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Hamiltonian formulation of (2+1)-dimensional QED on the light cone

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It is shown that an improper regularization in the light-cone quantization of field theories can introduce a violation of rotational invariance as well as spurious divergences. Several methods are developed to avoid or cure these problems, as required for a consistent renormalization procedure of gauge theories. Based on these methods, the light-cone Hamiltonian for (2+1)-dimensional QED is constructed. Extension to gauge theories in 3+1 physical dimensions is also discussed.

I. INTRODUCTION

One of the main advantages of light-cone quantization in field theory is its manifest invariance under a maximally large subgroup of the Lorentz group [1] which contains even certain boost transformations. The corresponding generators of these "simple" transformations are nondynamical operators; i.e., they do not involve any interaction terms. Such nondynamical symmetries can be preserved under a wide class of approximations [2], such as, e.g., cutoffs in the number of particles. This feature greatly simplifies the task of constructing the Hamiltonian formulation of a relativistic field theory.

The price to pay for having simple generators of boost transformations is the occurrence of complicated and dynamical generators for certain rotations, which implies that angular momentum is not manifestly conserved in light-cone quantization. We will show that this results in a divergent structure of even superrenormalizable theories.

Rotational invariance is not a natural symmetry in light-cone quantization procedure since it mixes longitudinal and transverse degrees of freedom. In particular, an improper treatment of the short-distance singularities due to regularization will result, in a violation of rotational invariance. In fact, most approximations or regularizations (if infinities are present) will spoil the invariance under rotations which mix the $\mathbf{x}=(x^-,x^1,x^2)$ and x^+ directions [3]. In this paper we will concentrate on this aspect.

We will discuss several complementary approaches to this problem. The first approach using Pauli-Villars (PV) regularization softens the short-distance singularities and thus solves the cause of the problem, since it regularizes symmetrically in longitudinal and transverse coordinates [4]. The second approach starts from naive light-cone quantization. Any violations of rotational invariance, e.g., due to an improper treatment of the short-distance singularities, are then canceled by adding explicitly rotationally noninvariant terms to the light-cone Hamiltonian.

The resulting regularization and renormalization program has *a priori* nothing to do with the usual renormalizations of mass and charge. As a matter of fact, while infinite mass and charge renormalization are often unnecessary in less than 3+1 dimensions, the problems which are discussed here appear in any number of dimensions (except in 1+1, where there are no spatial rotations).

In order to emphasize this point, we will work mostly in 2+1 dimensions. This will help separate light-cone specific divergences and renormalizations from the usual ones. An extension of the techniques developed here to 3+1 dimensions will be discussed at the end of this paper.

II. PAULI-VILLARS REGULARIZATION OF THE LIGHT-CONE QUANTIZED YUKAWA MODEL

As a simple example, which exhibits many of the light-cone-related problems, we first consider the light-cone quantized Yukawa model

$$\mathcal{L}=\bar{\psi}(i\partial\!\!\!/-m)\psi-\phi(\Box+\lambda^2)\phi+\gamma\bar{\psi}\psi\phi\ , \tag{2.1}$$

in 2+1 dimensions. It is easy to study the violation of rotational invariance in this model since it is, in contrast with, e.g., gauge theories in the light-cone gauge, described by a fully covariant Lagrangian, and therefore even off-shell Green's functions should exhibit covariance. In particular, one should be able to express the fermion self-energy in the covariant form

$$\Sigma(p^\mu)=(p\!\!\!/-m)f_1(p^2)+f_2(p^2)\ . \tag{2.2}$$

If we want to analyze whether the self-energy in light-cone quantization is of the form (2.2), we need to know $\Sigma(p^\mu)$ off mass shell. Furthermore, we have to know the full γ-matrix structure and not only matrix elements of the self-energy between spinors. As pointed out in Refs. [5,6] there are two equivalent approaches to light-cone quantization [7]. The most familiar one is based on canonical quantization on an $x^+=$const surface (Hamiltonian formulation). The lower components of the fermion field ψ are defined through a constraint equation [8]

$$\psi_{\text{lower}}=(\partial_-)^{-1}\gamma^+(\gamma_\perp\partial_\perp-i\gamma\phi+im)\psi\ . \tag{2.3}$$

After the canonical light-cone Hamiltonian has been constructed in terms of upper components of ψ only, the constraint equation is no longer needed unless one is considering the γ-matrix structure of off-shell Green's functions such as $\Sigma(p^\mu)$. If one uses the above constraint equation, a new issue arises—namely, the renormalization of the mass m in the constraint equation and how it relates to the kinetic mass in the canonical Hamiltonian. We will come back to this point in the section about the noncovariant counterterms.

The second approach starts from Feynman perturbation theory. After performing the p^- integrations first (using complex contour integration), one obtains expressions which resemble expressions known from old-fashioned time-ordered perturbation theory. After regularization (in order to avoid ambiguities from infinite normal-ordering terms), both methods yield to equivalent results for physical observables. However, the second method avoids the problem of reconstructing the γ-matrix structure and is therefore more appropriate for the investigation in this section. In addition, the full structure of off-shell Green's functions in Feynman perturbation theory is not destroyed by doing a simple integration.

After these remarks we can finally start calculating the fermion self-energy. Naive light-cone perturbation theory with a regulator field of mass Λ yields [9], at one loop,

$$\mathrm{tr}(\Sigma\gamma^+)=cp^+\int_0^1 dx\int_{-\infty}^{\infty}dk_\perp\frac{1-x}{x(1-x)p^2-m^2x-\lambda^2(1-x)-(k_\perp-xp_\perp^2)}-(\text{“}\lambda\to\Lambda\text{”})\ , \tag{2.4}$$

$$\mathrm{tr}(\Sigma\gamma^-)=\frac{c}{p^+}\int_0^1 dx\int_{-\infty}^{\infty}dk_\perp\frac{[m^2+(p_\perp-k_\perp)^2]/(1-x)}{x(1-x)p^2-m^2x-\lambda^2(1-x)-(k_\perp-xp_\perp)^2}-(\text{“}\lambda\to\Lambda\text{”})\ , \tag{2.5}$$

where $c=\gamma^2/\pi$. Adding

$$0=\frac{1}{1-x}\,\frac{x(1-x)p^2-m^2x-\lambda^2(1-x)-(k_\perp-xp_\perp)^2}{x(1-x)p^2-m^2x-\lambda^2(1-x)-(k_\perp-xp_\perp)^2}-(\text{“}\lambda\to\Lambda\text{”}) \tag{2.6}$$

to the integrand in Eq. (2.5), one finds

$$\begin{aligned}\mathrm{tr}(\Sigma\gamma^-)&=\frac{c}{p^+}\int_0^1 dx\int_{-\infty}^{\infty}dk_\perp\frac{xp^2+m^2-\lambda^2+(1-x)p_\perp^2}{x(1-x)p^2-m^2x-\lambda^2(1-x)-(k_\perp-xp_\perp)^2}-(\text{“}\lambda\to\Lambda\text{”})\\&=\frac{c}{p^+}\int_0^1 dx\int_{-\infty}^{\infty}dk_\perp\frac{(1-x)(p^2+p_\perp^2)-(d/dx)[x(1-x)p^2-m^2x-\lambda^2(1-x)]}{x(1-x)p^2-m^2x-\lambda^2(1-x)-(k_\perp-xp_\perp)^2}-(\text{“}\lambda\to\Lambda\text{”})\\&=\frac{p^-}{p^+}\mathrm{tr}(\Sigma\gamma^+)-\frac{c\pi}{2p^+}\{[(m^2)^{1/2}-(\lambda^2)^{1/2}-(\text{“}\lambda\to\Lambda\text{”})]\}\ .\end{aligned} \tag{2.7}$$

