressed relative to the soft contribution by a factor of e^2. Unfortunately, the transverse term in the integral in (38) has a logarithmic infrared divergence from the end point $q\to 0$. This arises because the dynamical screening of the magnetic interaction provided by the transverse effective propagator (28) is not sufficient to completely screen the divergence from the long-range static magnetic interaction. The logarithmic divergence is a big improvement over the quadratic infrared divergence that appears if the interaction rate Γ is calculated from the tree-level scattering diagram in Fig. 1. Nevertheless it indicates that the screening involves not only the scale eT but also the scale e^2T. Thus the resummation of hard thermal loops is not sufficient to calculate Γ to leading order in e. It is necessary to develop more powerful resummation methods that can handle the scale e^2T (Ref. [13]).

The soft contribution to $-dE/dx$ is obtained from (38) by inserting the factor ω/v inside the integral and imposing an upper limit q^* on the momentum transfer q. Since $|\omega|<vq$, ω is also restricted to be soft, and the Bose factor can be expanded as in (39). In the case of dE/dx, the T/ω term in (39) does not contribute because it gives a term in the integrand which is an odd function of ω and integrates to 0. The leading contribution comes instead from the 1/2 term in (39). As noted by Thoma and Gyulassy [4], the net result is that the soft contribution to dE/dx differs from $\Gamma(E)$ in (38) by a factor of $\omega^2/2vT$ in the integrand:

$$\left[-\frac{dE}{dx}\right]_{\mathrm{soft}}=\frac{e^2}{4\pi v^2}\int_0^{q^*}dq\, q\int_{-vq}^{+vq}d\omega\,\omega\left[\rho_l(\omega,q)+\left[v^2-\frac{\omega^2}{q^2}\right]\rho_t(\omega,q)\right]\ . \tag{40}$$

This result agrees up to a color factor and change in coupling constant with that of Thoma and Gyulassy [4] for the quark-gluon plasma. They calculated dE/dx for a heavy quark by computing the chromoelectric field induced by a classical source consisting of a colored point charge moving at fixed velocity. The agreement holds only after using the small ω approximation in (40), which is equivalent to a high-temperature approximation. A similar correspondence between a high-temperature approximation and a classical approximation has been observed by Heinz [16] in the case of the gluon polarization function and the color response function.

To further simplify (40), it is convenient to change integration variables from q and ω to q and $x=\omega/q$. The integral over q can then be evaluated analytically revealing a logarithmic dependence on q^*. In the logarithmic term, the integral over x can also be evaluated analytically. Our final result for the soft contribution to dE/dx is

$$\left[-\frac{dE}{dx}\right]_{\rm soft}=\frac{3e^2}{8\pi}m_\gamma^2\left\{\left[\frac{1}{v}-\frac{1-v^2}{2v^2}\ln\frac{1+v}{1-v}\right]\ln\frac{q^*}{m_\gamma}-\frac{1}{v^2}\int_0^v dx\,x^2\left[\ln\frac{3\pi x}{2}+\frac{1}{2}\ln[1+Q_l(x)^2]+Q_l(x)\left[\frac{\pi}{2}-\arctan[Q_l(x)]\right]\right]-\frac{1}{2v^2}\int_0^v dx\,x^2\frac{v^2-x^2}{1-x^2}\left[\ln\frac{3\pi x}{4}+\frac{1}{2}\ln[1+Q_t(x)^2]+Q_t(x)\left[\frac{\pi}{2}-\arctan[Q_t(x)]\right]\right]\right\}, \quad (41)$$

where the functions $Q_l(x)$ and $Q_t(x)$ are

$$Q_l(x)=\frac{1}{\pi}\left[-\ln\frac{1+x}{1-x}+\frac{2}{x}\right], \quad (42)$$

$$Q_t(x)=\frac{1}{\pi}\left[\ln\frac{1+x}{1-x}+\frac{2x}{1-x^2}\right]. \quad (43)$$

The remaining integrals over x in (41) must be evaluated numerically. Note that the dependence on the arbitrary scale q^* cancels between the hard contribution in (22) and the soft contribution in (41).

