Low-energy limit of two-scale field theories

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We present a full and self-contained discussion of the decoupling theorem applied to several general models in four-dimensional field theory. We compute in each case the low-energy effective action and show the explicit one-loop expressions for each of the effective parameters. We find that for suitable conditions one can always build an effective low-energy theory where the conditions of the decoupling theorem are satisfied.

I. INTRODUCTION

The majority of the models now in use to study the unification of the known forces involve particles with very different masses. They are quantum field theories with multiple mass scales, which we may call *multiscale field theories.* In this context the simplest model will, at least, have two mass scales: one of them corresponding to the unification with gravity, i.e., Planck's mass ($M_{\rm Pl}$), and another associated with the electroweak unification. Then, if one wants to relate the physics of the large-mass scales to the physics of the $W^{\pm}$ and Z^0, it is clear that one has to develop a field-theoretical mechanism for extracting the effective low-energy actions from theories which were intended to describe physics at $M_{\rm Pl}$. What one is after is a well-defined procedure for the extraction of the so-called effective theory from a larger underlying theory, whose range of validity extends to momenta much larger than the scales at which one expects to use the effective theory. There are, of course, many well-known examples of very successful effective theories, such as Maxwell's laws, Yukawa's meson theory, or the Fermi theory of weak interactions. Finally, the standard model of particle physics must be another low-energy effective action of some, still unknown, theory of everything.

The first tool proposed to study the effects of the large-scale physics on the physics of the low-mass scales, was the decoupling theorem of Appelquist and Carazzone,[1,2] which asserts that if there is decoupling, then the effects of the heavy degrees of freedom (HDF) on the physics of the low-mass scales are both calculable and innocuous. Nevertheless the theorem does not supply with any method for calculating the low-energy effective action or its corresponding effective parameters. Whatever that method, two features will characterize the resulting effective-field theory: (i) A more complete underlying theory does exist, which is valid up to energies much higher than those for which the effective theory is meant, and (ii) only the light degrees of freedom of the original theory will survive, as such, in the low-energy limit.

A problem emerges in two-scale models when computing quantum effects which mix the heavy and light sectors of the *full* theory: *if the tree-level action mixes the heavy and light degrees of freedom, then quantum corrections will, in principle, destroy the initial hierarchy or separation between light and heavy.* In trying to solve this problem, one has to be especially careful because one can find many new complications which can be even worse than what one was trying to solve. For example, a typical collection of "solutions" to this problem is (i) one is only able to impose *unnatural conditions* on the size of the original parameters, (ii) one could introduce a *fine-tuning* of the parameters order by order in perturbation theory, or (iii) given enough imagination, one can design a new global symmetry which forbids the dangerous tree-level interactions.

In this paper we will find that there exist *solutions* to the mixing problem, which are different (in fact, opposite) from the three quoted above. We will show that there is an appropriate method, due to Weisberger,[3,4] to deal with this class of multiscale theories, which may be used as a starting point for the construction of *"well-behaved"* effective theories. It is based on a straightforward interpretation of the Appelquist-Carazzone theorem, which allows, in principle, the computation of the effective action to all orders in perturbation theory.

The situation we are interested in describing is that of a theory with two widely different mass scales, i.e., with heavy particles of mass M and light particles of mass m, where $(m/M)\ll 1$, in the case when the external energies (E) are much lower than the heavy scale, but of the order of the light scale. In this case kinematics forbids the presence of external heavy particles, so that the relevant part of the action will only involve light degrees of freedom. Under these conditions, can we dispose of the heavy degrees of freedom altogether? A positive answer to this question amounts to assuring the existence of a theory of the light degrees of freedom alone, giving the same predictions than the full theory. Obviously, this is untenable in its strongest sense, but could we ensure it as a suitable approximation? The constructive answer to this question given by Weisberger starts by defining an effective action $\overline{\Gamma}$ for the light particles as follows: Define the full effective action from the generating functional for connected Green's functions Z as usual:

$$\Gamma[\phi,\pi]=Z[j,J]-\int d^4x\,(j\phi+J\pi)\ ,$$

where j and J are the external sources for ϕ and π. To avoid the presence of external heavy particles, set $J=0$, and to forbid π as an independent degree of freedom, take

$$\overline{\Gamma}[\phi]=\Gamma[\phi,\pi(\phi)]|_{\delta\Gamma/\delta\pi=J=0} .$$

Here $\pi(\phi)$ is the solution to $\delta\Gamma/\delta\pi=0$.