Obviously, two conditions, namely, $\int d\lambda^2\rho(\lambda^2)=0$ and

$$\int d\lambda^2\lambda^{D_1}\rho(\lambda^2)=\int d\lambda^2\sqrt{\lambda^2}\rho(\lambda^2)=0\ ,$$

have to be imposed on the spectral density function of the scalar field to cancel the noncovariant term in curly brackets. Thus at least two PV particles are needed. This is rather unpleasant and perhaps unexpected, since, in a manifestly covariant approach, the fermion self-energy in $(2+1)$-dimensional Yukawa theory is finite by power counting. As we have seen here, in light-cone quantization, Σ is linearly divergent and extra degrees of freedom have to be introduced to make it finite and covariant. As far as perturbation theory is concerned, for theories with a manifestly covariant Lagrangian, the violation of rotational invariance is in principle not a problem. In any Green's function one calculates only the "good components", such as $\mathrm{tr}\Sigma\gamma^+$, and uses general relations, such as Eq. (2.2), to construct the "bad components." In the above example one could recover rotational invariance by defining

$$\mathrm{tr}\Sigma\gamma^-=\frac{p^-}{p^+}\mathrm{tr}\Sigma\gamma^+\ . \tag{2.8}$$

However, this does not work in gauge theories in the light-cone gauge (or any noncovariant gauge), since there $\Sigma(p^\mu)$ does not have such a simple structure as in Eq. (2.2). Furthermore, in the Hamiltonian formalism, one does not calculate $\Sigma(p^\mu)$ but on-mass-shell matrix elements thereof. Thus, in general, it will be technically more difficult to develop an algorithm for extracting the noncovariant piece. Nevertheless, the noncovariant terms still have observable effects which allow one to extract them. We will discuss this point later in the context of $(2+1)$-dimensional QED (QED_{2+1}).

One should emphasize that the term which violates the rotational invariance depends only on the external p^+, but not on $p_\perp$ or p^2. Furthermore, a simple calculation shows that $\mathrm{tr}(\Sigma\gamma^+)$ and $\mathrm{tr}(\Sigma\gamma^\perp)$ do not contain such extra terms. This implies that we can write (if we do not regularize properly)

$$\Sigma^{\mathrm{LC}}(p^\mu)=\Sigma^{\mathrm{cov}}(p^\mu)+\mathrm{const}\times\frac{\gamma^+}{p^+}\ , \tag{2.9}$$

which breaks covariance [10]. This is a general result which also holds for higher loops [11]—provided all noncovariant terms have been removed from subloops—and for other field theories such as, e.g., QED in the light-cone gauge. This has various practical consequences. First, one might be able to remove this term by adding only one kind of counterterm to the Hamiltonian (i.e., by changing the mass of the fermion in the kinetic energy term only and not the one in the vertex). Second, this allows one to develop simple subtraction procedures in perturbative calculations to get rid of such terms [12].

A last point which we are going to make in the context of $(2+1)$-dimensional Yukawa theory concerns the "overregularization" of the theory. As we mentioned already, there are no PV particles necessary in covariant perturbation theory, whereas here we needed two of them for a proper regularization at one loop. At higher loops the situation becomes a little better. There one PV particle is necessary and sufficient (provided subloops are rendered covariant) to guarantee a covariant result as demonstrated in the example in Appendix B. For those renormalizable theories where PV regularization poses no extra problems, such as QED_{3+1}, this means that once appropriate one-loop counterterms are added, no further additional regularization is necessary to render the theory covariant.

However, in non-Abelian gauge theories, PV regularization violates gauge invariance and we would have to restore it by further counterterms. We also emphasize, and this can also be read off from the example in Appendix B, that dimensional regularization does not take care of the noncovariant terms. The reason for this is that dimensional regularization in the transverse coordinate does not regularize the longitudinal coordinates.

To the order g^2 there is still a loophole from our conclusion that naive light-cone quantization leads to violations of rotational invariance. The normal-ordering terms in the Hamiltonian formulation can be treated in various ways. For example, we were using Pauli-Villars regularization, which removes those ambiguous contributions, but other methods are possible. In fact, one can find a cutoff method in Ref. [13] that renders the one-loop self-energies in $(1+1)$-dimensional Yukawa theory finite and covariant. Since the normal-ordering terms are of the order g^2, this will affect higher loops only through one-loop insertions; i.e., one cannot use ambiguities in the normal ordering to remedy noncovariant contributions to order g^4 or higher. In Appendix B we will show that it is not sufficient to insert the one-loop counterterms into the two-loop self-energy to make Σ covariant to the order g^4. This means essentially that, even if an approach different from ours is capable of avoiding noncovariant counterterms at order g^2 (as in Ref. [13]), such trems might still be needed in higher orders.

III. HAMILTONIAN FORMULATION FOR QED_{2+1} IN THE LIGHT-CONE GAUGE (PAULI-VILLARS REGULARIZATION)

We start our considerations from the QED Lagrangian in two space and one time dimensions with a gauge-fixing term ($n_\mu A^\mu = A^+$):

$$\mathcal{L}=\mathcal{L}_{\text{ferm}}+j_\mu A^\mu-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\lim_{\xi\to\infty}\frac{\xi}{2}(n_\mu A^\mu)^2 . \tag{3.1}$$

For the purpose of PV regularization (as well as if one wants to introduce an IR regulator), it is necessary to specify how to introduce a mass for the A field. One might be tempted to add just a term such as $(\Lambda^2/2)A_\mu A^\mu$ to Eq. (3.1). However, since $A_\mu A^\mu = A^+A^- - A_\perp^2 = -A_\perp^2$ (note that $A^+=0$), this means that only the $\perp$ degrees of freedom become massive, whereas the longitudinal degrees of freedom remain massless. In terms of the photon propagator, this means

$$D_\Lambda^{\mu\nu} = -\lim_{\xi\to\infty}[(k^2-\Lambda^2)g^{\mu\nu}+\xi n^\mu n^\nu - k^\mu k^\nu]^{-1}$$
$$= -\frac{g^{\mu\nu}-\dfrac{k^\mu n^\nu + k^\nu n^\mu}{kn}+\dfrac{\Lambda^2 n^\mu n^\nu}{(nk)^2}}{k^2-\Lambda^2+i\epsilon} ; \tag{3.2}$$

i.e., even at the tree level, the photon propagator does not vanish for $\Lambda^2\to\infty$ and the "instantaneous" contribution

$$\lim_{\Lambda\to\infty} D_\Lambda^{\mu\nu} = \frac{n^\mu n^\nu}{(nk)^2} \tag{3.3}$$

remains. What one has to do, in addition to adding a $(\Lambda^2/2)A_\mu A^\mu$ term to $\mathcal{L}$, is to introduce a dynamical longitudinal degree of freedom: a scalar field ϕ of mass Λ which couples with strength $e\Lambda/k^+$ to the current j^+, i.e.,

$$\delta\mathcal{L}_{\text{long}} = -\phi(\Box+\Lambda^2)\phi + ie\Lambda\phi\frac{1}{n^\mu\partial_\mu}n^\mu j_\mu . \tag{3.4}$$

The effect of this scalar field can be absorbed into the photon propagator, yielding

$$\tilde{D}_\Lambda^{\mu\nu}(\text{eff})$$
$$= D_\Lambda^{\mu\nu} + D_\Lambda^{\mu\nu}(\text{longitudinal}) = -\frac{g^{\mu\nu}-\dfrac{n^\mu k^\nu + n^\nu k^\mu}{nk}}{k^2-\Lambda^2} . \tag{3.5}$$

Since, for on-shell Green's functions the $n^\mu k^\nu$ terms do not contribute [14], all S-matrix elements should exhibit rotational invariance, even for finite Λ^2.