The soft contribution to dE/dx in (41) can be written more compactly in the form

$$\left[-\frac{dE}{dx}\right]_{\rm soft}=\frac{e^4T^2}{24\pi}\left[\frac{1}{v}-\frac{1-v^2}{2v^2}\ln\frac{1+v}{1-v}\right]\times\left[\ln\frac{q^*}{eT}+A_{\rm soft}(v)\right], \quad (44)$$

where we have used $m_\gamma=eT/3$. The function $A_{\rm soft}(v)$ begins at 0.049 at $v=0$, increases to a maximum of 0.292 at $v=0.96$, and then decreases to 0.256 at $v=1$. If we wish to approximate the expression $\ln(q^*/eT)+A_{\rm soft}(v)$ in (44) by a simple logarithm $\ln(q^*/q_{\rm min})$ as in (1), the choice of $q_{\rm min}$ that minimizes the maximum error over the range $0<v<1$ is $q_{\rm min}=0.84eT$.

V. ALTERNATIVE CALCULATION OF THE SOFT CONTRIBUTION

In this section, we repeat the calculation of the soft contribution to dE/dx using a more intuitive approach. Our starting point is (9), which gives the contribution to dE/dx from $e^\pm\mu\to e^\pm\mu$ scattering. In the region of phase space where the exchanged photon in Fig. 1 has soft momentum, hard thermal loop corrections to the photon propagator are not higher order in e and must be resummed. We assume that the net effect of the resummation is that the photon propagator in Fig. 1 is replaced by the effective propagator $\Delta^{\mu\nu}(Q)$, as illustrated in Fig. 6. Our justification for this prescription is that it reproduces the results obtained in Sec. IV using the imaginary-time formalism.

This prescription has been used previously without justification by Baym *et al.* [9], who showed that it provides sufficient screening to eliminate infrared divergences from transport coefficients. However the prescription cannot be trivially derived from the Feynman rules of the real-time formalism for thermal field theory. It also cannot be derived easily from the corresponding result in the imaginary-time formalism. For example, Weldon's formula (6) gives an exact relation between the muon interaction rate $\Gamma(E)$ calculated from the tree-level diagram in Fig. 1 and the two-loop diagram for the muon self-energy $\Sigma(P)$ in Fig. 3. There is no corresponding exact relation between Γ calculated from the resummed diagram in Fig. 6 and the resummed diagram for Σ in Fig. 4. These quantities satisfy Weldon's formula only in the limit of soft momentum transfer. While our results suggest that the simple prescription of using an effective propagator for the photon is correct, a rigorous

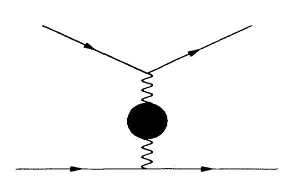

FIG. 6. Feynman diagram for the scattering process $e^-\mu\to e^-\mu$ with effective photon propagator.

justification must await the development of resummation methods for the real-time formalism that are as powerful as the resummation methods for the imaginary-time formalism developed in Ref. [5].

The calculation of the contribution to dE/dx from the Feynman diagram in Fig. 6 is most easily calculated in Coulomb gauge, where the only nonzero components of the effective propagator are given in (25) and (26). The matrix element in Coulomb gauge is

$$i\mathcal{M}=e^2\Delta_l(Q)\bar{u}(P',s')\gamma^0 u(P,s)\bar{u}(K',\lambda')\gamma^0 u(K,\lambda)+e^2\Delta_t(Q)(\delta^{ij}-\hat{q}^i\hat{q}^j)\bar{u}(P',s')\gamma^i u(P,s)\times\bar{u}(K',\lambda')\gamma^j u(K,\lambda)\ . \tag{45}$$

