Implications of unitarity for low-energy $W_L^\pm$, Z_L scattering

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We investigate the partial-wave scattering of longitudinally polarized $W^\pm$ and Z bosons for energies $M_W \ll \sqrt{s} \ll M_H$ using an Argand-diagram analysis. We find that, for large Higgs-boson masses M_H, partial-wave unitarity is badly violated at one loop at energies $\sqrt{s} \ll M_H$, and the standard model of the electroweak interactions becomes effectively strongly interacting in this sector. The implications of this and other similar results are summarized.

I. INTRODUCTION

The mass of the Higgs boson is at present an unknown parameter in the standard model of the electroweak interactions. If M_H is large, the quartic Higgs-boson coupling constant $\lambda = M_H^2/2v^2$ is also large, and the model will be strongly interacting in the Higgs sector. It is possible using unitarity arguments [1, 2] to delimit the range of Higgs-boson masses for which the standard model [1–8], and models with an extended Higgs sector [9, 10], remain weakly interacting, and treatable in low-order perturbation theory. In particular, a one-loop analysis [6] shows that M_H must be less than about 400 GeV if calculations of scattering processes at this level are to be reliable at energies of a few TeV. The mass limits are much stricter, $M_H \lesssim 155$ GeV, and also sharper theoretically, for a large class of models involving unification of the strong and electroweak interactions at (typical) energies of $\approx 10^{14}$ GeV [11]. However, within the context of the standard model, the only strict upper limit on the Higgs-boson mass, $M_H \lesssim 650$ GeV, follows from the so-called triviality bound in theories with elementary scalar fields [12], as implemented in nonperturbative lattice calculations [13]. One might nevertheless suppose—in the absence of other upper bounds on M_H—that the standard model is simply the low-energy limit of a deeper theory in which the role of the Higgs boson is played by a composite object with an effective mass or energy scale $M_H \gg 1$ TeV. The relevant question then shifts to that of determining the energy $\sqrt{s_{\rm cr}}$ at which the complete theory must depart from the standard model [14], for example, by becoming strongly interacting. We address that question here using the onset of large violations of partial-wave unitarity in the scattering of longitudinally polarized $W^\pm$ and Z bosons as a signal for the breakdown of low-order perturbation theory, hence the effective onset of a nonperturbative or strongly interacting regime. We make use in this analysis of recent complete calculations of the $W_L^\pm$, Z_L scattering amplitudes to one loop [4, 7, 15–17], and of an Argand-diagram analysis of the corresponding partial-wave amplitudes for $J = 0$ and $J = 1$.

The "low-energy" ($\sqrt{s} \ll M_H$) behavior of a model with a high mass scale M_H in the Higgs sector, and the possible observability of new effects in the scattering of $W_L^\pm$ and Z_L bosons, was investigated by Chanowitz and Gaillard [14] in lowest-order perturbation theory in the standard model. Other authors have considered models with more complicated Higgs sectors [10]. The method used to bound the perturbative region was that of Lee, Quigg, and Thacker [2], which consists of calculating the matrix of two-body $J = 0$ partial-wave $W_L^\pm$, Z_L scattering amplitudes, and imposing the minimal unitarity constraint $|a_0| \leq 1$ on the largest eigenamplitude. The result of Chanowitz and Gailland, that $|a_0| > 1$ for $s > 16\pi v^2$, where $v = 246$ GeV is the standard-model vacuum expectation value, is fairly typical. Hence lowest-order perturbation theory must fail and new, potentially observable effects must appear, below a critical energy $\sqrt{s_{\rm cr}} \approx 1.7$ TeV.

The lowest-order results are independent of M_H, a result that follows quite generally from the constraints on electroweak symmetry breaking and the chiral nature of the low-energy theory [18]. They do not, therefore, provide any dynamical information about the nature of electroweak symmetry breaking. The leading M_H-dependent corrections to the lowest-order result are of order $(s/v^2)^2 \ln(M_H/s)$. These were calculated in chiral perturbation theory by Cheyette and Gaillard [19], who found them to be large. Dobado and Herrero [20] subsequently used the complete one-loop calculation of ${\rm Re}\, T(W_L^+ W_L^- \to W_L^+ W_L^-)$ by Dawson and Willenbrock [4] to fix unknown constants in the one-loop chiral Lagrangean, and investigated the new unitarity constraints, using the complete matrix of $W_L^\pm$, Z_L, scattering amplitudes to that order in a loop expansion. Unfortunately, their analysis was confined to relatively low Higgs-boson masses, and unitarity violations only appeared for values of $\sqrt{s}$ for which the approximations used fail in the case of standard model. We correct those deficiencies here. Dobado and Herrero [20] also considered a more general low-energy parametrization of electroweak symmetry breaking.

II. PARTIAL-WAVE AMPLITUDES FOR $\sqrt{s} \ll M_H$

A. 2 →2 amplitudes

To probe the low-energy behavior of the standard model in the limit of very large Higgs-boson mass, we

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will use the constraints imposed by partial-wave unitarity in the scattering of longitudinally polarized $W^{\pm}$ and Z bosons. (The scattering of transversely polarized gauge bosons does not lead to any large, unitarity-violating scattering amplitudes because of the smallness of the gauge couplings; the scattering of longitudinally polarized bosons may be enhanced by powers of M_H^2/M_W^2 [21].) We work in the limit $M_W \ll \sqrt{s} \ll M_H$ and use the Goldstone-boson equivalence theorem [2, 14, 22–24]. This allows us to replace real, longitudinally polarized $W_L^{\pm}$ and Z_L bosons in the calculations of scattering amplitudes at energies $\sqrt{s} \gg M_W$ by the massless scalar fields $w^{\pm}$ and $z = w_3$ to which they reduce for vanishing gauge couplings $g, g' \to 0$. That is, $W_L^{\pm}$ and Z_L are replaced by the would-be Goldstone bosons of the complete broken-symmetry gauge theory. The scattering amplitudes for n external bosons of the two types are related by

$$T(W_L^{\pm}, Z_L, H) = (iC)^n T(w^{\pm}, z, H) + O(M_W^2/s), \tag{1}$$

where C depends in general on the renormalization scheme used in the calculation [23]. Using the scheme of Sirlin and Zucchini [25] and Marciano and Willenbrock [21], $C = 1$ up to second-order corrections in the gauge couplings [23]. We will neglect these corrections and take $g = g' = 0$. The effective theory is then described by the interaction Lagrangian

$$\begin{aligned} \mathcal{L}_I &= -\frac{\lambda}{4}(v^2 - h^2 - \mathbf{w}^2)^2 \\ &\xrightarrow[h\to H+v]{} -\lambda v^2 H^2 - \lambda v(2w^+w^- + zz + HH)H \\ &\qquad -\frac{1}{4}\lambda(2w^+w^- + zz + HH)^2. \end{aligned} \tag{2}$$

Here v is the Higgs-field vacuum expectation value, $v^2 = 1/\sqrt{2}G_F$, and λ is the Higgs-boson coupling. We note that $M_H^2 = 2\lambda v^2$, so the limit of large λ is the limit of large Higgs-boson mass.

The behavior of the standard model—or of any modification which preserves its low-energy structure and spectrum—is restricted for $\sqrt{s} \ll M_H$ by the low-energy theorems of Chanowitz and Gaillard [14] and Chanowitz, Golden, and Georgi [18]. In particular, the two-body scattering amplitudes for the $w^{\pm}$ and z bosons vanish for $s, t, u \to 0$, and are independent of M_H in leading order [see Eqs. (5)]. This structure is easily seen by noting that for λ or M_H large, the Higgs field h is effectively pinned at the value $h^2 = v^2 - \mathbf{w}^2$, and the full Lagrangean reduces to that for the nonlinear σ model,

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial_\mu \mathbf{w}\cdot\partial^\mu \mathbf{w} + \frac{1}{2}\frac{(\mathbf{w}\cdot\partial^\mu \mathbf{w})(\mathbf{w}\cdot\partial_\mu \mathbf{w})}{v^2 - \mathbf{w}^2}\quad, \tag{3}$$

a result valid to leading order in an expansion in derivatives ∂_μ, and to any order in $1/v^2$ at the tree level. The interaction terms involve two powers of momentum, and the two-body scattering amplitudes are therefore proportional to s, t, or u up to additive terms proportional to M_W^2/s, which we neglect. Loop corrections to $\mathcal{L}_{\text{eff}}$ introduce additional effective interactions which depend, however, on higher (even) derivatives of $\mathbf{w}$.

The low-energy limits of the two-body scattering amplitudes involving the scalars $w^{\pm}$ and z have been calculated to one loop from the Lagrangian in Eq. (2) by Dawson and Willenbrock [4, 15], and Passarino [7, 26]. Identical results were obtained in the limit of large M_H by Veltman and Ynduráin [16] and Bouamrane [17] starting from the full gauge theory, and by Dobado and Herrero [20] using the result of Ref. [4] for $W_L^+W_L^- \to W_L^+W_L^-$ to determine otherwise-unknown parameters in a calculation based on chiral perturbation theory. We will define the independent Feynmann amplitudes for scattering in the neutral two-body channels w^+w^-, w^-w^+, zz by

$$\begin{aligned} a(s,t,u) &= T(w^+w^- \to w^+w^-), \\ b(s,t,u) &= T(w^+w^- \to zz), \\ c(s,t,u) &= T(zz \to zz), \end{aligned} \tag{4}$$

where a, b, c are given in the limit $s \ll M_H^2$ by

$$\begin{aligned} a(s,t,u) &= \frac{1}{v^2}\left\{-u + \frac{1}{(4\pi v)^2}\left[\frac{5s^2+st}{6}\ln\frac{M_H^2}{-s} + \frac{5t^2+st}{6}\ln\frac{M_H^2}{-t} + \frac{u^2}{2}\ln\frac{M_H^2}{-u} + \left(\frac{9\pi}{2\sqrt{3}} - \frac{76}{9}\right)(s^2+t^2) - \frac{4}{9}u^2\right]\right\}, \\ b(s,t,u) &= \frac{1}{v^2}\left\{s + \frac{1}{(4\pi v)^2}\left[\frac{s^2}{2}\ln\frac{M_H^2}{-s} + \frac{2t^2+st}{6}\ln\frac{M_H^2}{-t} + \frac{2u^2+us}{6}\ln\frac{M_H^2}{-u} + \left(\frac{9\pi}{2\sqrt{3}} - \frac{74}{9}\right)s^2 - \frac{2}{9}(t^2+u^2)\right]\right\}, \\ c(s,t,u) &= \frac{1}{v^2}\left\{0 + \frac{1}{(4\pi v)^2}\left[s^2\ln\frac{M_H^2}{-s} + t^2\ln\frac{M_H^2}{-t} + u^2\ln\frac{M_H^2}{-u} + \left(\frac{9\pi}{2\sqrt{3}} - \frac{26}{3}\right)(s^2+t^2+u^2)\right]\right\}, \quad -s = e^{-i\pi}s. \end{aligned} \tag{5}$$