Having specified how to treat the A field, we can now proceed to construct the Hamiltonian. As a matter of convenience, we choose to represent the Hamiltonian using discrete light-cone quantization [15,16] (DLCQ). Except for the longitudinal field, this has been done already by Tang [17] for QED_{3+1} so that we do not have to go into the details. For one flavor of fermion ($b^\dagger$=fermion, $d^\dagger$=fermion) and one massive photon ($a^\dagger$=transverse photon, $c^\dagger$=longitudinal photon), one finds, in $2+1$ dimensions,

$$H = H_0 + V_{\text{flip}} + V_{\text{no flip}} + V_{\text{inst phot}}$$
$$+ V_{\text{long}} + V_{\text{inst ferm}} + V_{\text{NO}} , \tag{3.6}$$

where

$$H_0=\sum_{\mathrm{p}}\frac{1}{p}\left[\lambda^2+\left[\frac{p_\perp\pi}{L_\perp}\right]^2\right](a_{\mathrm{p}}^\dagger a_{\mathrm{p}}+c_{\mathrm{p}}^\dagger c_{\mathrm{p}})+\sum_{s,\mathrm{n}}\frac{1}{n}\left[m^2+\left[\frac{n_\perp\pi}{L_\perp}\right]^2\right](b_{s,\mathrm{n}}^\dagger b_{s,\mathrm{n}}+d_{s,\mathrm{n}}^\dagger d_{s,\mathrm{n}}) , \tag{3.7}$$

$$V_{\text{flip}}=\frac{em}{2\sqrt{\pi L_\perp}}\sum_{\mathrm{p,m,n}}\frac{a_{\mathrm{p}}}{\sqrt{p}}\left[(b_{\uparrow\mathrm{m}}^\dagger b_{\downarrow\mathrm{n}}-b_{\downarrow\mathrm{m}}^\dagger b_{\uparrow\mathrm{n}})\left[\frac{1}{m}-\frac{1}{n}\right]\delta^{(2)}_{\mathrm{n+p,m}}+(d_{\uparrow\mathrm{m}}^\dagger d_{\downarrow\mathrm{n}}-d_{\downarrow\mathrm{m}}^\dagger d_{\uparrow\mathrm{n}})\left[\frac{1}{n}-\frac{1}{m}\right]\delta^{(2)}_{\mathrm{n+p,m}}\right.$$
$$\left.+(b_{\uparrow\mathrm{m}}^\dagger d_{\uparrow\mathrm{n}}^\dagger-b_{\downarrow\mathrm{m}}^\dagger d_{\downarrow\mathrm{n}}^\dagger)\left[\frac{1}{m}+\frac{1}{n}\right]\delta^{(2)}_{\mathrm{n+m,p}}\right]+\mathrm{H.c.} , \tag{3.8}$$

$$V_{\text{no flip}}=\frac{e}{2L_\perp}\left[\frac{\pi}{L_\perp}\right]^{1/2}\sum_{s,\mathrm{p,m,n}}\frac{a_{\mathrm{p}}}{\sqrt{p}}\left[2\frac{p_\perp}{p}-\frac{n_\perp}{n}-\frac{m_\perp}{m}\right]\{b_{s\mathrm{m}}^\dagger b_{s\mathrm{n}}\delta^{(2)}_{\mathrm{n+p,m}}-d_{s,\mathrm{m}}^\dagger d_{s\mathrm{n}}\delta^{(2)}_{\mathrm{n+p,m}}+b_{s\mathrm{m}}^\dagger d_{-s\mathrm{n}}^\dagger\delta^{(2)}_{\mathrm{n+m,p}}\}+\mathrm{H.c.} , \tag{3.9}$$

$$V_{\text{inst phot}}=\frac{e^2}{\pi L_\perp}\sum_{s,t,\mathrm{k,l,m,n}}\{[k-m|n-l](b_{s\mathrm{k}}^\dagger d_{-t\mathrm{l}}^\dagger b_{s\mathrm{m}}d_{-t\mathrm{n}}-\tfrac{1}{2}b_{s\mathrm{k}}^\dagger b_{t\mathrm{l}}^\dagger b_{s\mathrm{m}}b_{t\mathrm{n}}-\tfrac{1}{2}d_{s\mathrm{k}}^\dagger d_{t\mathrm{l}}^\dagger d_{s\mathrm{m}}d_{t\mathrm{n}})$$
$$+[k-m|l+n](d_{s\mathrm{k}}^\dagger d_{t\mathrm{l}}d_{s,\mathrm{m}}b_{-t\mathrm{n}}+t_{s\mathrm{k}}^\dagger b_{t\mathrm{l}}b_{s\mathrm{m}}d_{-t\mathrm{n}})+\mathrm{H.c.}-b_{s\mathrm{k}}^\dagger d_{-s\mathrm{l}}^\dagger b_{t\mathrm{m}}d_{-t\mathrm{n}}[k+l|m+n]\} , \tag{3.10}$$

$$V_{\text{long}}=\frac{e\lambda}{\sqrt{\pi L_\perp}}\sum_{s,\mathrm{k,l,m}}\frac{1}{l^{3/2}}c_l ec[(b_{s\mathrm{k}}^\dagger b_{s,\mathrm{m}}-d_{s\mathrm{k}}^\dagger d_{s,\mathrm{m}})\delta^{(2)}_{\mathrm{k,m}+l}+b_{s\mathrm{k}}^\dagger d_{-s\mathrm{m}}^\dagger\delta^{(2)}_{l,\mathrm{k+m}}]+\mathrm{H.c.} , \tag{3.11}$$

$$V_{\text{inst ferm}}=\frac{e^2}{4\pi L_\perp}\sum_{s,\mathrm{p,q,m,n}}\frac{1}{\sqrt{pq}}\left\{a_{\mathrm{p}}^\dagger a_{\mathrm{q}}\left[b_{s\mathrm{m}}^\dagger b_{s\mathrm{n}}+d_{s\mathrm{m}}^\dagger d_{s\mathrm{n}}\right][\{p+m|q+n\}-\{p-n|q-m\}]\right.$$
$$+a_{\mathrm{p}}^\dagger a_{\mathrm{q}}^\dagger b_{s\mathrm{m}}d_{-s\mathrm{n}}\{p-m|-q+n\}+\mathrm{H.c.}$$
$$+a_{\mathrm{p}}^\dagger a_{\mathrm{q}}b_{s\mathrm{m}}d_{-s\mathrm{n}}[\{p-m|q+n\}-\{p-n|q+m\}]+\mathrm{H.c.} .$$
$$\left.+a_{\mathrm{p}}a_{\mathrm{q}}[b_{s\mathrm{m}}^\dagger b_{s\mathrm{n}}+d_{s\mathrm{m}}^\dagger d_{s\mathrm{n}}]\{p+n|-q+m\}+\mathrm{H.c.}\right\} . \tag{3.12}$$

Here

$$p,q=2,4,6,\ldots,\quad k,l,m,n=1,3,5,\ldots,\quad p_\perp,q_\perp,k_\perp,l_\perp,m_\perp,n_\perp=0,\pm1,\pm2,\ldots,\quad s,t=\uparrow,\downarrow , \tag{3.13}$$

$$\{m|n\}=\delta^{(2)}_{\mathrm{m,n}}\frac{1}{m},\quad [m|n]=\delta^{(2)}_{\mathrm{m,n}}\frac{1}{m^2} . \tag{3.14}$$

V_{NO} represents the normal-ordering terms which are part of the $O(e^2)$ contributions to the self-energies. Since they arise from instantaneous interactions, they are independent of particle masses and thus vanish in PV regularization [18].