This matrix element must be squared, averaged over the initial muon spin s, and summed over the other spins. Evaluating the resulting Dirac traces and using the conditions $M,p \gg T$ to simplify the expression, we get

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2=16e^4\{|\Delta_l(Q)|^2E^2(kk'+\mathbf{k}\cdot\mathbf{k}')+2\,\text{Re}[\Delta_l(Q)\Delta_t(Q)^*]E[k(\mathbf{p}\cdot\mathbf{k}'-\mathbf{p}\cdot\hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\mathbf{k}')+k'(\mathbf{p}\cdot\mathbf{k}-\mathbf{p}\cdot\hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\mathbf{k})]+|\Delta_t(Q)|^2[2(\mathbf{p}\cdot\mathbf{k}-\mathbf{p}\cdot\hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\mathbf{k})(\mathbf{p}\cdot\mathbf{k}'-\mathbf{p}\cdot\hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\mathbf{k}')-(kk'-\mathbf{k}\cdot\mathbf{k}')(p^2-\mathbf{p}\cdot\hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\mathbf{p})]\}\ . \tag{46}$$

Using the momentum δ function in (9) to integrate over $\mathbf{p}'$ and using $M,p \gg T$ to reduce the energy δ function to $\delta(\omega-\mathbf{v}\cdot\mathbf{q})$, the soft contribution to the energy loss reduces to

$$\left[-\frac{dE}{dx}\right]_{\text{soft}}=\frac{\pi}{4vE^2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k}n_F(k)\int\frac{d^3k'}{(2\pi)^3}\frac{1}{k'}[1-n_F(k')]\delta(\omega-\mathbf{v}\cdot\mathbf{q})\omega\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2\theta(q^*-q)\ . \tag{47}$$

Note that $\sum|\mathcal{M}|^2$ in (46) is symmetric under the interchange of $\mathbf{k}$ and $\mathbf{k}'$. The integrand in (47) can therefore be symmetrized by replacing the thermal distribution factors $n_F(k)[1-n_F(k')]$ by $[n_F(k)-n_F(k')]/2$.

We now make use of the restriction of the momentum transfer q to the soft region $q<q^*$. The leading-order contributions to the integrals in (47) come from the region where k and k' are hard (i.e., of order T) while their difference $\omega=k'-k$ is soft (i.e., of order eT). The thermal distribution factor can be expanded to lowest order in ω:

$$\frac{n_F(k)-n_F(k')}{2}=-\frac{\omega}{2}n_F'(k)+\cdots\ . \tag{48}$$

After changing the integration variable $\mathbf{k}'$ in (47) to $\mathbf{q}$ and using the approximation $k'=k-\hat{\mathbf{k}}\cdot\mathbf{q}$, the energy loss reduces to

$$\left[-\frac{dE}{dx}\right]_{\text{soft}}=\frac{\pi}{8vE^2}\int\frac{d^3q}{(2\pi)^3}(\mathbf{v}\cdot\mathbf{q})^2\theta(q^*-q)\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2}[-n_F'(k)]\delta(\mathbf{v}\cdot\mathbf{q}-\hat{\mathbf{k}}\cdot\mathbf{q})\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2\ . \tag{49}$$

The matrix element in (46) can also be simplified by setting $\mathbf{k}'=\mathbf{k}$:

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2=32e^4E^2k^2\{|\Delta_l(Q)|^2+2\,\text{Re}[\Delta_l(Q)\Delta_t(Q)^*](\mathbf{v}\cdot\hat{\mathbf{k}}-\mathbf{v}\cdot\hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\hat{\mathbf{k}})+|\Delta_t(Q)|^2(\mathbf{v}\cdot\hat{\mathbf{k}}-\mathbf{v}\cdot\hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\hat{\mathbf{k}})^2\}\ . \tag{50}$$