It will be convenient to eliminate the identical-particle restriction on the zz phase space by replacing the state $|zz\rangle$ by $\frac{1}{\sqrt{2}}|zz\rangle$. With this convention, the full neutral scattering matrix in the w,z sector is given in terms of a,b,c by

$$\mathbf{T} = \begin{array}{c} |w^+w^-\rangle \\ \frac{1}{\sqrt{2}}|zz\rangle \\ |w^-w^+\rangle \end{array} \overset{\begin{array}{ccc} |w^+w^-\rangle & \frac{1}{\sqrt{2}}|zz\rangle & |w^-w^+\rangle \end{array}}{\begin{pmatrix} a(s,t,u) & \frac{1}{\sqrt{2}}(s,t,u) & a(s,u,t) \\ \frac{1}{\sqrt{2}}b(s,t,u) & \frac{1}{2}c(s,t,u) & \frac{1}{\sqrt{2}}b(s,u,t) \\ a(s,u,t) & \frac{1}{\sqrt{2}}b(s,u,t) & a(s,t,u) \end{pmatrix}}. \tag{6}$$

It is evident from Eq. (2) that the interactions of the Goldstone bosons are SO(3) symmetric, with w^+, $z = w_3$, w^- the $I_3 = 1, 0, -1$ components of an SO(3) "isospin" vector. The matrix $\mathbf{T}$ can therefore be diagonalized by a transformation to a basis of $I_3 = 0$, $I = 0, 1, 2$ states. The diagonalized scattering amplitudes T^I in that basis are

$$\begin{aligned} T^0 &= \tfrac{1}{3}[\, a(s,t,u) + a(s,u,t) + b(s,t,u) \\ &\quad + b(s,u,t) + \tfrac{1}{2}c(s,t,u)], \\ T^1 &= \tfrac{1}{2}[a(s,t,u) - a(s,u,t)], \\ T^2 &= \tfrac{1}{3}[\, \tfrac{1}{2}a(s,t,u) + \tfrac{1}{2}a(s,u,t) - b(s,t,u) \\ &\quad - b(s,u,t) + c(s,t,u)]. \end{aligned} \tag{7}$$

The partial-wave amplitudes $a_{IJ}(s)$, normalized so that the exact amplitudes have the form

$$a_{IJ}(s) = \frac{1}{2ip_i}(\eta_{IJ} e^{2i\delta_{IJ}} - 1), \tag{8}$$

are related to T^I by the partial-wave projection

$$a_{IJ} = \frac{1}{32\pi}\left(\frac{4p_i p_f}{s}\right)^{1/2} \int_{-1}^{1} d\cos\theta \, T^I(s,\cos\theta) P_J(\cos\theta). \tag{9}$$

Since the symmetry of the Jth partial-wave state is given by $(-1)^{I+J}$, and states containing identical bosons must be symmetric under interchange, the lowest nonzero eigenamplitudes for the different isospins are a_{00}, a_{11}, and a_{20}. These are given to $O(s^2/v^4)$ by the partial-wave projections of the expressions in Eqs. (4) and (6):

$$a_{0,0}(s) = \frac{s}{32\pi v^2}\left[2 + \frac{s}{(4\pi v)^2}\left(\frac{25}{9}\ln\frac{M_H^2}{s} + \frac{33\pi}{2\sqrt{s}} - \frac{1673}{54}\right)\right] + i\left(\frac{s}{16\pi v^2}\right)^2, \tag{10a}$$

$$a_{1,1}(s) = \frac{s}{96\pi v^2}\left[1 - \frac{s}{(4\pi v)^2}\left(\frac{9\pi}{2\sqrt{3}} - \frac{143}{18}\right)\right] + i\left(\frac{s}{96\pi v^2}\right)^2, \tag{10b}$$

$$a_{2,0}(s) = -\frac{s}{32\pi v^2}\left[1 - \frac{s}{(4\pi v)^2}\left(\frac{10}{9}\ln\frac{M_H^2}{s} + \pi\sqrt{3} - \frac{631}{108}\right)\right] + i\left(\frac{s}{32\pi v^2}\right)^2. \tag{10c}$$

Note that $a_{1,1}$ is independent of the Higgs-boson mass, while $a_{0,0}$ and $a_{2,0}$ are not.

The partial-wave amplitudes for $J > 1$ have no contributions of $O(s/v^2)$, and are real to the order calculated. In particular, for $J = 2$,

$$\begin{aligned} a_{0,2}(s) &= \frac{1}{18\pi}\left(\frac{s}{16\pi v^2}\right)^2\left(\ln\frac{M_H^2}{s} - \frac{89}{15} + \frac{27\pi}{10\sqrt{3}}\right), \\ a_{2,2}(s) &= \frac{1}{45\pi}\left(\frac{s}{16\pi v^2}\right)^2\left(\ln\frac{M_H^2}{s} - \frac{3287}{240} + \frac{27\pi}{4\sqrt{4}}\right). \end{aligned} \tag{11}$$

The $J = 2$ amplitudes first become complex when calculated to three loops.

It is important to recall the constraint on s used in the derivations of the results above, specifically $M_W^2 \ll s \ll M_H^2$. There is no range of s in which the results are valid unless M_H (or the scale of symmetry breaking) is much larger than M_W, a condition which requires that M_H be at least in the TeV range. The upper limit on s restricts the analysis to energies well below the Higgs pole (or the energy at which the Higgs model must change, e.g., in a composite theory). We can obtain more precise estimates of this upper limit by starting with the tree-level amplitudes for $2 \to 2$ scattering given by the complete Lagrangian in Eq. (2), expanding the Higgs propagators which appear in these amplitudes in powers of s/M_H^2, and retaining the first corrections to the leading terms in Eq. (4). For example, at the tree level,

$$\begin{aligned} T(w^+w^- \to w^+w^-) &= -4\lambda + 4\lambda^2 v^2\left(\frac{1}{M_H^2 - s} + \frac{1}{M_H^2 - t}\right) \\ &= \frac{1}{v^2}\left(-u + \frac{s^2}{M_H^2} + \frac{t^2}{M_H^2} + \cdots\right), \end{aligned} \tag{12}$$

where we have used the relations $M_H^2 = 2\lambda v^2$ and $s + t + u = 4M_W^2$, and have dropped terms of orders M_W^2/M_H^2 and M_W^2/s. The resulting corrections to the partial-wave isospin amplitudes are given to leading order in s by

$$\begin{aligned} &\frac{\delta a_{0,0}}{a_{0,0}} = \frac{11}{6}\frac{s}{M_H^2}, \quad \frac{\delta a_{2,0}}{a_{2,0}} = -\frac{2}{3}\frac{s}{M_H^2}, \\ &\frac{\delta a_{1,1}}{a_{1,1}} = -\frac{s}{M_H^2}, \\ &\frac{\delta a_{0,2}}{a_{0,2}} = \frac{48\pi^2}{5}\frac{v^2}{M_H^2}\left(\ln\frac{M_H^2}{s} - 1.0361\right)^{-1}, \\ &\frac{\delta a_{2,2}}{a_{2,2}} = 24\pi^2\frac{v^2}{M_H^2}\left(\ln\frac{M_H^2}{s} - 1.4528\right)^{-1}. \end{aligned} \tag{13}$$

If we require that the correction be less than $\frac{1}{4}$ of the leading term, $\sqrt{s}/M_H$ must be less than 0.37 (0.61) if the $I = 0$ (2), $J = 0$ amplitudes are to be reliable, and less than 0.5 if the $I = J = 1$ amplitude is to be reliable. For $J = 2$, the ratios above depend on $\sqrt{s}$ and M_H independently. As examples of the corresponding restrictions, we note that, for $M_H = 5$ TeV (10 TeV), $\sqrt{s}$ must be less than 1.88 TeV (5.31 TeV) for $I = 0$, and less than 0.77 TeV (3.63 TeV) for $I = 2$. We will use these restrictions on the range of s in our later analysis.

B. 2 →4 amplitudes

We have also calculated the complete matrix of inelastic $2 \to 4$ amplitudes in the neutral channels to order $1/v^4$ (tree level), and have estimated the sum of the squares of the amplitudes in the $I = 0, 2$, $J = 0$ states integrated over the final phase space [27]. The results give, as we shall see, an independent (but weak) upper bound $\sqrt{s_{\rm cr}}$ on the energy above which low-order perturbation theory ceases to be valid. Moreover, we find that we can ignore inelastic processes when studying unitarity constraints on the perturbative elastic scattering amplitudes for $\sqrt{s} \ll \sqrt{s_{\rm cr}}$ ($\ll M_H$).