We leave the explicit construction of the PV-regularized Hamiltonian to the Appendix A. For perturbative calculations we will weight the contributions from the various electrons and photons (physical and PV) with coefficients c_i^2 and c_j^2, which are later determined such that all unwanted terms vanish. E.g., the $O(e^2)$ contributions to the self-energy of a transverse photon with momentum **p** are ($\hat{p}_\perp=p_\perp\pi/L_\perp$)

$$\delta E_{\mathrm{p}}^{\text{trans}}=\frac{e^2}{4\pi L_\perp}\frac{1}{p}\sum_i c_i^2\sum_n\frac{m_i^2\left[\frac{1}{n}+\frac{1}{p-n}\right]^2+\left[\frac{2\hat{p}_\perp}{p}-\frac{\hat{n}_\perp}{n}-\frac{\hat{p}_\perp-\hat{n}_\perp}{p-n}\right]^2}{\frac{\lambda^2+\hat{p}_\perp^2}{p}-\frac{m_i^2+\hat{n}_\perp^2}{n}-\frac{m_i^2+(\hat{p}_\perp-\hat{n}_\perp)^2}{p-n}} . \tag{3.15}$$

In order to obtain finite results in the continuum limit, we have to require $\sum_i c_i^2=0$. This allows us to simplify the numerator by using the replacement

$$\left[\hat{n}_\perp-\hat{p}_\perp\frac{n}{p}\right]^2\to -m_i^2+\frac{\lambda^2}{p}\left[\frac{1}{n}+\frac{1}{p-n}\right]^{-1} ,$$

i.e.,

$$\delta E_{\mathbf{p}}^{\text{trans}}=\frac{e^2}{4\pi L_\perp}\frac{1}{p}\sum_i c_i^2\sum_n \frac{4m_i^2+\lambda^2\frac{(p-2n)^2}{p^2}}{n(p-n)\left\{\frac{\lambda^2}{p}-\left[\frac{1}{n}+\frac{1}{p-n}\right]\left[\left[\hat{n}_\perp-\hat{p}_\perp\frac{n}{p}\right]^2+m_i^2\right]\right\}}$$

$$=\frac{e^2}{4\pi L_\perp}\frac{1}{p}\sum_i c_i^2\frac{4m_i^2+\lambda^2\left[1-8\frac{n}{p}\left[1-\frac{n}{p}\right]\right]}{n(p-n)\left\{\frac{\lambda^2}{p}-\left[\frac{1}{n}+\frac{1}{p-n}\right]\left[\left[\hat{n}_\perp-\hat{p}_\perp\frac{n}{p}\right]^2+m_i^2\right]\right\}}+\delta E_{\mathbf{p}}^{\text{long}}\ , \tag{3.16}$$

where we have already separated the self-energy of a longitudinal photon:

$$\delta E_{\mathbf{p}}^{\text{long}}=\frac{e^2}{4\pi L_\perp}\frac{1}{p}\sum_i e_i^2\sum_n \frac{4\lambda^2/p^2}{\frac{\lambda^2}{p}-\left[\frac{1}{n}+\frac{1}{p-n}\right]\left[\left[\hat{n}_\perp-\hat{p}_\perp\frac{n}{p}\right]^2+m_i^2\right]}\ . \tag{3.17}$$

In the continuum limit the self-energies of longitudinal and transverse photons must be equal—otherwise, rotational invariance is broken. To analyze this condition further, we transform this term into an integral,

$$\delta E^{\text{trans}}-\delta E^{\text{long}}\to\frac{e^2}{4\pi^2}\frac{1}{p}\sum_i c_i^2\int_0^1 dx\int_{-\infty}^{\infty}dk_\perp\frac{4m_i^2+\lambda^2[1-8x(1-x)]}{\lambda^2x(1-x)-m_i^2-k_\perp^2}=-\frac{e^2}{\pi}\frac{1}{p}\sum_i c_i^2(m_i^2)^{1/2}\ , \tag{3.18}$$

and our second PV condition has to be $\sum_i c_i^2(m_i^2)^{1/2}=0$.

We have performed similar calculations for the on-shell self-energy of an electron. Since this is a gauge-invariant quantity, we can require that our calculation in the light-cone gauge and light-cone quantization reproduces the covariant result obtained in Feynman gauge and $(2+1)$-dimensional symmetrical integration. An alternative approach, which will be elaborated on in more detail in the next section, is to calculate the one-loop corrections to the Compton cross section and compare with well-known results. Both methods lead to the same condition, namely,

$$\sum_j c_j^2=0,\quad \sum_j c_j^2(\lambda_j^2)^{1/2}=0\ . \tag{3.19}$$

For practical calculations it is useful to reduce the number of PV conditions. To achieve this one can add a counterterm to the Hamiltonian which cancels those terms which are multiplied to $c_i^2(m_i^2)^{1/2}$ and $c_j^2(\lambda_j^2)^{1/2}$ in the self-energies of photons and electrons, respectively. At one loop this reduces, by construction, the number of PV conditions required. However, and this is a highly nontrivial result, numerical calculations of the self-energies as well as the example in Appendix B show that this is also true for higher loops; i.e., the second PV particle is only necessary at one loop. Once we avoid it by adding a suitable one-loop counterterm, there is only one PV particle needed at two loops and most probably (we have not checked this numerically) also for higher loops.

There might be various reasons for this special behavior at one loop. First of all, there are ambiguities in how to treat normal-ordering divergences which are of $O(e^2)$ and contribute only to the one-loop self-energies. Second, power counting in light-cone coordinates is different from the usual covariant power counting [19]. One has to count separately powers in $k_\perp$ and $1/k^+$ in order to properly estimate the degree of divergence. Here it turns out that the strongest divergence (e.g., a linear $k_\perp$ divergence in $2+1$ dimensions) occurs only at the one-loop level. The situation here is similar to scalar QED in equal-time quantization.

IV. RENORMALIZATION USING NONCOVARIANT COUNTERTERMS

QED_{2+1} is superrenormalizable, and only two graphs are superficially divergent in Feynman perturbation theory (the one- and two-loop vacuum polarizations are finite if gauge-invariant regularization is used). However, the presence of terms which break rotational invariance has forced us to introduce four PV particles (two photons and two electrons); i.e., the Fock space content of the theory has increased considerably. Even after calculating the one-loop counterterms by hand, one has to deal with one PV photon and one PV electron; i.e., the number of degrees of freedom still increases by a factor of 4 compared to the unregularized theory.

Furthermore, practical calculations require in general some approximations which in general lead to further violations of rotational invariance [20]. In this work we deal only with those violations of rotational invariance which are induced by an improper treatment of the high-energy degrees of freedom (large $k_\perp$, small x) if no PV regularization, or anything equivalent, is applied. (The methods, which we are going to develop for the latter problem, should, however, also be applicable for approximation-induced effects.)