The integrals over the angles of $\mathbf{k}$ are

$$\int\frac{d\Omega}{4\pi}\delta(\omega-\hat{\mathbf{k}}\cdot\mathbf{q})=\frac{1}{2q}\ , \tag{51}$$

$$\int\frac{d\Omega}{4\pi}\delta(\omega-\hat{\mathbf{k}}\cdot\mathbf{q})\left[\mathbf{v}\cdot\hat{\mathbf{k}}-\frac{\omega}{q}\hat{\mathbf{q}}\cdot\hat{\mathbf{k}}\right]=0\ , \tag{52}$$

$$\int\frac{d\Omega}{4\pi}\delta(\omega-\hat{\mathbf{k}}\cdot\mathbf{q})\left[\mathbf{v}\cdot\hat{\mathbf{k}}-\frac{\omega}{q}\hat{\mathbf{q}}\cdot\hat{\mathbf{k}}\right]^2=\frac{1}{4q}\left[1-\frac{\omega^2}{q^2}\right]\left[v^2-\frac{\omega^2}{q^2}\right]\ , \tag{53}$$

where $\omega=\mathbf{v}\cdot\mathbf{q}$ and $d\Omega$ is the angular integration element for $\mathbf{k}$. The integral over the magnitude of $\mathbf{k}$ is

$$\int_0^\infty dk\,k^2[-n_F'(k)]=\frac{\pi^2T^2}{6}\ . \tag{54}$$

Changing the remaining integration variables in (49) from q and $\cos\theta=\hat{\mathbf{v}}\cdot\hat{\mathbf{q}}$ to q and $\omega=\mathbf{v}\cdot\mathbf{q}$, the soft contribution to dE/dx reduces to

$$\left[-\frac{dE}{dx}\right]_{\text{soft}}=\frac{e^4T^2}{24\pi v^2}\int_0^{q^*}dq\int_{-vq}^{+vq}d\omega\,\omega^2\left[|\Delta_l(Q)|^2+\frac{1}{2}\left[1-\frac{\omega^2}{q^2}\right]\left[v^2-\frac{\omega^2}{q^2}\right]|\Delta_t(Q)|^2\right]\ . \tag{55}$$

This agrees with the result (40) in the imaginary-time formalism after inserting (33) and (34) for the spectral functions ρ_l and ρ_t in (40) and setting $m_\gamma = eT/3$.

VI. COMPLETE LEADING-ORDER RESULT FOR dE/dx

The complete result for dE/dx to leading order in e and T/M for a heavy lepton propagating through a hot QED plasma is the sum of the hard contribution in (23) and the soft contribution in (44):

$$-\frac{dE}{dx} = \frac{e^4T^2}{24\pi}\left[\frac{1}{v} - \frac{1-v^2}{2v^2}\ln\frac{1+v}{1-v}\right] \times \left[\ln\frac{E}{M} + \ln\frac{1}{e} + A(v)\right], \quad (56)$$

where $A(v) = A_{\text{hard}}(v) + A_{\text{soft}}(v)$. The dependence on the arbitrary scale q^* that separates the hard and soft regions of the momentum transfer q cancels, leaving a logarithm of $1/e$. This logarithm is simply an indication that dE/dx receives contributions from momentum transfers ranging from the hard scale T down to the soft scale eT. The dependence of the function $A(v)$ on the velocity v is shown in Fig. 7, together with hard and soft contributions. $A(v)$ increases from $A(0)=1.288$ at $v=0$ to a maximum of 1.478 at $v=0.88$ and then decreases to $A(1)=1.328$ at $v=1$.