We have calculated the low-energy $2 \to 4$ amplitudes two ways. We first calculated the relevant tree graphs for the Lagrangian in Eq. (2) to effective order λ^2 (where $\lambda v H = \frac{1}{\sqrt{2}}\lambda^{1/2} M_H H \to O(\lambda^{1/2})$ in the trilinear couplings using the new computer programs DIAGRAMMAR, CALCULATE, and DRAW [28]. These construct, evaluate formally, and draw Feynman graphs for general Lagrangians. We then expanded the Higgs-boson denominators in powers of s/M_H^2 using the algebraic-manipulation program REDUCE to obtain the final amplitudes of order s/v^4. We also calculated the $2 \to 4$ amplitudes directly using the σ-model Lagrangian in Eq. (3), expanded to order $1/v^4$:

$$\mathcal{L}_{\rm eff} \approx \frac{1}{2}\partial \mathbf{w} \cdot \partial \mathbf{w} + \frac{1}{8v^2}\partial \mathbf{w}^2 \partial \mathbf{w}^2 + \frac{1}{8v^4}\mathbf{w}^2 \partial \mathbf{w}^2 \partial \mathbf{w}^2 + \cdots . \tag{14}$$

The effective interaction in Eq. (14) gives the tree-level graphs for $2 \to 4$ processes shown in Fig. 1. The 6-point vertex in Fig. 1(a) arises from the large-M_H limit of the Feynman graphs in Fig. 2 calculated using the 3- and 4-particle couplings in Eq. (2). The exchange and jetlike graphs in Figs. 1(b) and 1(c) involve the reduced 4-point amplitudes discussed in the previous section [and given by the first interaction term in Eq. (14)], connected by $w^\pm$ or z propagators. The amplitudes in all cases are of order s/v^4; 4-body phase space adds a factor s^2 in cross sections [or unitarity sums, see Eqs. (21) and (23)], so the amplitudes are counted as order s^2/v^4 as we will use them.

We will use the following notation:

$$s_{ij} = (p_i + p_j)^2 \ , \quad t_{ij} = (p_i - p_j)^2 \ ,$$
$$s_{ijk} = (p_i + p_j + p_k)^2 \ , \quad t_{ijk} = (p_i - p_j - p_k)^2 \ , \tag{15}$$

where the labels $i = 1, 2$ and $i = 3, \ldots, 6$ refer to the initial and final particles in the orders listed. The Feynman amplitudes for the various $2 \to 4$ processes are then given, up to an overall factor $1/v^4$, by

$$\begin{aligned}
w^+w^- \to w^+w^-w^+w^- :\ & 4(s + t_{135} - s_{34} - s_{56}) - t_{134}^{-1}(t_{13} + s_{34})(t_{26} + s_{56}) \\
& - t_{135}^{-1}(t_{13} + s_{36})(t_{24} + s_{45}) - t_{135}^{-1}(t_{13} + t_{15})(t_{24} + t_{26}) \\
& - t_{145}^{-1}(t_{15} + s_{45})(t_{26} + s_{36}) - t_{156}^{-1}(t_{15} + s_{56})(t_{24} + s_{34}) \\
& - s_{345}^{-1}(s + t_{26})(s_{34} + s_{45}) - s_{346}^{-1}(s + t_{15})(s_{34} + s_{36}) \\
& - s_{356}^{-1}(s + t_{24})(s_{36} + s_{56}) - s_{456}^{-1}(s + t_{13})(s_{45} + s_{56}) \ , \\[1ex]
w^+w^- \to w^+w^-zz :\ & s + t_{13} + t_{24} + s_{34} + 2s_{56} - t_{134}^{-1}s_{56}(t_{13} + s_{34}) - (t_{135}^{-1} + t_{136}^{-1})t_{13}t_{24} \\
& - t_{156}^{-1}s_{56}(t_{24} + s_{34}) - (s_{345}^{-1} + s_{346}^{-1})s s_{34} - s_{356}^{-1}s_{56}(s + t_{24}) - s_{456}^{-1}s_{56}(s + t_{13}), \\[1ex]
w^+w^- \to zzzz :\ & -(t_{134}^{-1} + t_{156}^{-1})s_{34}s_{56} - (t_{136}^{-1} + t_{145}^{-1})s_{36}s_{45} - (t_{135}^{-1} + t_{146}^{-1})s_{35}s_{46}, \\[1ex]
zz \to w^+w^-w^+w^- :\ & 3s - s_{35} - s_{46} - (t_{134}^{-1} + t_{156}^{-1})s_{34}s_{56} - (t_{136}^{-1} + t_{145}^{-1})s_{36}s_{45} \\
& - s_{345}^{-1}s(s_{34} + s_{45}) - s_{346}^{-1}s(s_{34} + s_{36}) - s_{356}^{-1}s(s_{56} + s_{36}) - s_{456}^{-1}s(s_{56} + s_{45}) \ , \\[1ex]
zz \to w^+w^-zz :\ & s + t_{134} + t_{156} - s_{34} - s_{56} - (t_{135}^{-1} + t_{145}^{-1})t_{15}t_{26} - (t_{136}^{-1} + t_{146}^{-1})t_{16}t_{25} \\
& - (s_{356}^{-1} + s_{456}^{-1})s s_{56} \ , \\[1ex]
zz \to zzzz :\ & 0.
\end{aligned} \tag{16}$$

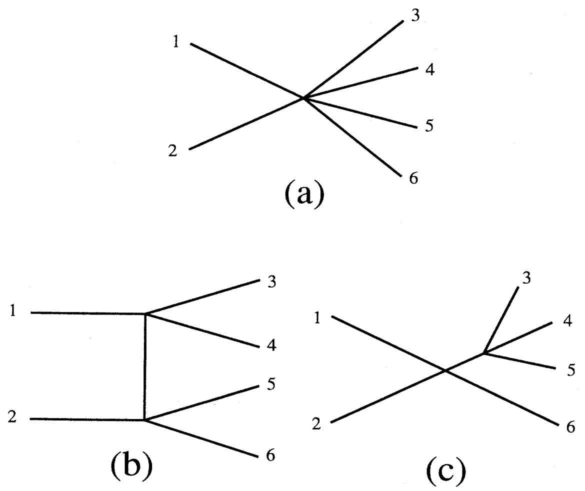

FIG. 1. The tree-level diagrams that contribute to the $2 \to 4$ scattering processes for the chiral Lagrangian in Eqs. (3) and (14): (a) the six-particle vertex of order v^{-4}; (b) the w or z exchange; and (c) the jetlike graphs of order v^{-4}.

The matrix of $J = 0$ partial-wave amplitudes can be projected out of the Feynman amplitudes above by averaging over the directions of the momentum of particle 1 in the center-of-mass system, $\mathbf{p}_2 = -\mathbf{p}_1$, with the final momenta held fixed:

$$\mathbf{M}^{J=0}(\sqrt{s};\ \mathbf{p}_3, \ldots, \mathbf{p}_6) = \frac{1}{4\pi} \int d\Omega(\hat{\mathbf{p}}_1) \mathbf{M}_{\text{Feynman}}. \tag{17}$$

To obtain partial-wave amplitudes normalized properly for the considerations below, $\mathbf{M}^{J=0}$ must be multiplied by a kinematic factor $(p_i/4\sqrt{s})^{1/2}$, $p_i = |\mathbf{p}_1| = |\mathbf{p}_2|$, and by diagonal matrices of statistical factors $\mathbf{N_f}$, $\mathbf{N_i}$ where the elements of $\mathbf{N_f}$ and $\mathbf{N_i}$ are products of factors $1/\sqrt{n!}$ for each set of n identical particles in the final or the initial state. With the inital and final states listed in the orders used above,

$$\mathbf{N_f} = \text{diag}\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{6}}\right), \ \mathbf{N_i} = \text{diag}\left(1, \frac{1}{\sqrt{2}}\right). \tag{18}$$

Finally, the initial w^+w^-, zz states can be combined into states of definite isospin $I = 0, 2$ by multiplying on the right by the matrix

$$\mathbf{C} = \begin{pmatrix} \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \tag{19}$$

with row labels w^+w^-, zz and column labels $I = 0$, $I = 2$. The result is a 4×2 matrix which gives the transition amplitudes from $J = 0$ states with $I = 0, 2$ to the final states $w^+w^-w^+w^-$, w^+w^-zz, $zzzz$,

$$\mathbf{a}^{J=0}(\sqrt{s};\ \mathbf{p}_3, \ldots, \mathbf{p}_6) = \left(\frac{p_i}{4\sqrt{s}}\right)^{1/2} \mathbf{N_f} \mathbf{M}^{J=0} \mathbf{N_i} \mathbf{C}. \tag{20}$$

With the normalization above, $\mathbf{a} = \mathbf{S}^{2\to 4}/2i$, where $\mathbf{S}$ is the S matrix, and the cross sections for the $2 \to 4$ processes with $J = 0$ and $I = 0$ or 2 are just the squares of the corresponding columns in $\mathbf{a}^{J=0}$, integrated over the 4-body phase space and multiplied by $1/p_i^2$.

We will need the (formal) sum

$$\begin{aligned} \sum |a_{I,J=0}^{2\to 4}|^2 &= p_i^2 \frac{d\sigma_{I,J=0}^{2\to 4}}{d\Omega^*} = \frac{p_i^2}{4\pi} \sigma_{I,J=0}^{2\to 4} \\ &= \frac{1}{4\pi} \int \prod_{j=3}^{6} \frac{d^3 p_i}{(2\pi)^3 2E_j} (2\pi)^4 \delta^4(P' - P) \\ &\quad \times |\mathbf{a}_{\mathbf{I},\mathbf{J}=\mathbf{0}}|^2 \end{aligned} \tag{21}$$

in our analysis of unitarity constraints on low-energy $w^{\pm}$, z scattering. Here $\mathbf{a}_{\mathbf{I},\mathbf{J}}$ is the isospin-I column of $\mathbf{a}^{J=0}$. It is clear from Eq. (16) that there are large cancellations in the partial-wave amplitudes since the variables s_{ij}, s_{ijk} are positive while the variables t_{ij} and t_{ijk} can be negative. The projected amplitudes are all of order s/v^4. We have found by explicit calculation of representative contributions to $\mathbf{a}$ that the coefficient functions are ~ 1 when averaged over phase space, so a reasonable estimate for the sum in Eq. (21) is simply $(s/v^4)^2$ times the volume of phase space. This estimate gives

$$\sum |a_{I,J}^{2\to 4}|^2 = \frac{1}{6\pi^2} \left(\frac{s}{16\pi v^2}\right)^4 K. \tag{22}$$

with K expected to be ~ 1. In fact, for the sum of the amplitudes corresponding to the six-particle vertex in Fig. 1(a), $K = 55/48$ for $I = 0$ and $K = 59/240$ for $I = 2$. We do not expect the inclusion of all the terms in Eq. (17) to change these results drastically.