Using the light-cone power-counting rules, one shows that light-cone QED in $3+1$ and $2+1$ dimensions is renormalizable [19]. This implies that the violations of rotational invariance (which in our case are induced by an improper handling of arbitrarily high energies) can be compensated by a redefinition of terms in the Hamiltonian. In general, such renormalization procedures can be quite lengthy since, at least in principle, the e^- masses which appear in the kinetic energy and in the vertex, the various e^- charges, and the various photon masses can all require different renormalizations i.e., instead of three renormalization constants (m,λ,e), we would have to deal with nine $(m_{\rm kin}, m_{\rm vertex}, e_{\rm flip}, e_{\rm no\ flip} e_{\rm no\ flip}, e_{\rm inst\ phot}, e_{\rm inst\ ferm}, \lambda_{\rm long}, \lambda_{\rm trans}, \lambda_{\rm vertex})$. However, practical calculations [21] have shown that violations of rotational invariance in the light-cone (LC) gauge occur only in two-point functions and there only in a very specific form [22]: namely,

$$\Sigma=\Sigma^{\rm PV}+\frac{\gamma^+}{p^+}c_1, \quad \Pi^{\mu\nu}=\Pi^{\mu\nu}_{\rm PV}+\delta^{\mu\perp}\delta^{\nu\perp}c_2 , \tag{4.1}$$

for electron and photon self-energies, respectively; i.e., the deviations from the PV regularized results, which lead to rotational invariant observables, can be parametrized by only two additional constants c_1, c_2. The burden of fitting nine renormalization constants has thus been reduced to fitting five. In practice, one adds two extra counterterms:

$$\begin{aligned}\delta H^{(1)}&=\sum_{s,n}\frac{b^\dagger_{sn}b_{sn}+d^\dagger_{sn}d_{sn}}{n}\delta m^2_{\rm kin} ,\\ \delta H^{(2)}&=\sum_{\rm p}\frac{a^\dagger_{\rm p}a_{\rm p}}{p}\delta\lambda^2_{\rm trans} ,\end{aligned} \tag{4.2}$$

to the Hamiltonian and adjusts $\delta m^2_{\rm kin}$ and $\delta\lambda^2_{\rm trans}$ such that rotational invariance is restored (this point will be discussed below). The next step, which is not necessary in QED$_{2+1}$, would then be the usual mass and charge renormalization [23,24].

The constants $\delta m^2_{\rm kin}$ and $\delta\lambda^2_{\rm trans}$ are determined as follows. Fixing $\delta\lambda^2_{\rm trans}$ is rather easy: One diagonalizes the Hamiltonian (within some approximations such as, e.g., the cutoff in Fock space) for a given $\delta\lambda^2_{\rm trans}$ and compares the physical masses (eigenvalues of the Hamiltonian) of longitudinal and transverse photons. $\delta\lambda^2_{\rm trans}$ is then tuned until these eigenvalues coincide.

For $\delta m^2_{\rm kin}$ two methods are suggested. The first method is based on the fact that instantaneous e^- exchange becomes singular for small p^+ transfer (e.g., in Compton backscattering). This is, of course, an unphysical singularity which has to be canceled by noninstantaneous e^- exchange. At the tree level it is crucial for the cancellation that the kinetic mass of an electron (m in H_0 [Eq. (3.7)]) equals the vertex mass (m in $V_{\rm flip}$ [Eq. (3.8)]). At one loop the interaction will renormalize $m_{\rm kin}$ and $m_{\rm vertex}$ differently and one can easily convince oneself that the cancellation will be spoiled unless one renormalizes $m_{\rm kin}$ differently from $m_{\rm vertex}$. This defines already the renormalization procedure, namely, tuning $m^2_{\rm kin}$ until finiteness of the Compton backscattering amplitude for zero p^+ transfer is achieved.

The second method uses the degeneracy of the positronium spectrum due to rotational invariance. A glance at the Hamiltonian [Eq. (3.6)] shows that, for zero perpendicular momenta, an annihilation of an e^+e^- pair into a transverse photon is possible if and only if both have a parallel spin, but not for the $S=1$, $S_z=0$ state. Another annihilation process is possible via longitudinal or instantaneous photons, but only from the $S=1$, $S_z=0$ state. In the first case the vertex mass appears, whereas in the second it does not. For degeneracy of the $S_z=0,\pm 1$ states, it is important that both interactions have the same strength. Again, this is achieved at the tree level by choosing $m_{\rm kin}=m_{\rm vertex}$, but if loops are taken into account, the condition changes. Degeneracy of the $S_z=0,\pm 1$ states in the ground state of positronium can thus be used as a renormalization condition.

The first method seems to be superior from a practical point of view, since it requires one to look at the $e^-\gamma$ system only and not at $e^-e^+\gamma$ states, as in the second method. However, from a practical point of view, we are interested in the positronium spectrum; i.e., we diagonalize the Hamiltonian. The second method thus requires only little effort to implement—namely, diagonalizing H for two spin configurations and repeating this a few times (to fit δm^2_k iteratively). Furthermore, and this will also be of practical importance, the renormalization constants will thus be evaluated automatically to the same loop order and with the same approximations as the actual positronium calculations are done.

V. EXTENSION TO 3+1 DIMENSIONS

In the Pauli-Villars approach and for those theories considered in this work Yukawa theory and QED), an extension to $3+1$ dimensions is straightforward. The only difference will be that more coefficients have to be renormalized and that there will be in general an infinite renormalization.

In practice, the following steps have to be performed. If one wants to render all loops covariant, i.e., even the one-loop graphs, using PV, there will be three PV conditions for photons and electrons: namely [25],

$$\begin{aligned}&\int d\lambda^2\rho(\lambda^2)=0 ,\\ &\int d\lambda^2\lambda^2\rho(\lambda^2)=0 ,\\ &\int d\lambda^2\lambda^2\ln\lambda^2\rho(\lambda^2)=0 ,\end{aligned} \tag{5.1}$$

which is awkward from a numerical point of view. Thus one should only use the improved version of the PV approach, where the one-loop counterterms are constructed "by hand" and only one PV condition has to be imposed for higher loops. The number of degrees of freedom will thus be the same as in a covariant approach (e.g., Euclidean integration) with PV regularization. We were not able to prove this to all orders. However, numerical results in QED$_{3+1}$ for $(g-2)$ of an electron to three loops and a test involving Ward identities up to two loops have been positive.

The method of noncovariant counterterms might also be very useful. For example, if one uses a kinetic energy cutoff, further violations of rotational invariance are induced. As far as the self-energies are concerned, the algorithm described in Sec. IV would automatically remedy this without further effort. However, there might be additional corrections in the vertices which need further consideration. For a discussion at the one-loop level, see Ref. [5].

The extension to non-Abelian gauge theories involves additional problems. All methods discussed in this work violate local gauge invariance, at least in intermediate steps. For QED this is not a problem since, e.g., the PV regularization preserves the Ward identities. In QCD this is not the case and one has to add further gauge-breaking counterterms which restore gauge invariance. [26]

VI. SUMMARY AND CONCLUSION

Naive light-cone quantization without careful regularization or noncovariant renormalization violates rotational invariance. Even dimensional regularization in the transverse direction is not sufficient to guarantee rotational invariance without noncovariant counterterms. In theories with a covariant Lagrangian, we have demonstrated this by investigating the covariant structure of self-energies. In the case of a noncovariant Lagrangian (QED) in the light-cone gauge), the Lorentz transformation properties of Green's functions are nontrivial and therefore possible violations of Lorentz invariance are not obvious.

However, these effects must show up in the calculation of physical processes. To study them it is convenient to select those processes which are sensitive to violation of its covariant structure as well as technically rather easy to deal with. In QED the degeneracy of the triplet positronium state with parallel and antiparallel spin as well as Compton backscattering are such processes.

The violation of rotational invariance is not limited to one loop, although one might expect this since normal-ordering ambiguities arise only in one-loop self-energies. In fact, unless regularized properly, the normal-ordering contributions lead to violation of rotational symmetry. However, those terms are not the only source of violations of this kind, as our explicit two-loop calculations show. The induced divergences are less severe there, though.

We have discussed two basic methods to restore rotational invariance, the Pauli-Villars method and the method of noncovariant counterterms. Both methods seem to require a large number of additional degrees of freedom or counterterms. However, because of the specific structure of rotational invariance violation in light-cone quantization—the worst problems are restricted to one loop and only certain components of two-point functions (the γ^+ component of the fermion self-energy and the $\perp\perp$ components of the vacuum polarization) are affected. This allows us to optimize these methods considerably. We give analytic expressions for one-loop counterterms. As a result, the PV approach then requires only one ghost per particle to offset the violations of rotational symmetry at higher loops.