The energy loss dE/dx as a function of the heavy-quark velocity v is shown as a solid line in Fig. 8, where it is compared with the results of previous calculations adapted for the QED plasma. The dotted curve in Fig. 8 is the analog of Bjorken's [1] estimate (1) for the quark-gluon plasma:

$$-\frac{dE}{dx} \simeq \frac{e^4T^2}{24\pi}\left[\frac{1}{v} - \frac{1-v^2}{2v^2}\ln\frac{1+v}{1-v}\right]\ln\frac{q_{\max}}{q_{\min}}. \quad (57)$$

Following Bjorken, we set $q_{\max} = \sqrt{4TE}$ and $q_{\min} = \Lambda$,

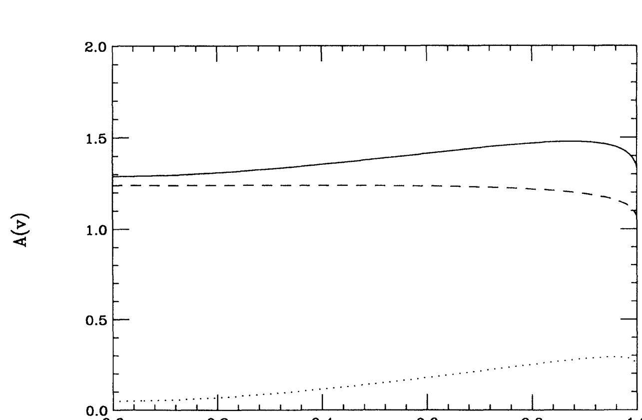

FIG. 7. The function $A(v)$ defined in (56) as a function of the muon velocity v (solid curve). It is the sum of $A_{\text{hard}}(v)$ (dashed curve) and $A_{\text{soft}}(v)$ (dotted curve).

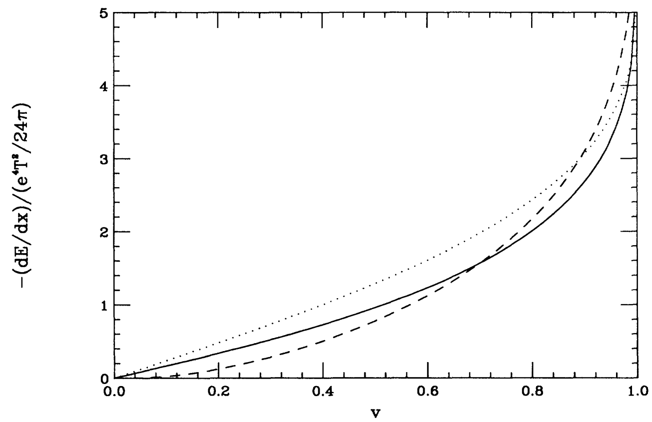

FIG. 8. Energy loss dE/dx of the muon as a function of its velocity v. The complete leading-order result (solid line) is compared to previous calculations by Thoma and Gyulassy (dashed curve) and Bjorken (dotted curve).

but we choose Λ to be the inverse Debye screening length: $\Lambda = \sqrt{3}m_\gamma$. The dashed curve is the analog of Thoma and Gyulassy's [4] result for the quark-gluon plasma. It is the soft contribution to dE/dx given in (41) with $q^* = 4Tp/(E-p+4T)$. Unlike the complete leading-order result in (56), the approximations of Bjorken and of Thoma and Gyulassy depend not only on v but also logarithmically on the ratio M/T. For the purpose of illustration in Fig. 7, we have set $M/T=10$. Both of the previous calculations differ significantly from the complete leading-order result. Bjorken's approximation significantly overestimates dE/dx over most of the range of velocity. The calculation of Thoma and Gyulassy underestimates dE/dx for $v<0.7$ and overestimates it for $v>0.7$.

As $v \to 0$, the factor in the first set of large parentheses in (56) approaches $2v/3$, implying a small positive energy loss proportional to v. This is clearly incorrect on physical grounds, since a heavy particle with kinetic energy much less than T should on the average gain energy from a collision. The energy loss dE/dx should therefore change sign at some velocity on the order of the thermal velocity $\sqrt{T/M}$. Our result (56) does not have this behavior, because it was calculated under the assumption $v \gg T/E$. Fortunately, the method described in this paper can equally well be used to calculate dE/dx in the limit $v \to 0$. An arbitrary scale q^* is introduced to separate the hard and soft regions of momentum transfer. The contribution from hard momentum transfer $q > q^*$ is calculated as in Sec. III, except that the assumption $|\mathbf{v}-\mathbf{v}'| \ll v$ is removed and the limit $v \to 0$ is taken. The result is

$$\left[-\frac{dE}{dx}\right]_{\text{hard}}^{v\to 0} = -\frac{e^4T^3}{12\pi Mv}\left[\ln\frac{4T}{q^*} + 1 - \gamma + \frac{\zeta'(2)}{\zeta(2)}\right]. \quad (58)$$