The sum in Eq. (22) is bounded above by unitarity [see Eq. (23) below]. The limiting value of $\frac{1}{4}$ is reached for $\sqrt{s} = 2.44$ TeV for $K = 1$. For $\sqrt{s} < 1.2$ TeV, the region we will mainly be concerned with, the (estimated) sum is less than 3.4×10^{-3} in magnitude, and decreases as $(\sqrt{s})^8$ for smaller values of $\sqrt{s}$. Because the sum is so small in the region of interest, we have not bothered to make a complete numerical calculation of K.

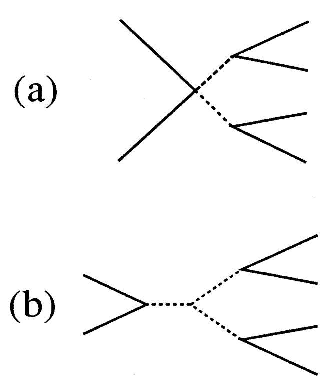

FIG. 2. The Feynman graphs containing intermediate Higgs-boson propagators which reduce to the 6-particle vertex in Fig. 1(a) in the limit $M_H^2 \gg s$. The graphs involve the Yukawa couplings in the Lagrangian in Eq. (2).

C. Unitarity restrictions on perturbative amplitudes

The unitarity of the partial-wave S matrix requires that the exact partial-wave scattering amplitudes satisfy the relation

$$\left|a_{IJ}^{2\to2} - \frac{i}{2}\right|^2 + \sum_{n>2} \left|a_{IJ}^{2\to n}\right|^2 = \frac{1}{4}\,. \tag{23}$$

This is just the condition that the two-body elastic scattering amplitudes lie on a circle of radius $\eta_{IJ}/2$ centered at $(0, \frac{1}{2})$ in the complex plane, with

$$\eta_{IJ} = \left(1 - 4\sum_{n>2}\left|a_{IJ}^{2\to n}\right|^2\right)^{1/2}\,. \tag{24}$$

This gives the familiar Argand diagram for the elastic scattering amplitudes shown in Fig. 3. We have seen above that the $2 \to 4$ (and presumably the remaining $2 \to n$) contributions to the sum in Eq. (24) are negligible for the range of $\sqrt{s}$ with which we are primarily concerned. As a result, the exact $2 \to 2$ amplitudes must lie *on* the "unitarity circle" with radius $\frac{1}{2}$ to very good approximation. This condition will be violated when the perturbation series is truncated. In this section, we will formulate various conditions which allow us to decide when an apparent violation of unitarity is "too large" for the truncated series, that is, when higher-order terms *must* be included to obtain a reliable result. We will be concerned primarily with the convergence of the perturbation series, not with the absolute magnitude of the scattering amplitudes. The perturbation series may fail to converge rapidly even when the amplitudes are small.

We will write a given partial-wave amplitude calculated to n loops, i.e., to order v^{-2n-2}, as $\sum_{i=0}^{n} \mathbf{a}_i$, with the i-loop amplitudes $\mathbf{a}_i = (\mathrm{Re}\, a_i,\ \mathrm{Im}\, a_i)$ regarded as vectors in the complex plane. We will work here with the one-loop approximation, $\mathbf{a} \approx \mathbf{a}_0 + \mathbf{a}_1$. A typical situation is sketched in Fig. 4: the approximate amplitude remains near the unitarity circle for some range of $\sqrt{s}$, then departs markedly from it. We will denote by $\Delta\mathbf{a}$ the minimum-length vector which can be added to $\mathbf{a}_0 + \mathbf{a}_1$ to bring the resultant vector to the unitarity circle. The vector $\Delta\mathbf{a}$ necessarily lies along the radial direction as indicated in the figure, and is less in magnitude than the sum of all the higher-order terms. (Recall that we are assuming that $1 - \eta_{IJ} \approx 2\sum_{n>2}\left|a_{IJ}^{2\to n}\right|^2 \ll 1$.)

It will be important to the following discussion to note that we are *not* trying to write relations which hold to a given order in $1/v^2$: the perturbation series breaks down precisely when higher-order terms are *not* smaller than the low-order terms. We are instead interested in criteria which quantify when the departures of $\mathbf{a}_0 + \mathbf{a}_1$ from the unitarity circle—which are clearly evident in an Argand diagram, and involve $\mathbf{a}_0 + \mathbf{a}_1$ and $\Delta\mathbf{a}$ (or $\mathbf{a}_2, \mathbf{a}_3, \ldots$) *linearly*—are "too large" for the amplitudes as actually calculated. In fact, when the unitarity relation, Eq. (23), is rewritten as

$$\mathrm{Im}\, a_{IJ}^{2\to2} = \left|a_{IJ}^{2\to2}\right|^2 + \sum_{n>2}\left|a_{IJ}^{2\to n}\right|^2 \tag{25}$$

and required to hold order-by-order in perturbation theory, it becomes simply an equation for $\mathrm{Im}\, a_n$ in terms of lower-order amplitudes. In our case

$$\mathrm{Im}\, a_n^{2\to2} = \sum_{j=0}^{n-1} a_j^{2\to2} a_{n-j}^{2\to2\,*} + \sum_{2\le m\le \frac{n+1}{2}} \sum_{j=0}^{n+1-2m} a_j^{2\to2m} a_{n+1-2m-j}^{2\to2m\,*}\,. \tag{26}$$

In particular, to two loops,

$$\mathrm{Im}\, a_0 = 0,\quad \mathrm{Im}\, a_1 = |a_0|^2,\quad \mathrm{Im}\, a_2 = 2a_0 \mathrm{Re}\, a_1\,. \tag{27}$$

We will use these relations below.

Two constraints on the scattering amplitude are immediately evident from Fig. 4, specifically that $|\mathbf{a}| \le 1$

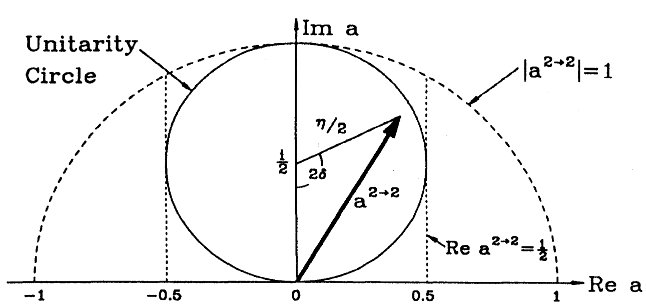

FIG. 3. The Argand diagram for two-body elastic-scattering amplitudes showing the unitarity circle within which the amplitude vectors $\mathbf{a}$ must lie, and the upper bounds $|\mathbf{a}_{IJ}^{2\to2}| = 1$ and $|\mathrm{Re}\, a_{IJ}^{2\to2}| = \frac{1}{2}$. For negligible inelastic processes, $\mathbf{a}^{2\to2}$ must terminate *on* the circle.

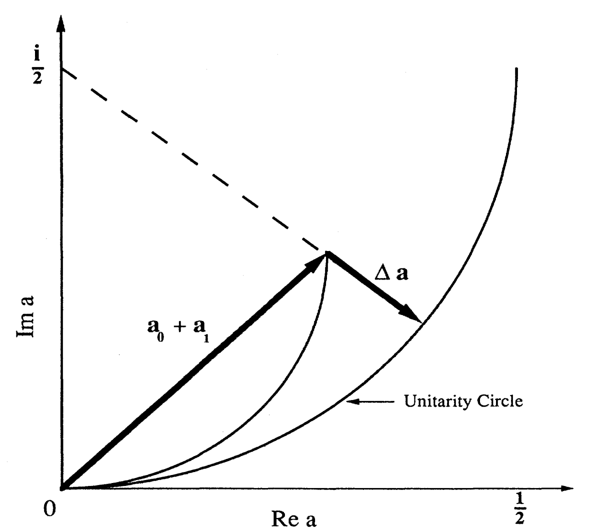

FIG. 4. Argand diagram indicating a typical behavior for a one-loop-order perturbative approximation to a (diagonal) elastic-scattering amplitude, $\mathbf{a}_0+\mathbf{a}_1$. $\Delta\mathbf{a}$ is the shortest vector which can be added to $\mathbf{a}_0 + \mathbf{a}_1$ to reach the unitarity circle, so $|\sum_{i=2}^{\infty} \mathbf{a}_i| \le |\Delta\mathbf{a}|$.

and that $|\mathrm{Re}\,\mathbf{a}| \le \frac{1}{2}$. Both have been used to obtain limits on the validity of low-order perturbation theory for the approximations $\mathbf{a} \approx \mathbf{a}_0$ and $\mathbf{a} \approx \mathbf{a}_0 + \mathbf{a}_1$. Both are minimal constraints when applied to perturbative amplitudes. Some allowance must clearly be made for small violations of the relations by a truncated perturbation series, and neither condition covers the possibility that $|\mathbf{a}_0 + \mathbf{a}_1|$ and $|a_0 + \mathrm{Re}\,a_1|$ have acceptable magnitudes while the vector $(\mathbf{a}_0 + \mathbf{a}_1)$ lies unacceptably far inside or outside the unitarity circle. We will use $\Delta\mathbf{a}$ to quantify such violations of unitarity.