The method of noncovariant counterterms requires only two additional counterterms (compared to a manifest covariant approach): namely, a mass term for transverse photons and an additional correction to the fermion mass term which appears at spin-flip photon-electron vertices. To fix the additional constants, one has to specify the renormalization conditions. This can be achieved by considering the degenerate ground state of positronium as well as the degeneracy of the longitudinal and transverse photons.

While finishing this work, we received several papers dealing with similar subjects [5,13]. Like our work, Ref. [5] considers QED in the $A^+=0$ gauge using canonical quantization on an $x^+=$const hyperplane. They concentrate on $3+1$ dimensions in one loop; i.e., in addition to the "light-cone artifacts" (to be worked out in the following sections), they have to handle the usual infinities of QED_{3+1}. Furthermore, they use a method different from ours to renormalize the noncovariant terms arising in a perturbative treatment of the radiative corrections. Since a direct comparison of results was not possible, we do not know whether the differences in the methods have any physical consequences. Another method which is similar to ours has become known as the light-front Tamm-Dancoff approach [13]. These authors concentrate in their renormalization procedure more on effects introduced by approximations (such as, e.g., cutoffs in the Fock-space expansion) and on a systematic improvement in the context of such approximation schemes. It might be that the renormalization programs in Refs. [5,13] are capable to cover the problems addressed in our work. However, our approach offers an alternative formulation to these works, and more importantly, the problems discussed in this work have nothing to do with the usual divergences in QED_{3+1} or approximations such as a cutoff in the Fock expansion. They are artifacts which are induced by an improper regularization of continuum light-cone field theory and occur even in superrenormalizable theories in a loop expansion.

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APPENDIX A: PAULI-VILLARS REGULARIZED HAMILTONIAN FOR QED_{2+1}

As discussed in the section about Pauli-Villars regularization, one Pauli-Villars condition

$$\int dm^2 \rho_e(m^2)=0 , \tag{A1}$$

$$\int d\lambda^2 \rho_\gamma(\lambda^2)=0 , \tag{A2}$$

for electrons and photons, respectively, is sufficient to guarantee covariant regularization in all calculations beyond one loop—provided all one-loop subgraphs have been rendered covariant (e.g., by constructing the necessary one-loop counterterms). One can easily convince oneself that the sum rules (A1) and (A2) can be achieved by introducing one additional electron field and photon field, respectively, which are quantized with the wrong metric. One way to do so in practice is to introduce an extra factor of $\sqrt{-1}$ for all heavy-photon vertices and another factor of $\sqrt{-1}$ for all heavy-electron pair creation and annihilation vertices. In addition, the heavy electron has to be quantized as a boson.

In practice, this implies

$$H^{\rm PV}=H_0+V_{\rm flip}+V_{\rm no\ flip}+V_{\rm long}+V_{\rm inst\ ferm}+V_{\rm 1\ loop}\ ,$$

where

$$H_0=\sum_{\mathbf{p}}\frac{1}{p}\left[\left[\frac{p_\perp\pi}{L_\perp}\right]^2+\lambda^2\right](a^\dagger_{\mathbf{p}}a_{\mathbf{p}}+c^\dagger_{\mathbf{p}}c_{\mathbf{p}})+\sum_{s\mathbf{n}}\frac{1}{n}\left[\left[\frac{n_\perp\pi}{L_\perp}\right]^2+m_e^2\right](b^\dagger_{s,\mathbf{n}}b_{s,\mathbf{n}}+d^\dagger_{s,\mathbf{n}}d_{s,\mathbf{n}})$$
$$+\sum_{\mathbf{p}}\frac{1}{p}\left[\left[\frac{p_\perp\pi}{L_\perp}\right]^2+\Lambda^2\right](A^\dagger_{\mathbf{p}}A_{\mathbf{p}}+C^\dagger_{\mathbf{p}}C_{\mathbf{p}})+\sum_{s\mathbf{n}}\frac{1}{n}\left[\left[\frac{n_\perp\pi}{L_\perp}\right]^2+M^2\right](B^\dagger_{s,\mathbf{n}}B_{s,\mathbf{n}}+D^\dagger_{s,\mathbf{n}}D_{s,\mathbf{n}})\ , \quad \text{(A4)}$$

$$V_{\rm flip}=\frac{e}{2\sqrt{\pi L_\perp}}\sum_{\mathbf{pmn}}\frac{a_{\mathbf{p}}+i\,\mathbf{A}_{\mathbf{p}}}{\sqrt{p}}\left[[m_e(b^\dagger_{\uparrow\mathbf{m}}b_{\downarrow\mathbf{n}}-b^\dagger_{\downarrow\mathbf{m}}b_{\uparrow\mathbf{n}})+M(B^\dagger_{\uparrow\mathbf{m}}B_{\downarrow\mathbf{n}}-B^\dagger_{\downarrow\mathbf{m}}B_{\uparrow\mathbf{n}})]\left[\frac{1}{m}-\frac{1}{n}\right]\delta^{(2)}_{n+p,m}\right.$$
$$-[m_e(d^\dagger_{\uparrow\mathbf{m}}d_{\downarrow\mathbf{n}}-d^\dagger_{\downarrow\mathbf{m}}d_{\uparrow\mathbf{n}})+M(D^\dagger_{\uparrow\mathbf{m}}D_{\downarrow\mathbf{n}}-D^\dagger_{\downarrow\mathbf{m}}D_{\uparrow\mathbf{n}})]\left[\frac{1}{m}-\frac{1}{n}\right]\delta^{(2)}_{n+p,m}$$
$$\left.+[m_e(b^\dagger_{\uparrow\mathbf{m}}d^\dagger_{\uparrow\mathbf{n}}-b^\dagger_{\downarrow\mathbf{m}}d^\dagger_{\downarrow\mathbf{n}})+iM(B^\dagger_{\uparrow\mathbf{m}}D^\dagger_{\uparrow\mathbf{n}}-B^\dagger_{\downarrow\mathbf{m}}D^\dagger_{\downarrow\mathbf{n}})]\left[\frac{1}{m}+\frac{1}{n}\right]\delta^{(2)}_{n+m,p}\right]+\text{“H.c.”}\ , \quad \text{(A5)}$$

$$V_{\rm no\ flip}=e\left[\frac{\pi}{L_\perp}\right]^{1/2}\frac{1}{2L_\perp}\sum_{s\mathbf{pmn}}\frac{a_{\mathbf{p}}+iA_{\mathbf{p}}}{\sqrt{p}}\left[\frac{2p_\perp}{p}-\frac{n_\perp}{n}-\frac{m_\perp}{m}\right]$$
$$\times[(b^\dagger_{s\mathbf{m}}b_{s\mathbf{n}}+B^\dagger_{s\mathbf{m}}B_{s\mathbf{n}})\delta^{(2)}_{n+p,m}-(d^\dagger_{s\mathbf{m}}d_{s\mathbf{n}}+D^\dagger_{s\mathbf{m}}D_{s\mathbf{n}})\delta^{(2)}_{n+p,m}$$
$$+(b^\dagger_{s\mathbf{m}}d^\dagger_{-s\mathbf{n}}+iB^\dagger_{s\mathbf{m}}D^\dagger_{-s\mathbf{n}})\delta^{(2)}_{p,n+m}]+\text{“H.c.”}\ , \quad \text{(A6)}$$