In general, this action is very complicated and nonlocal. However, this exact effective theory has many irrelevant terms which will be eliminated by establishing a systematic expansion in powers of E/M, with E being any combination of the parameters having the size of the low-energy scale. We will expand $\overline{\Gamma}$ in powers of E/M and keep terms up to zeroth order in E/M. Obviously, this truncation is a very excellent approximation at ordinary energies. [Note that the coefficients in this series will in general be functions of $\ln(M/\mu)$]. We will refer to this procedure as the *large hierarchy limit,* and we will call its result simply by $\overline{\Gamma}$. The decoupling theorem states that the vertices it contains can be identified with those of an equivalent theory containing only light particles. The identification is completed by comparing the parameters of this equivalent theory with those in $\overline{\Gamma}$, which can be done order by order in $\hbar$.

The above situation is somewhat puzzling in that the parameters of $\overline{\Gamma}$, while being functions of the large scale through the combination $\ln(M/\mu)$, should be identified with those of an equivalent theory that by construction are independent of M. We will show that the solution to this paradox is that these parameters behave as the bare parameters of the equivalent theory, in other words, that the explicit dependence on $\ln(M/\mu)$ cancel against the implicit dependence coming from their definition in terms of the parameters of the original theory. As a bonus, we will get that once the hierarchy is established at the tree level at the high-mass scale, it remains stable under radiative corrections at all scales.

We will present a study of the application of techniques based on the above methods to the computation of effective low-energy theories and will illustrate their use for several examples of physical relevance. In particular, and for each case, we will carry out the computation of the effective parameters in the one-loop approximation, explicitly showing how decoupling takes place.

The paper is organized as follows. We start with a real, two-scale, scalar theory having the double reflection invariance, $\pi\rightarrow-\pi$, $\phi\rightarrow-\phi$. In the following section, we consider the former model with the symmetry of the heavy scale $\pi\rightarrow-\pi$ being spontaneously broken. The most general, renormalizable, four-dimensional two-scalar model will be studied in the third section. It is important to consider these models because they do not have any higher symmetry that accounts for cancellations of the dangerous contributions.

The last two sections consider two-scale globally supersymmetric theories in four spacetime dimensions. The simplest of them are the Wess-Zumino model (Sec. V) and the softly broken Wess-Zumino model (Sec. VI). The method is extended to them by developing some new techniques; we solve superfield equations of motion and give a compact formula (written in the superspace-superfield formalism) for the perturbative evaluation of the one-loop effective action. We study softly broken supersymmetric theories in this paper because they are the principal candidates to give a supersymmetric version of the standard model of particle physics. We finish the paper by offering some conclusions.

II. SIMPLE TWO-SCALE MODEL

In this section we will show in detail how to carry out the integration of the heavy degrees of freedom, using Weisberger's method in the simplest two-scale model: real, four-dimensional $\lambda\phi^4$, with two fields and reflection symmetry. The action is then

$$S(\pi,\phi)=\int d^4x\left[\tfrac{1}{2}(\partial\pi)^2+\tfrac{1}{2}(\partial\phi)^2-V(\pi,\phi)\right] , \qquad (2.1)$$

with the potential given by

$$V(\pi,\phi)=\Lambda+\frac{M^2}{2}\pi^2+\frac{m^2}{2}\phi^2+\frac{\lambda_1}{24}\pi^4+\frac{\lambda_2}{24}\phi^4+\frac{\lambda_3}{4}\pi^2\phi^2 .$$