The vector $\Delta\mathbf{a}$ is the vector of *minimum length* which can be added to $\mathbf{a}_0 + \mathbf{a}_1$ to bring the elastic scattering amplitude to the unitarity circle. If the scattering amplitude calculated to one loop is to be reliable numerically, we must certainly require that

$$|\Delta\mathbf{a}| \ll |\mathbf{a}_0 + \mathbf{a}_1|. \tag{28}$$

If we are to have a satisfactorily convergent perturbation series, we must require in addition that

$$|\Delta\mathbf{a}| \ll |\mathbf{a}_1| \ll |\mathbf{a}_0| \text{ and } |\Delta\mathbf{a}| \ll \tfrac{1}{2}. \tag{29}$$

The inequality $|\mathbf{a}_1| \ll |\mathbf{a}_0|$ gives a standard condition for the validity of the one-loop truncation of the perturbation series. The inequality $|\Delta\mathbf{a}| \ll |\mathbf{a}_1|$ sharpens the condition by requiring that the minimum higher-order contribution be small compared to $|\mathbf{a}_1|$. The final constraint compares $|\Delta\mathbf{a}|$ with the radius of the unitarity circle, and is included to cover cases in which the first conditions are satisfied, but $\mathbf{a}_0 + \mathbf{a}_1$ is still unacceptably far from the circle.

The magnitude of $\Delta\mathbf{a}$ is given by

$$\begin{aligned} |\Delta\mathbf{a}| &= \left|\tfrac{1}{2} - \sqrt{(a_0 + \mathrm{Re}\,a_1)^2 + (\tfrac{1}{2} - \mathrm{Im}\,a_1)^2}\,\right| \\ &= \left|\tfrac{1}{2} - \sqrt{\tfrac{1}{4} + 2a_0\mathrm{Re}\,a_1 + |a_1|^2}\,\right|, \end{aligned} \tag{30}$$

where we have used Eq. (27) in writing the second form. If we expand formally, we see that $|\Delta\mathbf{a}|$ involves terms of two-loop and higher orders, but reemphasize that we will *not* be able to expand when the approximate amplitudes are large.

The condition $|\mathbf{a}_1| \ll |\mathbf{a}_0|$ is frequently replaced in tests of the convergence of perturbation series by the condition

$$|\mathrm{Re}\,a_2| \ll |\mathrm{Re}\,a_1| \ll |\mathrm{Re}\,a_0| = |a_0|; \tag{31}$$

that is, the series for the real part of $\mathbf{a}$ is required to converge rapidly. Similarly, since $\mathrm{Im}\,a_0 = 0$, we should require for the imaginary part that

$$|\mathrm{Im}\,a_2| \ll \mathrm{Im}\,a_1, \tag{32}$$

or, using Eq. (27), that

$$2|\mathrm{Re}\,a_1| \ll |a_0|. \tag{33}$$

This inequality is stronger than the inequality $|\mathrm{Re}\,a_1| \ll |a_0|$ in Eq. (31), and may also be stronger than the condition $|\mathbf{a}_1| \ll |a_0|$ in Eq. (29). If a_0 and $\mathrm{Re}\,a_1$ have opposite signs, $\mathrm{Im}\,a$ vanishes to two loops when equality is reached in Eq. (32), and becomes negative for $2|\mathrm{Re}\,a_1| > |a_0|$ in violation of the relation $\mathrm{Im}\,a \ge 0$, Eq. (25), and the inequality should be strictly enforced. (We encountered this situation in an earlier analysis of the high-energy limits of $W_L^\pm$, Z_L, H scattering in the standard model [6].)

We cannot obtain limits such as those in Eq. (31) and (32) using $\mathrm{Re}\,\Delta a$ and $\mathrm{Im}\,\Delta a$ because the direction of the actual higher-order displacement vector $\delta\mathbf{a} = \mathbf{a} - \mathbf{a}_0 - \mathbf{a}_1$ is not known, and generally is not along $\Delta\mathbf{a}$. However, we can estimate $\mathrm{Re}\,a_2$ on the assumption that $|\mathbf{a}_3| \ll 1$, i.e., that $\mathbf{a}_2$ is the last significant term in the perturbation series. Using the known value of $\mathrm{Im}\,a_2$ and the condition that the vector $\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2$ lie on—or very near—the unitarity circle, we find that $|\mathrm{Re}\,a_2|$ has the (approximate) value

$$|\mathrm{Re}\,a_2|_{\mathrm{approx}} = \left|\,|a_0 + \mathrm{Re}\,a_1| - \sqrt{\tfrac{1}{4} - \left(\tfrac{1}{2} - \mathrm{Im}\,a_1 - \mathrm{Im}\,a_2\right)^2}\,\right|. \tag{34}$$

For this to be consistent with the assumption of rapid convergence, that is, $|\mathbf{a}_3| \ll |\mathbf{a}_2| \ll |\mathbf{a}_1|$, we should have

$$|\mathrm{Re}\,a_2|_{\mathrm{approx}} \ll |\mathrm{Re}\,a_1|. \tag{35}$$

If the inequality is not strongly satisfied, there is no reason to expect higher-order contributions to $\mathbf{a}$ to be small.

To summarize, we have the following sets of conditions for satisfactory convergence of the perturbation series, *all* of which must be satisfied: (A) $|\Delta\mathbf{a}| \ll |\mathbf{a}_1| \ll |\mathbf{a}_0|$ (evidence of convergence of the perturbation series); (B) $|\Delta\mathbf{a}| \ll |\mathbf{a}_0 + \mathbf{a}_1|$ (necessary if $\mathbf{a}_0 + \mathbf{a}_1$ is to give a satisfactory approximation for $\mathbf{a}$); (C) $|\boldsymbol{\Delta}\mathbf{a}| < \frac{1}{2}$ (magnitude of $\boldsymbol{\Delta}\mathbf{a}$ small compared to the radius of the unitarity circle); (D) $|\mathrm{Im}\,a_2| \ll |\mathrm{Im}\,a_1|$ or $|2\,\mathrm{Re}\,a_1| \ll |a_0|$, with $(\mathrm{sign}\ a_0)(a_0 + 2\,\mathrm{Re}a_1) \gtrsim 0$ (convergence of the series for $\mathrm{Im}\,a$, positivity of $\mathrm{Im}\,a$); (E) $|\mathrm{Re}\,a_2|_{\mathrm{approx}} \ll |\mathrm{Re}\,a_1| \ll |a_0|$ (evidence, with C, that $\mathbf{a}_2$ can be ignored).

Sets (A)–(C) involve the minimum-length vector $\Delta\mathbf{a}$ and, hence, give quite general constraints, but ones which may be weaker than the conditions in (D) and (E). Which constraints are most useful will vary from problem to problem. Just how much smaller "$\ll$ " signifies will also depend somewhat on the context, and on how conservative one wishes to be. We have nevertheless found the constraints to be quite useful in quantifying what is immediately evident from an Argand plot of an approximate scattering amplitude, namely the onset of an unacceptably large violation of elastic unitarity [29]. We will apply these conditions to the analysis of the low-energy scattering amplitudes in Eq. (10c) in the next section.

III. CONSTRAINTS ON LOW-ENERGY $W^\pm$, Z SCATTERING

A. Argand-diagram analysis

The $I = 0, 1, 2$ "isospin" amplitudes $a_{I,J}$ for the elastic scattering of low energy W's and Z's given in Eqs.

(10) are plotted on Argand diagrams as function of $\sqrt{s}$ for $M_H = 3, 5$, and 10 TeV in Figs. 5(a)–5(c). The results of our analysis of the leading inelastic-scattering processes—the $2 \rightarrow 4$ processes—show that the $2 \rightarrow 2$ elastic-scattering amplitudes must lie *on* the unitarity circle to very good approximation for the range of $\sqrt{s}$ shown in Fig. 5. This constraint is clearly violated by unacceptable amounts (and in different ways) by the one-loop approximations for a_{00} and a_{20}. In contrast, a_{11} remains close to the unitarity circle for $\sqrt{s} \lesssim 4$ TeV. Since a_{11} is independent of M_H except through the restriction $\sqrt{s} \ll M_H$ required for the validity of our approximations, we obtain no useful low-energy constraints from this amplitude.

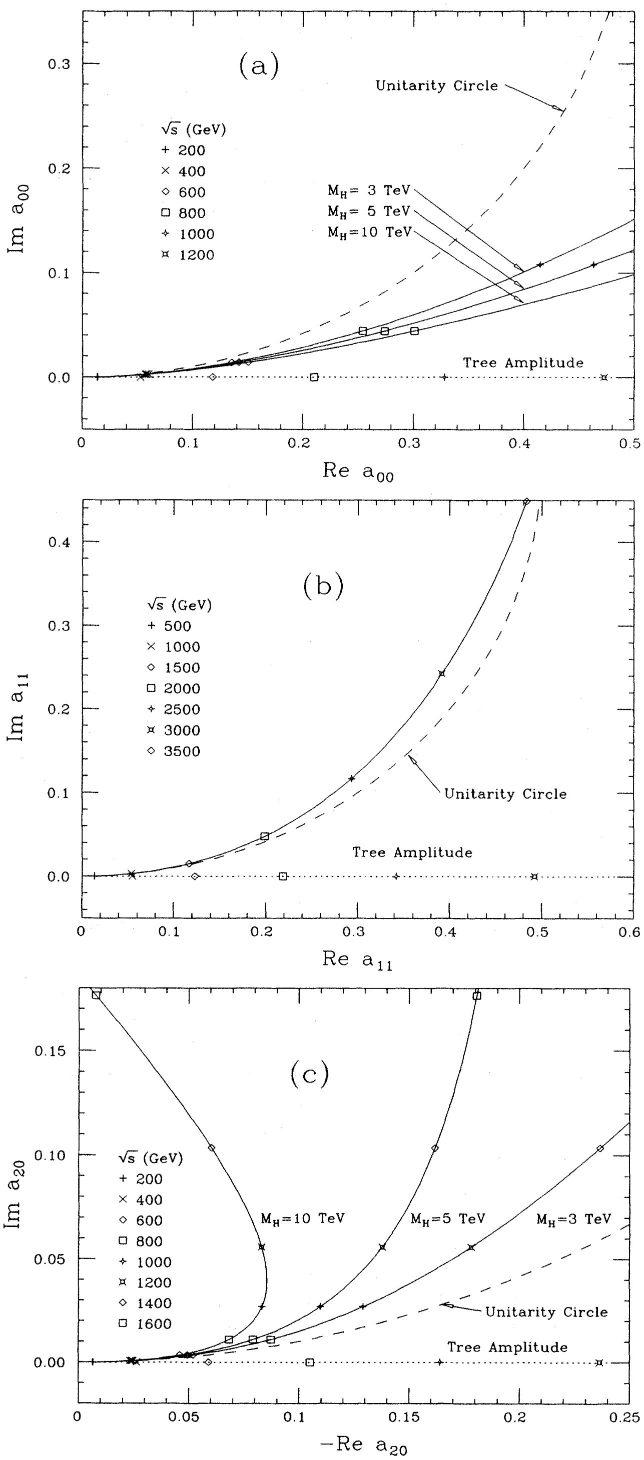

FIG. 5. Argand diagram giving the behavior of the one-loop elastic-scattering amplitudes $a_{I,J}$ of Eqs. (10) as functions of $\sqrt{s}$ for several values of M_H. (a) $I = J = 0$. (b) $I = J = 1$. (c) $I = 2$, $J = 0$.