$$V_{\rm long}=\frac{e}{\sqrt{\pi L_\perp}}\sum_{s\mathbf{kpm}}\frac{1}{p^{3/2}}(\lambda c_{\mathbf{p}}+i\Lambda C_{\mathbf{p}})[(b^\dagger_{s\mathbf{k}}b_{s\mathbf{m}}+B^\dagger_{s\mathbf{k}}B_{s\mathbf{m}})\delta^{(2)}_{m+p,k}$$
$$-(d^\dagger_{s\mathbf{k}}d_{s\mathbf{m}}+D^\dagger_{s\mathbf{k}}D_{s\mathbf{m}})\delta^{(2)}_{m+p,k}+(b^\dagger_{s\mathbf{k}}d^\dagger_{-s\mathbf{m}}+iB^\dagger_{s\mathbf{k}}D^\dagger_{-s\mathbf{m}})\delta^{(2)}_{k+m,p}]+\text{“H.c.”}\ , \quad \text{(A7)}$$

$$V_{\rm ins\ ferm}=e^2\frac{1}{4\pi L_\perp}\sum_{s}\sum_{pqmn}\frac{1}{\sqrt{pq}}[(a^\dagger_{\mathbf{p}}+iA^\dagger_{\mathbf{p}})(a_{\mathbf{p}}+iA_{\mathbf{p}})(b^\dagger_{s\mathbf{m}}b_{s\mathbf{n}}+B^\dagger_{s\mathbf{m}}B_{s\mathbf{n}}+d^\dagger_{s\mathbf{m}}d_{s\mathbf{n}}+D^\dagger_{s\mathbf{m}}D_{s\mathbf{n}})\delta^{(2)}_{\mathbf{p+m,q+n}}$$
$$\times(\{p+m|q+n\}-\{p-n|q-m\})$$
$$-(a^\dagger_{\mathbf{p}}+iA^\dagger_{\mathbf{p}})(a^\dagger_{\mathbf{q}}+iA^\dagger_{\mathbf{q}})(d_{-s\mathbf{n}}b_{s\mathbf{m}}+iD_{-s\mathbf{n}}B_{s\mathbf{m}})\delta^{(2)}_{\mathbf{p+q,m+n}}\{p-m|-q+n\}+\text{“H.c.”}$$
$$+(a^\dagger_{\mathbf{p}}+iA^\dagger_{\mathbf{p}})(a_{\mathbf{q}}+iA_{\mathbf{q}})(d_{-s\mathbf{n}}b_{s\mathbf{m}}+iD_{-s\mathbf{n}}B_{s\mathbf{m}})\delta^{(2)}_{\mathbf{p,q+m+n}}$$
$$\times(\{p-n|q+m\}-\{p-m|q+n\})+\text{“H.c.”}$$
$$+(a_{\mathbf{p}}+iA_{\mathbf{p}})(a_{\mathbf{q}}+iA_{\mathbf{q}})(b^\dagger_{s\mathbf{m}}b_{s\mathbf{n}}+B^\dagger_{s\mathbf{m}}B_{s\mathbf{n}}+d^\dagger_{s\mathbf{m}}d_{s\mathbf{n}}+D^\dagger_{s\mathbf{m}}D_{s\mathbf{n}})$$
$$\times\delta_{\mathbf{m,p+q+n}}\{p+n|-q+m\}+\text{“H.c.”}]\ , \quad \text{(A8)}$$

$$V_{1\ \text{loop}}=\frac{e^2}{4\pi L_\perp}\sum_p \frac{a_p^\dagger a_p}{p}\left[\sum_{i,n}\frac{\lambda^2\left[8\frac{n}{p}\left[1-\frac{n}{p}\right]-1\right]-4m^2}{\frac{\lambda^e+\hat{p}_\perp^2}{p}-\frac{m^2+\hat{n}_\perp^2}{n}-\frac{m^2+(\hat{p}_\perp-\hat{n}_\perp)^2}{p-n}}-(\text{“}m\to M\text{”})\right]$$
$$+\text{“}a_p^\dagger a_p\to A_p^\dagger A_p,\lambda^2\to\Lambda^2\text{”}$$
$$+\frac{e^2}{4\pi L_\perp}\sum_{n,s}\frac{b_{sn}^\dagger b_{sn}+d_{sn}^\dagger d_{sn}}{n}\left[\sum_p \frac{\lambda^2-s[m^2+\hat{n}_\perp^2]\frac{p}{n}+2\hat{n}_\perp\hat{p}_\perp}{\frac{m^2+\hat{n}_\perp^2}{n}-\frac{\lambda^2+\hat{p}_\perp}{p}-\frac{m^2+(\hat{n}_\perp-\hat{p}_\perp)^2}{n-p}}\frac{1}{p(p-n)}-(\text{“}\lambda^2\to\Lambda^2\text{”})\right]$$
$$+\text{“}b^\dagger b\to B^\dagger B,d^\dagger d\to D^\dagger D,m^2\to M^2\text{”}\ . \tag{A9}$$

The conventions are the same as in Eqs. (3.6)–(3.12). a_p, A_q, c_p, C_q, $B_{s,m}$, $D_{s,n}$ obey usual boson commutation relations, $b_{s,m}d_{s,n}$ fermion anticommutation relations. "H.c." indicates Hermitian conjugation only for field operators, not for c numbers; i.e., iA_p+ "H.c." $=iA_p+iA_p^\dagger$. Of course, H is thus not Hermitian, but this should not influence unitarity below the production threshold for the heavy photons and electrons. There is no instantaneous photon-exchange term since those terms cancel among the light and heavy photons [27]. The one-loop counterterms have been constructed such that they, together with one-loop corrections induced by H, avoid all one-loop self-energies which would be proportional to $\int dm^2\rho(m^2)(m^2)^{1/2}$ or $\int d\lambda^2\rho(\lambda^2)(\lambda^2)^{1/2}$ in the continuum limit. Without the one-loop counterterms, more Pauli-Villars particles would be necessary to make all such terms vanish.

APPENDIX B: TWO-LOOP SELF-ENERGY IN ($D_\perp+2$)-DIMENSIONAL YUKAWA THEORY

In light-cone perturbation theory (LCPT), the strongest divergences (quadratic in 3+1) occur at the one-loop level. Thus one might be tempted to expect that the violations of rotational invariance occur also only at one loop. This is not true as the following simple example shows.

We consider a massless fermion coupled to a massive scalar boson via a Yukawa interaction term. As a specific example, we evaluate explicitly the rainbow graph (Fig. 1) contribution to the γ^+ component of the one-shell fermion self-energy. If we choose vanishing $p_\perp$ for the incoming electron, i.e., $p_{\text{in}}^-=p_\perp^2/p^+=0$, it follows from (2.2) that this component should be zero.