The contribution from the soft region $q < q^*$ is calculated as in Sec. IV or V, with the result

$$\left[-\frac{dE}{dx}\right]_{\text{soft}}^{v\to 0} = -\frac{e^4T^3}{12\pi M v}\left[\ln\frac{q^*}{\sqrt{3}m_\gamma}-\frac{1}{2}\right]. \quad (59)$$

Upon adding the hard and soft contributions, the q^* dependence cancels and the total energy loss is

$$\left[-\frac{dE}{dx}\right]^{v\to 0} = -\frac{e^4T^3}{12\pi M v}\left[\ln\frac{4\sqrt{3}}{e}+\frac{1}{2}-\gamma+\frac{\zeta'(2)}{\zeta(2)}\right]. \quad (60)$$

The energy loss is negative as it should be for a particle with subthermal energy. The factor of $1/v$ gives rise to a divergence as $v\to 0$ which is purely kinematical; the time rate of energy loss dE/dt approaches a constant as $v\to 0$. We can estimate the crossover velocity where the energy loss changes sign by equating the $v\to 0$ limit of (56) with the magnitude of (60). The result is $v=\sqrt{3T/M}$, which for $M/T=10$ is 0.55.

The formula (56) for the energy loss also breaks down in the opposite limit as $v\to 1$. The general form for the upper limit on the momentum transfer in (17) is $q_{\max}=2k(1+k/E)/(1-v+2k/E)$. The correction to the factor of $1-v$ in the denominator is of order T/E. When $1-v$ becomes of order $(T/M)^2$, or equivalently when the energy E becomes of order M^2/T, this correction cannot be ignored, and the formula for the energy loss becomes very complicated. It simplifies again in the limit $E\gg M^2/T$, in which case the upper limit on momentum transfer simplifies to $q_{\max}=E$. The hard contribution to the energy loss is given by (17) with v set equal to 1, except that the upper limit $q_{\max}=2k/(1-v)$ of the integral over q is replaced by $q_{\max}=E$:

$$\left[-\frac{dE}{dx}\right]_{\text{hard}}^{v\to 1} = \frac{e^4T^2}{48\pi}\left[\ln\frac{2TE}{(q^*)^2}+\frac{8}{3}-\gamma+\frac{\zeta'(2)}{\zeta(2)}\right]. \quad (61)$$

The soft contribution is simply the $v\to 1$ limit of (41):

$$\left[-\frac{dE}{dx}\right]_{\text{soft}}^{v\to 1} = \frac{e^4T^2}{24\pi}\left[\ln\frac{q^*}{eT}+0.256\right]. \quad (62)$$

Upon adding (61) and (62), the dependence on q^* cancels and the total energy loss is

$$\left[-\frac{dE}{dx}\right]^{v\to 1} = \frac{e^4T^2}{48\pi}\left[\ln\frac{E}{e^2T}+2.031\right]. \quad (63)$$

The crossover energy between the regions of validity of (56) and (63) can be estimated by equating the $v\to 1$ limit of (56) with (63). This gives $E\simeq 0.54M^2/T$. If $M/T=10$, the crossover velocity is 0.98. In summary, we have computed the energy loss for a heavy lepton with mass $M\gg T$ in three regimes of velocity or energy. For subthermal velocities $v\ll\sqrt{3T/M}$, it is negative and given by (60). In the intermediate region where $v\gg\sqrt{3T/M}$ and $E<0.54M^2/T$, the energy loss is given by (56). In the ultrarelativistic region $E>0.54M^2/T$, it is given by (63).