We will choose the parameters such that ϕ is light, and π is the heavy degree of freedom. There is no symmetry breaking, and as is well known, to one loop there is no wave-function renormalization either. The model has quadratic and quartic divergences which tend to destabilize the original size of the tree-level parameters. This can be seen in the renormalization-group equations for the parameters of the full action which we will need later on in our calculation. (We use throughout this paper the total derivative symbol $\mu d/d\mu$ to denote the total derivative of a quantity keeping the *bare* parameters fixed. This should not be confused with the partial derivative $\mu\,\partial/\partial\mu$ keeping the *renormalized* parameters fixed and which occurs in the renormalization-group equations.) These are

$$\begin{aligned}
\mu\frac{d\Lambda}{d\mu}&=\frac{\hbar}{32\pi^2}(M^4+m^4) ,\\
\mu\frac{d\lambda_1}{d\mu}&=\frac{\hbar}{16\pi^2}3(\lambda_1^2+\lambda_3^2) ,\\
\mu\frac{d(M^2)}{d\mu}&=\frac{\hbar}{16\pi^2}(\lambda_1M^2+\lambda_3m^2) ,\\
\mu\frac{d\lambda_2}{d\mu}&=\frac{\hbar}{16\pi^2}3(\lambda_2^2+\lambda_3^2) ,\\
\mu\frac{d(m^2)}{d\mu}&=\frac{\hbar}{16\pi^2}(\lambda_3M^2+\lambda_2m^2) ,\\
\mu\frac{d\lambda_3}{d\mu}&=\frac{\hbar}{16\pi^2}(\lambda_1\lambda_3+\lambda_2\lambda_3+4\lambda_3^2) .
\end{aligned} \qquad (2.2)$$

The fact that m^2 gets contributions of order M^2 through these quantum corrections is a well-known form of the hierarchy problem. We will explicitly show how the *true effective mass* of the light degrees of freedom does not suffer from this disease.

We study this model here because the results that we find extend to more complex situations. It was already treated by Weisberger in Ref. 3, and we get the same formal results here as he did, but we go one step beyond and provide a novel interpretation for them. In the remainder of this section, we will explain, step by step, the method and its implementation.

At the tree level, we have for π the equation of motion

$$\left[\Box+M^2+\frac{\lambda_1}{6}\pi^2+\frac{\lambda_3}{2}\phi^2\right]\pi=J\ .$$

The source J will vanish for momenta where $p^2 \ll M^2$. The solution for the range of momenta we are considering is $\pi(\phi)=0$. Hence the effective action $\bar{\Gamma}[\phi]$ for the light degrees of freedom is

$$\bar{\Gamma}[\phi]=\int d^4x\left[\frac{1}{2}(\partial\phi)^2-\Lambda-\frac{m^2}{2}\phi^2-\frac{\lambda_2}{4!}\phi^4\right],$$

so that the effective parameters are given by

$$\phi_0^*=\phi,\quad \Lambda_0^*=\Lambda,\quad m_0^*=m,\quad \lambda_0^*=\lambda_2\ ,$$

at the classical level. Trivially, this is a theory of light degrees of freedom if the original dimensionful parameters Λ and m are small.

To actually carry out the calculation of the full effective action $\bar{\Gamma}[\phi]$ to order $\hbar$, we must compute the quantum corrections to the tree-level action and then solve in the appropriate kinematic regime $p^2_{\rm ext} \ll M^2$ the equation of motion of the heavy degree of freedom in terms of the light degree of freedom: $\pi=\pi(\phi)$. One then takes the limit $m/M\to 0$ and finally eliminates the HDF from the action. While doing this one also carries out the renormalization of the theory by canceling out the infinities in the usual fashion.

The one-loop contribution to the effective action is obtained by using the well-known steepest-descent procedure (cf., e.g., Ref. 5):

$$\Gamma^{(1)}[\pi,\phi]=\frac{i\hbar}{2}\int d^4x\int\frac{d^4k}{(2\pi)^2}\mathrm{tr}\ln[k^2-\mathcal{M}^2-U(\pi,\phi)]\ ,$$

where $\mathcal{M}^2$ is the mass matrix, and U is the matrix obtained by taking the second derivatives of the scalar potential with respect to the fields: i.e.,

$$U(\pi,\phi)=\frac{1}{2}\begin{bmatrix}\lambda_1\pi^2+\lambda_3\phi^2 & 2\lambda_3\pi\phi\\ 2\lambda_3\pi\phi & \lambda_2\phi^2+\lambda_3\pi^2\end{bmatrix}.$$