It is evident from Fig. 5(a) that the one-loop corrections to $\mathrm{Re}\, a_{0,0}$ are positive and increase with increasing Higgs-boson mass. We would therefore expect to be able to obtain a stricter limit on the energy $\sqrt{s_{\rm cr}}$ at which the perturbative expansion breaks down than is given by the minimal tree-level constraint $|\mathrm{Re}\, a_{0,0}^{(0)}| < 1$ [14], $\sqrt{s_{\rm cr}} = 1.74$ TeV ($I = 0$). While the corrections to $\mathrm{Re}\, a_{2,0}$ are negative, the departure of $a_{2,0}$ from the unitarity circle shown in Fig. 5(c) is quite dramatic for large Higgs-boson masses, and we would again expect the one-loop constraints on $\sqrt{s_{\rm cr}}$ to be stronger than the tree-level constraint, $\sqrt{s_{\rm cr}} = 2.47$ TeV ($I = 2$).

The extent to which the one-loop amplitudes $a_{0,0}$ and $a_{2,0}$ in Eq. (10c) satisfy the various criteria in Eq. (A)–(E) is shown in Figs. 6(a)–6(f). In Figs. 6(a) and 6(b) we show the ratios $|\mathbf{a}_1/\mathbf{a}_0|$ and $|\Delta\mathbf{a}/\mathbf{a}_1|$. For a satisfactorily convergent perturbation series, we should have $|\Delta\mathbf{a}| \ll |\mathbf{a}_1| \ll |\mathbf{a}_0|$, with $|\Delta\mathbf{a}| \ll \frac{1}{2}$. If "$\ll$" is interpreted as a ratio of $\frac{1}{2}$, the inequalities in the first set are violated by the $I = 0$ amplitudes for Higgs-boson masses $M_H \gtrsim 3$ TeV and $\sqrt{s} > \sqrt{s_{\rm cr}}$, with $\sqrt{s_{\rm cr}} \lesssim 1.1$ TeV. The constraints from the $I = 2$ amplitudes are less stringent. There are no constraints for either isospin for $M_H < 3$ TeV. In particular, the curves for $M_H = 3$ TeV terminate in the range of $\sqrt{s}$ shown when $\sqrt{s}/M_H$ reaches the maximum values allowed by Eqs. (13); the curves for smaller M_H end at lower values of $\sqrt{s}$.

Even for a minimal restriction with "$\ll$" interpreted as equality, e.g., for *no* evidence of convergence of the perturbation series, the inequalities above are still violated for $I = 0$ at energies above a critical energy $\sqrt{s_{\rm cr}} \lesssim 1.5$ TeV provided $M_H \geq 5$ TeV. However, there are now no constraints (within our approximations [30]) for $M_H \lesssim 4$ TeV. The corresponding constraints from the $I = 2$ amplitudes are again much weaker, with no constraint from the ratio $|\Delta\mathbf{a}/\mathbf{a}_1|$. However, the condition $|\Delta\mathbf{a}/(\mathbf{a}_0 + \mathbf{a}_1)| \ll 1$ necessary for the complete one-loop amplitude to give a reasonable approximation to the full scattering amplitude *is* violated for $I = 2$, as is shown in Fig. 6(c), with $\sqrt{s_{\rm cr}} \lesssim 1.7$ TeV for $M_H > 5$ TeV, and no restriction for $M_H \lesssim 4$ TeV. Finally, the condition $|\Delta\mathbf{a}| \ll \frac{1}{2}$ is violated by the $I = 0$ amplitude for $M_H > 5$ TeV, with $\sqrt{s_{\rm cr}} \lesssim 1.4$ TeV, as shown in Fig. 6(d).

The foregoing analysis illustrates rather well the content of conditions (A)–(C) above: the perturbation series must converge (A), the full one-loop amplitude must be much larger than its *minimum* higher-order correction if the one-loop approximation is to be valid (B), and $|\Delta\mathbf{a}|$ must be smaller than about $\frac{1}{2}$, the radius of the unitarity circle. (D) and (E) deal with the imaginary and the

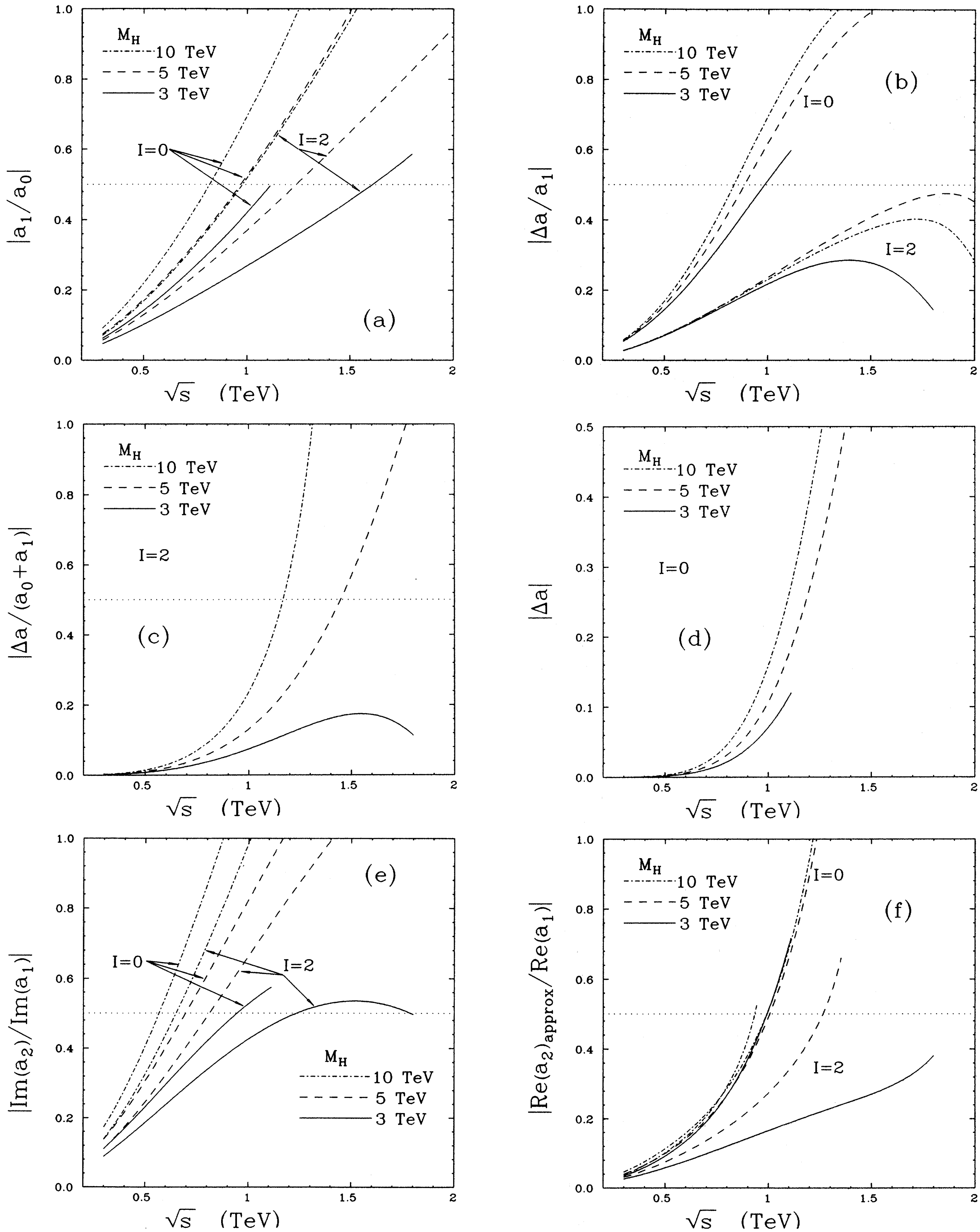

FIG. 6. Applications of the quantitative tests, in Eqs. (36) for the breakdown of perturbation theory to the $J = 0$, $I = 0, 2$ elastic-scattering amplitudes in Eqs. (10): (a) $|\mathbf{a}_1/\mathbf{a}_0| \ll 1$; (b) $|\boldsymbol{\Delta}\mathbf{a}/\mathbf{a}_1| \ll 1$; (c) $|\boldsymbol{\Delta}\mathbf{a}/(\mathbf{a}_0+\mathbf{a}_1)| \ll 1$ ($I = 2$ only); (d) $|\boldsymbol{\Delta}\mathbf{a}| \leq \frac{1}{2}$ ($I = 0$ only); (e) $|\mathrm{Im}\, a_2| \ll |\mathrm{Im}\, a_2|$; (f) $|\mathrm{Re}\, a_2|_{\mathbf{approx}} \ll |\mathrm{Re}\, a_1|$.

estimated real part of the two-loop amplitude.