It is useful to compare our result for the energy loss in an ultrarelativistic plasma with the result for a nonrelativistic electron plasma [17]. For a particle with velocity v that is much greater than the thermal velocity $\sqrt{T/M}$, the energy loss is

$$-\frac{dE}{dx} = \frac{e^2\omega_p^2}{4\pi v^2}\left[\ln\frac{m_e v^2 E}{\omega_p M}+A\right], \quad (64)$$

where ω_p is the plasma frequency, m_e is the electron mass, and A is of order 1. Note that the multiplicative factor of $1/v^2$ gives rise to a large energy loss at small velocities, while the factor of $\ln(E/M)$ gives rise to large energy loss as $v\to 1$. The minimum energy loss occurs at some intermediate velocity, just as in the Bethe-Bloch formula for scattering of charged particles in ordinary matter. In contrast, the magnitude of the energy loss in the ultrarelativistic plasma increases monotonically with v as shown in Fig. 8.

We have concentrated on the energy loss dE/dx for a heavy lepton propagating through a hot QED plasma in order to illustrate the calculational methods required. The same methods can be used to calculate the energy loss for a heavy quark propagating through a hot quark-gluon plasma, a problem of direct relevance to heavy-ion collisions [1,2]. The explicit calculation of dE/dx to leading order in the QCD coupling constant g_s will be presented elsewhere [11]. Here we simply outline the calculations that are required. We represent the heavy quark by the symbol Q, while light quarks and gluons are represented by q and g. The heavy quark loses energy by scattering off of thermal quarks and gluons via the processes $qQ\to qQ$ and $gQ\to gQ$. As in the QED calculation, we introduce an arbitrary scale q^* to separate the hard and soft regions of the momentum transfer. The soft momentum-transfer contribution has already been calculated by Thoma and Gyulassy [4]. It can also be calculated in the imaginary-time formalism as in Sec. IV by evaluating the Feynman diagram in Fig. 4 with the effective photon propagator replaced by an effective gluon propagator. The result is identical to (41) except that it is multiplied by a color factor 4/3, e is replaced by g_s, and m_γ is replaced by the thermal rest mass of the gluon: $m_g=\sqrt{1+n_f/6}\,g_sT/\sqrt{3}$.

The hard momentum-transfer contribution to dE/dx is calculated using the tree-level Feynman diagrams. The contribution from scattering from light quarks $qQ\to qQ$ comes from the diagram in Fig. 1 with the photon line replaced by a gluon. The calculation is identical to that in Sec. III except for color factors, and the result for each of the n_f flavors of light quarks is (22) multiplied by a color factor of 2/3 and with e replaced by g_s. The only contribution to dE/dx that requires additional calculation is that from the scattering of gluons with hard momentum transfer: $gQ\to gQ$. The Feynman diagrams include the Compton scattering diagrams analogous to those in Fig. 2 together with an additional diagram containing a three-gluon vertex. While the two Compton scattering diagrams cancel in QED, the cancellation is upset in QCD by color factors. Thus all three diagrams contribute to dE/dx, although the logarithmic dependence on q^* comes only from the three-gluon vertex diagram. The evaluation of these diagrams will complete the calculation of dE/dx for a heavy quark to leading order in g_s. The calculations are in progress and the results, together with phenomenological implications for heavy-ion collisions, will be presented elsewhere [11].

Note added. After this work was completed, we became aware of related work by Svetitsky [18] on the diffusion of heavy quarks in a quark-gluon plasma. Svetitsky's drag coefficient is $A(p^2)=(-dE/dx)/p$. His calculation was complementary to that of Thoma and Gyulassy [4] in that he correctly calculated the hard contribution to dE/dx, but he used an *ad hoc* cutoff prescription to remove the infrared divergences at soft momentum transfer. The methods described in this paper can also be used to give a complete calculation to leading order in g_s of the diffusion coefficients for heavy quarks in the quark-gluon plasma.

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