In order to separate one- and two-loop effects, we allow the masses of the inner (λ) and the outer boson (Λ) in the diagram to be different from each other. This also makes it easy to regularize the inner loop "sufficiently" while leaving the outer loop unregularized for the moment. Applying LCPT, one easily finds [28] (up to the same constants)

$$\Sigma(p^\mu)=C\int d^{D_\perp}k_\perp\int_0^1\frac{dx}{x(1-x)^2}\,\frac{\not{p}_1}{\frac{k_\perp^2}{1-x}+\frac{k_\perp^2+\lambda^2}{x}}\,I^{1\ \text{loop}}(p_1)\,\frac{\not{p}_1}{\frac{k_\perp^2}{1-x}+\frac{k_\perp^2+\lambda^2}{x}}\ , \tag{B1}$$

where

$$p_1^+=p^+(1-x),\quad p_1^-=\frac{k_\perp^2+\lambda^2}{xp^+},\quad p_{1\perp}=-k_\perp \tag{B2}$$

and

$$I^{1\ \text{loop}}(p_1)=\frac{\not{p}_1}{1-x}\int d^{D_\perp}q_\perp\int_0^1\frac{dy}{y(1-x)}\int d\Lambda^2\frac{(1-y)\rho(\Lambda^2)}{p_1^--\frac{q_\perp^2+\Lambda^2}{y(1-x)}-\frac{(k_\perp+q_\perp)^2}{(1-y)(1-x)}}\ . \tag{B3}$$

Here we have already used $\int d\Lambda^2\rho(\Lambda^2)=0$, $\int d\Lambda^2\rho(\Lambda^2)(\Lambda^2)^{D_\perp/2}=0$ to case $I^{1\ \text{loop}}$ into a rotationally invariant form [29]. Using [note that p_1 is on energy shell; see Eq. (B2)]

$$p_1^2=-(1-x)\left[\frac{k_\perp^2}{1-x}+\frac{k_\perp^2+\lambda^2}{x}\right]\ , \tag{B4}$$

one finds

$$\text{tr}(\Sigma\gamma^-)=C\int d^{D_\perp}k_\perp\int d^{D_\perp}q_\perp\int_0^1\frac{dx}{x^2(1-x)^2}\int_0^1\frac{dy}{y}\int d\Lambda^2\rho(\Lambda^2)\frac{k^2+\lambda^2}{\frac{k^2}{1-x}+\frac{k^2+\lambda^2}{x}}\,\frac{1}{\frac{k^2+\lambda^2}{x}+\frac{q^2+\Lambda^2}{y(1-x)}+\frac{(k+q)^2}{(1-y)(1-x)}}$$
$$=C\pi^{D_\perp}\Gamma(1-D_\perp)\,\frac{\pi}{\sin\pi(D_\perp/2)}\int\frac{d\Lambda^2\rho(\Lambda^2)}{(\Lambda^2)^{1-D_\perp}}\neq 0\ . \tag{B5}$$

First and most important, the γ^+ component of Σ is nonzero, and rotational invariance is thus violated since $p^-=0$. Second, the result is independent of the outer boson mass λ; i.e., a Pauli-Villars regularization [with condition $\int d\lambda^2 \rho(\lambda^2)=0$] would have rendered $\mathrm{tr}(\Sigma\gamma^-)$ zero.

This is a rather typical result for higher-loop graphs and implies the following. Once one has (over)regularized the short-distance singularities so much that one can handle the one-loop singularities in a rotationally invariant way (as in PV), then the (milder) higher-loop singularities should be no problem any more if one uses the same (over)regularized versions of the theory there.

It is however not sufficient to add only a one-loop counterterm and add no two-loop counterterms at all. Although one might be tempted to do so, because, e.g., in $2+1$ dimensions this would not introduce additional infinities, this violates rotational invariance by a finite term (for $D_\perp=1$).

Since we started from a manifestly covariant Lagrangian (Yukawa), one would usually expect from a formalism that is supposed to yield covariant observables that it is already covariant if one considers any single Feynman diagram. Of course, in principle, this is not necessary, and one might imagine a situation where several Feynman graphs each of them noncovariant and of the same order in g add up to a covariant answer. In such a situation one would not have to worry since, strictly speaking, only the sum of all Feynman diagrams (plus, of course, subloop insertions and renormalization) contributing to a certain observable has a physical meaning. In order to convince ourselves that rotational invariance is indeed violated [30], we numerically evaluated the residual g^4 contributions (including all one-loop insertions which are necessary to render the one-loop graph finite and covariant) and convinced ourselves that they do not restore rotational invariance.

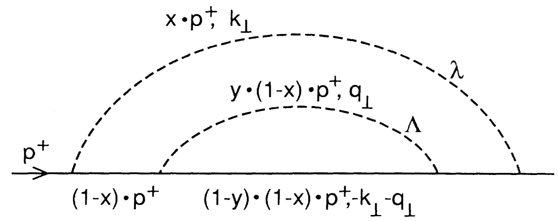

FIG. 1. Rainbow diagram contribution to the two-loop fermion self-energy in the Yukawa model.

[1] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949); H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) **112**, 94 (1978).
[2] A simple and well-known example is rotational invariance in the context of equal-time quantized field theory. The angular momentum operator is a sum of single-particle operators, and, generally, approximation schemes preserve rotational invariance.
[3] The situation here is very similar to boost transformations in equal-time quantization which mix **x** and x^0.
[4] In general, as we will see later, one has to impose more Pauli-Villars conditions than usual to treat the short-distance singularities properly.
[5] D. Mustaki, S. Pinsky, J. Shigemitsu, and K. Wilson, Phys. Rev. D **43**, 3411 (1991).
[6] C. B. Thorn, Phys. Rev. D **20**, 1934 (1979).
[7] This remark should also remove any confusion about what we mean by the term "light-cone quantization." Sometimes this is confused with an infinite-momentum boost, which is not what we are using in this work.
[8] D. Mustaki, Phys. Rev. D **42**, 1184 (1990).
[9] Note that we have already introduced one PV regulator field with mass Λ in order to avoid the ambiguity in the treatment of normal-ordering divergences.
[10] One might object that it is not the γ-matrix structure of some off-shell Green's functions, but rather S-matrix elements that matter. We evaluated the one-loop corrections to the decay $\phi\to\bar\psi\psi$ (assuming $\lambda>2m$) in the rest frame of the meson ϕ using only one PV regulator. It turns out that the angular distribution of the $\bar\psi\psi$ pairs is not rotational invariant, as one might have already expected from (2.3).
[11] Unfortunately, we were not able to prove this in general. It is, however, plausible, since the worst singularities in LC loop calculations occur usually for the γ^+ component where they lead to a violation of rotational invariance. We have verified the statement for all two-loop diagrams.
[12] M. Burkardt and A. Langnau (unpublished).
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[16] It should be emphasized that we could have chosen another basis set for the expansion of the fields. In fact, we are always free to replace sums by integrals and thus obtain a continuum formulation. Our choice here does not restrict generality.
[17] A. C. Tang, S. J. Brodsky, and H. C. Pauli, Phys. Rev. D (to be published); A. Tang, Ph.D. thesis, Report No. SLAC-PUB-351, 1990.
[18] In the next section we will construct noncovariant counterterms which display the same structure as the normal-ordering terms; i.e., in the latter approach we can also omit V_{NO} since taking it into account just corresponds to a redefinition of parameters.
[19] S. J. Brodsky, R. Roskies, and R. Suaya, Phys. Rev. D **8**, 3574 (1973).
[20] This is similar to approximations used in the equal-time approximation which lead to violations of boost invariance.
[21] A. Langnau, Ph.D. thesis, Heidelberg, 1991.
[22] This is true if all noncovariant terms have been removed in subloops.

[23] Note that no wave-function renormalization is necessary in the Hamiltonian approach since wave functions are automatically normalized. The normalization constants cancel, because of the Ward identities, in all physical observables.

[24] At this point we would like to remind the reader about the discussion involving the constraint equation (2.3). As we have just pointed out, the kinetic mass and vertex mass will, in general, require different renormalizations. This implies that the mass in (2.3) will also be renormalized differently from the kinetic mass. However, since it will not be needed in the following, we will not work out the relation between the renormalizations of these various masses.

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[26] B. Warr (private communication).

[27] Note that the original instantaneous photon-exchange term (3.10) is independent of the photon mass.

[28] Here we have used the Brodsky trick to combine instantaneous fermions with those fermion lines which extend only over one intermediate state by putting the corresponding fermions on energy shell in the numerator.

[29] In fact, $I^{1\ \mathrm{loop}}$ can be written as $I^{1\ \mathrm{loop}}=\not{p}_1 f(p_1^2)$.

[30] Unless one allows for Pauli-Villars regulators in the outer loop or adds noncovariant terms of the order g^4.