In Fig. 6(e), we show the ratios $|\mathrm{Im}\, a_2/\mathrm{Im}\, a_1|$ of the (exact) two-loop contributions to $\mathrm{Im}\, a$ to the one-loop contributions for $I = 0$ and 2. The limits on $\sqrt{s_{\rm cr}}$ are somewhat more restrictive than those above. For a ratio of one-half, $\sqrt{s_{\rm cr}} \lesssim 0.9$ TeV for $M_H > 3$ TeV, or $\sqrt{s_{\rm cr}} < 0.7$ TeV for $M_H > 5$ TeV. For ratio of unity (*no* evidence of convergence of the perturbation series), $\sqrt{s_{\rm cr}} \lesssim 1.2$ TeV for $M_H > 5$ TeV. The estimates of $|\mathrm{Re}\, a_2/\mathrm{Re}\, a_1|$, based on the assumption that $|\mathbf{a}_3|$ is negligible, are shown in Fig. 6(f). The limits on $\sqrt{s_{cr}}$ are again more restrictive than those from Figs. 6(a)–6(d).

These results are summarized in Figs. 7(a)–7(c), where we show the critical energies obtained from the conditions $|\mathbf{a}_1| \ll |\mathbf{a}_0|$, $|\mathbf{\Delta a}| \ll |\mathbf{a}_1|$, and $|\mathrm{Im}\, a_2| \ll |\mathrm{Im}\, a_1|$, as functions of M_H. The results all indicate that perturbation theory will fail to converge satisfactorily, and the low energy standard model will in effect become strongly interacting at energies not much above 1 TeV for Higgs-boson masses greater than 5 TeV. There are essentially no unitarity constraints within the range of validity of our approximations for $M_H < 3$ TeV. The strongest constraints are those in Fig. 7(c) from $\mathrm{Im}\, a_2$.

B. Comments and conclusions

The results above are more restrictive, and more comprehensive in their tests of unitarity violation, than those of previous analyses of low-energy $W_L^\pm, Z_L$ scattering [7, 14, 19, 20]. The qualitative results are the same: the standard model must become strongly interacting at relatively low energies if the Higgs-boson mass is large, certainly, in our view, for $\sqrt{s} \gtrsim 1.2$ TeV for $M_H > 5$ TeV, and more optimistically, for $\sqrt{s} \gtrsim 0.7$ TeV. Since our strongest limit comes from the known two-loop contribution to $\mathrm{Im}\, a$ for $I = 0$, and a similar limit from the estimated value of $\mathrm{Re}\, a_2$, we regard it as unlikely that the limit can be extended much by going to higher orders in perturbation theory.

Our estimate of the contributions of $2 \to 4$ processes to the unitarity sum gives an independent determination of the energy by which the standard model becomes strongly interacting, $\sqrt{s_{\rm cr}} \lesssim 2.4$ TeV. This is not as strong as the limits from elastic scattering, but is independent of M_H for $M_H \gg \sqrt{s_{\rm cr}}$. The $2 \to 4$ processes contribute to $\mathrm{Im}\, a$ at the three-loop level; to reduce them to an acceptable size while treating $\mathrm{Re}\, a$ to the same order would require at least a four-loop calculation. Because of the s^4 variation of the $2 \to 4$ contributions to the unitarity sum, it is unlikely in any case that the limit could be extended much.

Earlier work [1–8] on *high-energy* $W_L^\pm, Z_L, H$ scattering, with $M_W \ll M_H \ll \sqrt{s}$, has shown that the standard model with an elementary Higgs field (or a high scale of compositeness) violates unitarity badly in low-order perturbation theory for rather low Higgs-boson masses, e.g., for $M_H > 350$–400 GeV if the theory is to remain valid up to energies of a few TeV [6], or for $M_H \gtrsim 155$ GeV in a large class of unified theories with unification scales of order 10^{14} GeV. Nonperturbative lat-

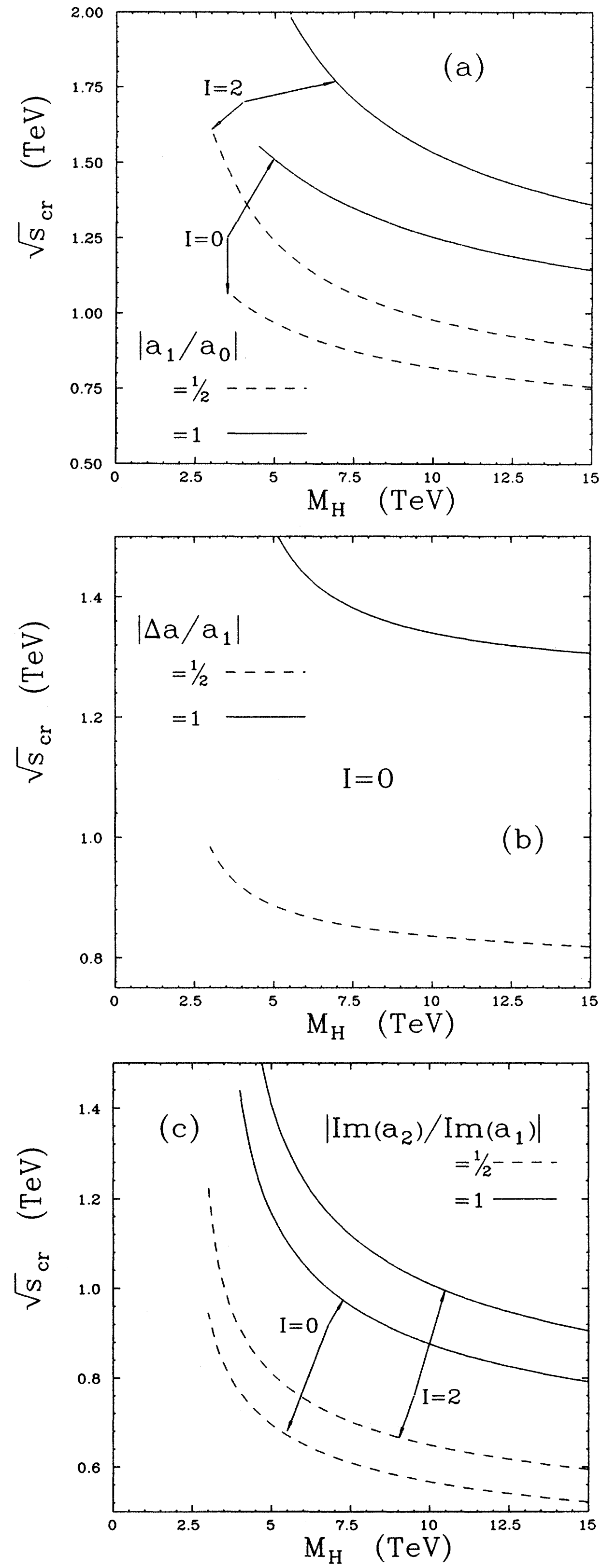

FIG. 7. Behavior of the critical energy $\sqrt{s_{\rm cr}}$ as a function of M_H obtained using the conditions (a) $|\mathbf{a}_1/\mathbf{a}_0| = \frac{1}{2}, 1$; (b) $|\mathbf{\Delta a}/\mathbf{a}_1| = \frac{1}{2}, 1$; (c) $|\mathrm{Im}\, a_2/\mathrm{Im}\, a_1| = \frac{1}{2}, 1$.

tice calculations [13] give upper bounds of around 650 GeV on M_H from the triviality limit. These bounds, and the restrictions obtained here, are summarized schematically in Fig. 8.

Unitarity analyses of $W_L^\pm, Z_L$ scattering which use the Goldstone boson equivalence theorem hold for $\sqrt{s} \gg M_W$. The region of the Higgs pole is excluded in either high-energy ($\sqrt{s} \gg M_H$) or low-energy ($\sqrt{s} \ll M_H$) analyses done to fixed order in the Higgs-boson self-coupling, such as that here. Values of M_H below the experimental lower limit are not of interest. The remaining regions in Fig. 8 are as follows.

(1) $M_H \lesssim 400$ GeV. The standard model has no significant unitarity violations in low-order perturbation theory for $\sqrt{s}$ of a few TeV [6]. If the standard model is embedded in a unified theory with a (typical) unification scale around 10^{14} GeV, the upper bound on M_H for a weakly interacting model decreases as shown to $M_H \lesssim 155$ GeV.

(2) 400 GeV $< M_H <$ 1 TeV: Large unitarity violations appear at the one-loop level for $\sqrt{s} \gg M_H$ [6], and the Higgs sector of the standard model is effectively strongly interacting even though $|\mathbf{a}|$ need not be large.

(3) $M_H >$ 1 TeV: The triviality bound for an elementary Higgs boson is violated. The Higgs boson may be composite, or electroweak symmetry breaking may be dynamical. In either case, a model with an effective Higgs boson is strongly interacting whether $\sqrt{s} \gg M_H > 1$ TeV, or (as here) $\sqrt{s} \gtrsim 0.7 - 1.2$ TeV with $M_H > 3 - 5$ TeV.

(4) $\sqrt{s} < \sqrt{s_{\rm cr}} \ll M_H$, with $\sqrt{s_{\rm cr}}$ dependent on M_H: There are no significant low-energy violations of unitarity at one loop. The theory is still strongly interacting at high energies.

The possibilities for observing strong interactions in $W_L^\pm, Z_L$ scattering, and of distinguishing the standard model from other models with the same low-energy limit, have been considered by a number of authors (see, e.g., Refs. 2, 3, 7, 14, and 31). We will conclude by noting simply that we have sharpened the ranges in $\sqrt{s}$ and M_H in which unitarity-based analyses are relevant.