Performing the integration over momenta using dimensional regularization, and after renormalizing using the modified minimal-subtraction ($\overline{\rm MS}$) scheme, we get

$$\Gamma[\pi,\phi]\equiv S(\pi,\phi)+\Gamma^{(1)}[\pi,\phi]+O(\hbar^2)\ ,$$

where all the quantum corrections are in $\Gamma^{(1)}[\pi,\phi]$, and the parameters and fields in $\Gamma[\pi,\phi]$ are renormalized. Considering only the terms with mass dimension ≤ 4 in $\Gamma^{(1)}[\pi,\phi]$ gives the generic expansion

$$\Gamma^{(1)}[\pi,\phi]=-\frac{\hbar}{32\pi^2}\int d^4x\sum_{m,n=0}^{m+n=4}a_{mn}\pi^m\phi^n\ . \tag{2.3}$$

In our model the values of the coefficients are

$$a_{00}=\frac{M^4}{2}\ln\frac{M^2}{\mu^2}+\frac{m^4}{2}\ln\frac{m^2}{\mu^2}\ ,$$

$$a_{20}=\frac{1}{2}\lambda_1M^2\ln\frac{M^2}{\mu^2}+\frac{1}{2}\lambda_3m^2\ln\frac{m^2}{\mu^2}\ ,$$

$$a_{02}=\frac{1}{2}\lambda_3M^2\ln\frac{M^2}{\mu^2}+\frac{1}{2}\lambda_2m^2\ln\frac{m^2}{\mu^2}\ ,$$

$$a_{40}=\frac{1}{8}\lambda_1^2\ln\frac{M^2}{\mu^2}+\frac{1}{8}\lambda_3^2\ln\frac{m^2}{\mu^2}\ ,$$

$$a_{04}=\frac{1}{8}\lambda_3^2\ln\frac{M^2}{\mu^2}+\frac{1}{8}\lambda_2^2\ln\frac{m^2}{\mu^2}\ ,$$

$$a_{22}=\frac{1}{4}\lambda_1\lambda_3\ln\frac{M^2}{\mu^2}+\frac{1}{4}\lambda_2\lambda_3\ln\frac{m^2}{\mu^2}+\lambda_3^2\left[\frac{m^2}{m^2-M^2}\ln\frac{m^2}{\mu^2}+\frac{M^2}{M^2-m^2}\ln\frac{M^2}{\mu^2}\right].$$

The next step is to eliminate π from the action. To do it, we solve for $\pi(\phi)$ through its one-loop equation of motion and take the limit $(\Box/M^2)$ and $(m^2/M^2)\to 0$. The result is $\pi=0$ and

$$\bar{\Gamma}(\phi)=\int d^4x\left\{\frac{1}{2}(\partial\phi)^2-\left[\Lambda+\frac{\hbar}{64\pi^2}\left[M^4\ln\frac{M^2}{\mu^2}+m^4\ln\frac{m^2}{\mu^2}\right]\right]-\frac{1}{2}\left[m^2+\frac{\hbar}{32\pi^2}\left[\lambda_3M^2\ln\frac{M^2}{\mu^2}+\lambda_2m^2\ln\frac{m^2}{\mu^2}\right]\right]\phi^2\right.$$
$$\left.-\frac{1}{24}\left[\lambda_2+\frac{\hbar}{32\pi^2}3\left[\lambda_3^2\ln\frac{M^2}{\mu^2}+\lambda_2^2\ln\frac{m^2}{\mu^2}\right]\right]\phi^4\right\}. \tag{2.4}$$