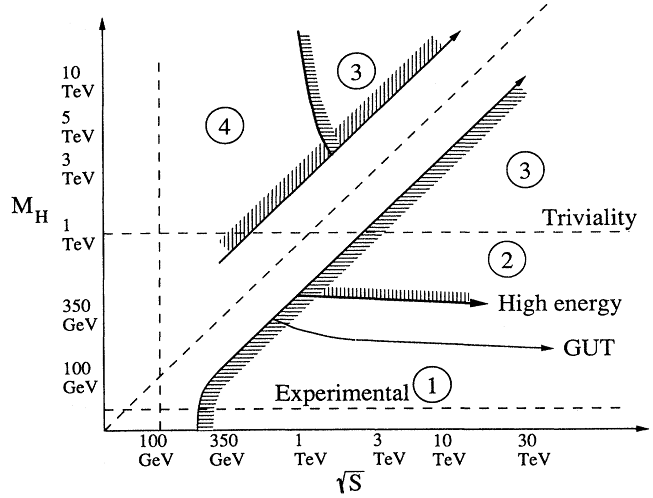

FIG. 8. Schematic summary of various constraints on perturbation theory and the standard-model Higgs-boson mass. The significance of the different regions is discussed in the text.

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[1] D.A. Dicus and V.S. Mathur, Phys. Rev. D **7**, 3111 (1973).
[2] B.W. Lee, C. Quigg, and H.B. Thacker, Phys. Rev. Lett. **38**, 883 (1977); Phys. Rev. D **16**, 1519 (1977).
[3] G. Passarino, Phys. Lett. **156B**, 231 (1985). This paper defines the coupling $G \approx G_F/\sqrt{2} = 1/2v^2$ in terms of the decay rate for $H \to zz$ rather than the vacuum expectation value v or the muon decay constant G_F, absorbing what would normally be regarded as radiative corrections to $\Gamma(H \to zz)$ in the definition of G. This leads to some differences in detail from the results of Refs. [4], [6], and [21].
[4] S. Dawson and S. Willenbrock, Phys. Rev. Lett. **62**, 1232 (1989).
[5] W. Marciano, G. Valencia, and S. Willenbrock, Phys. Rev. D **40**, 1725 (1989).
[6] L. Durand, J.M. Johnson, and J.L. Lopez, Phys. Rev. Lett. **64**, 1215 (1990).
[7] G. Passarino, Nucl. Phys. **B343**, 31 (1990).
[8] E. Lendvai, G. Pócsik, and T. Torma, ITP-Budapest Report No. 475, 1990 (unpublished).
[9] H. Hüffel and C. Pócsik, Z. Phys. C **8**, 13 (1981); R. Robinett, Phys. Rev. D **34**, 182 (1986).
[10] R. Casalbuoni, D. Dominici, R. Gatto, and C. Giunti, Phys. Lett. B **178**, 235 (1986); R. Casalbuoni, D. Dominici, F. Feruglio, and R. Gatto, *ibid.* **191**, 409 (1987); Nucl. Phys. **B299**, 117 (1988).
[11] L. Durand and J.L. Lopez, Phys. Lett. B **217**, 463 (1989); Phys. Rev. D **40**, 207 (1989); J.L. Lopez, Ph.D. thesis, University of Wisconsin–Madison, 1989.
[12] R. Dashen and H. Neuberger, Phys. Rev. Lett. **50**, 1897 (1983). For a review, see D.J.E. Callaway, Phys. Rep. **167**, 241 (1988).
[13] A. Hasenfratz, K. Jansen, C.B. Lang, T. Neuhaus, and H. Honeyama , Phys. Lett. B **199**, 531 (1987); J. Kuti, L. Lin, and Y. Shen, Phys. Rev. Lett. **61**, 678 (1988); M.

Lüscher and P. Weisz, Phys. Lett. B **212**, 472 (1989); A. Hasenfratz, K. Jansen, J. Jersák, C.B. Lang, T. Neuhaus, and H. Honeyama, Nucl. Phys. **B317**, 81 (1989).

[14] M.S. Chanowitz and M.K. Gaillard, Nucl. Phys. **B261**, 379 (1985).

[15] S. Dawson and S. Willenbrock, Phys. Rev. D **40**, 2880 (1989). The fraction 26/9 in the last line of Eq. (3.5) of this paper should be 26/3, as in the last line of our Eq. (5).

[16] M. Veltman and F. Ynduráin, Nucl. Phys. **B235**, 1 (1989) .

[17] R. Bouamrane, University of Michigan Report UM-TH-89-14, 1989 (unpublished).

[18] M. Chanowitz, M. Golden, and H. Georgi, Phys. Rev. D **36**, 1490 (1987).

[19] O. Cheyette and M.K. Gaillard, Phys. Lett. B **197**, 205 (1987).

[20] A. Dobado and M.J. Herrero, Phys. Lett. B **228**, 495 (1989).

[21] See, e.g., W. Marciano and S. Willenbrock, Phys. Rev. D **37**, 2509 (1988).

[22] J. Cornwall, D. Levin, and G. Tiktopoulos, Phys. Rev. D **10**, 1145 (1974); C. Vayonakis, Lett. Nuovo Cimento **17**, 383 (1976); G. Gounaris, R. Kögeler, and H. Neufeld, Phys. Rev. D **34**, 3257 (1986); Y. Yao and C. Yuan, Phys. Rev. D **38**, 2237 (1988).

[23] J. Bagger and C. Schmidt, Phys. Rev. D 41, 264 (1990); K. Aoki, Kyoto University Report No. RIFP-705, 1987 (unpublished); in *Physics at TeV Scale*, Proceedings of the Meeting, Tsukuba, Japan, 1988, edited by K. Hidaka and K. Hikasa (KEK, Tsukuba, 1988).

[24] H. Veltman, Phys. Rev. D **41**, 2294 (1990).

[25] A. Sirlin and R. Zucchini, Nucl. Phys. **B266**, 389 (1986).

[26] With the correction noted in Ref. 15, the low-energy results in Refs. 7 and 15–17 for the $J = 0$ amplitudes for $w^+w \to w^+w^-$, $w^+w^- \to zz$, and $zz \to zz$ agree, provided the coupling constant G in Ref. 7 is identified with $G = g^2/8M_W^2 = 1/2v^2 = G_F/\sqrt{2}$, and not with G_F as in that reference. However, the final results for the $I = 0, 2,\ J = 0$ amplitudes in Eq. (5.27) of Ref. 7 are incorrect. This is evident most obviously from the fact that $\mathrm{Im}\, a_{I,0}^{(1)} \neq \left(a_{I,0}^{(0)}\right)^2$ for the expressions given there.

[27] J.M. Johnson, Ph.D. thesis, University of Wisconsin–Madison, 1990.

[28] See Ref. 27, Appendix D, for a description of the programs and examples of their application. The programs are available on request from Dr. J.M. Johnson, Department of Physics, Wayne State University, Detroit, MI 48202.

[29] Passarino, [7], discussses three criteria for the breakdown of perturbative unitarity. His condition (i) is equivalent to $|a_0 + \mathrm{Re}\, a_1| < 1$ for $|\mathrm{Re} a_1| < |a_0|$, the only case of interest. This condition is equivalent to constraint (C) above, $|\Delta \mathbf{a}| < \frac{1}{2}$, for the most favorable value ($\frac{1}{2}$) for $\mathrm{Im} a_1$, and is otherwise somewhat weaker. His discussion of condition (ii)—which he does not us in his subsequent analysis—involves an unfortunate misinterpretation of the constraint introduced in Ref. [6], that the vector $\mathbf{a}_0 + \mathbf{a}_1$ lies acceptably close to the unitarity circle. This is the condition that $\Delta\mathbf{a}$ be small enough to be eliminated by (small) two-loop corrections to the perturbation series as in Eq. (29) above. Passarino interprets the constraint instead as the condition that $|\mathbf{a}_0 + \mathbf{a}_1 - \frac{i}{2}|^2 \leq \frac{1}{4}$, i.e., that the vector $\mathbf{a}_0 + \mathbf{a}_1$ lies on or inside the unitarity circle [$\mathbf{a}_0 + \mathbf{a}_1$ can perfectly well lie outside the circle; see, e.g., Fig. 5(a)], and derives his Eq. (3.7) from that condition. The important point is to quantify how far *off* the circle $\mathbf{a}_0 + \mathbf{a}_1$ can lie before this approximation to $\mathbf{a}$ becomes suspect; Eq. (3.7) does not provide a criterion for making this judgment. The final condition (iii) in Ref. [7] is obtained by converting an *identity* valid through two loops, $\mathrm{Im} a_{IJ}^{2\to 2} = |a_P^{2\to 2} IJ|^2$, to an inequality. The content of the identity is given in Eqs. (26) and (27) above. The inequality actually used in Ref. [7] is equivalent to $\mathrm{Im}\, a_2 = 2a_0 \mathrm{Re}\, a_1 \leq |\mathbf{a}_0| + |\mathbf{a}_1|$ which gives a restriction on the amplitudes only for $a_0 \mathrm{Re}\, a_1 > 0$. Condition (D) above is much stronger, and follows directly from Eqs. (27) and the convergence condition in Eq. (32).

[30] It is at this point that the analysis of the standard model in Ref. 20 goes awry. The result in Eq. (13) shows that the low-energy expansion for the standard model holds without significant correction only for $s \ll \frac{6}{11} M_H^2$ from Eq. (13), and the Higgs pole is of course encountered for $s = M_H^2$. These points are ignored in Ref. 20. In fact, as implied by the analysis above, and as is evident in Figs. 5(a) and 5(c), there is no significant unitarity violation in the standard model for the allowed range of $\sqrt{s}$ and the values of $M_H \leq 1.3$ TeV considered in Ref. 20. The low-energy expansion can of course be applied to other models as in Ref. 20, but the foregoing remarks should suggest some caution in the interpretation of the results. We remark also that one criterion used for unitarity violation in Ref. 20 is incorrect as stated: the phase shift $\delta_{0,0}$ *can* decrease with energy after reaching a maximum; causality restricts only the *rate* of decrease.

[31] J.F. Donoghue and C. Ramirez, Phys. Lett. B **234**, 361 (1990), and the references therein.