In spite of the appearance of M^2 in the above action, its form is typical of the action for a light degree of freedom. This can be shown in a variety of forms, the simplest of which we show here. Write

$$\bar{\Gamma}(\phi)=\int d^4x\sum_n\hat{O}_n\phi^n\ .$$

Now by scale invariance we have $\mu d\bar{\Gamma}/d\mu=0$, implying that $\mu\, d\hat{O}_n/d\mu=0$; therefore, the careted parameters behave as a sort of bare parameters of the effective action. (Notice that there is a straightforward modification when wave-function renormalization does not vanish.) All these parameters are of the form $\hat{O}=O_0^*+O_ML+O_ml$, where O_0^* is the tree-level value, $L=\ln(M/\mu)$, and $l=\ln(m/\mu)$. Note also that the one-loop contribution is such that O_m is M independent. Next, we introduce which will be the effective parameters of the light degrees of freedom $O^*_{1\ \rm loop}$ through the definition

$$\hat{O}=O^*_{1\ \rm loop}+O_ml\ , \tag{2.5}$$

where

$$O^*_{1\,\text{loop}} \equiv O^*_0 + O_M L \ . \tag{2.6}$$

The important point here is that we can split the effective action into two terms:

$$\overline{\Gamma}[\phi] = S^*[\phi] + \Delta\overline{\Gamma}[\phi] \ ,$$

where

$$S^*[\phi] \equiv \int d^4x \sum_n (O^*_{1\,\text{loop}})_n \phi^n \ ,$$
$$\Delta\overline{\Gamma}[\phi] \equiv \int d^4x \sum_n (O_m)_n l \phi^n \ . \tag{2.7}$$

$S^*[\phi]$ can be considered as the classical renormalized action of the light degree of freedom; then one can explicitly check that $\Delta\overline{\Gamma}[\phi]$ contains the one-loop quantum corrections produced by this action. Finally, the renormalization-group equations (RGE's) the light degree of freedom $(\mu dO^*_{1\,\text{loop}}/d\mu)_n = (O_m)_n$ are automatically satisfied by *construction*. This can be seen from Eqs. (2.5) and (2.6), which imply

$$O^*_{1\,\text{loop}} = \hat{O}^* - O_m L \ , \tag{2.8}$$

where

$$\hat{O}^* \equiv O^*_0 + (O_M + O_m)L \ , \tag{2.9}$$

which are the physical parameters and are therefore fixed[6] under the renormalization group, as can be checked by explicit computation *on using the renormalization-group equations* (2.2). (We use a rather cumbersome notation to help keep track at a glance of the origin and meaning of the different parameters.) The RGE's for the light degree of freedom are obtained directly from Eq. (2.8).

For the model we are analyzing in this section, the above parameters are explicitly given by

$$A^*_{1\,\text{loop}} = \Lambda + \frac{\hbar}{64\pi^2} M^4 \ln\frac{M^2}{\mu^2} \ ,$$
$$m^{*2}_{1\,\text{loop}} = m^2 + \frac{\hbar}{32\pi^2} M^2 \lambda_3 \ln\frac{M^2}{\mu^2} \ , \tag{2.10}$$
$$\lambda^*_{1\,\text{loop}} = \lambda_2 + \frac{\hbar}{32\pi^2} 3\lambda_3^2 \ln\frac{M^2}{\mu^2} \ .$$

They can be rewritten as

$$\Lambda^*_{1\,\text{loop}} = \hat{\Lambda}^* - \frac{\hbar}{64\pi^2} m^4 \ln\frac{M^2}{\mu^2} \ ,$$
$$m^{*2}_{1\,\text{loop}} = \hat{m}^{*2} - \frac{\hbar}{32\pi^2} m^2 \lambda_2 \ln\frac{M^2}{\mu^2} \ , \tag{2.11}$$
$$\lambda^*_{1\,\text{loop}} = \hat{\lambda}^* - \frac{\hbar}{32\pi^2} 3\lambda_2^2 \ln\frac{M^2}{\mu^2} \ ,$$

where the careted parameters are defined as

$$\hat{\Lambda}^* \equiv \Lambda + \frac{\hbar}{64\pi^2}(M^4 + m^4)\ln\frac{M^2}{\mu^2} \ ,$$
$$\hat{m}^{*2} \equiv m^2 + \frac{\hbar}{32\pi^2}(\lambda_3 M^2 + \lambda_2 m^2)\ln\frac{M^2}{\mu^2} \ ,$$
$$\hat{\lambda}^* \equiv \lambda_2 + \frac{\hbar}{32\pi^2} 3(\lambda_3^2 + \lambda_2^2)\ln\frac{M^2}{\mu^2} \ .$$

With these definitions, (2.10) and (2.11) coincide. As one can see, the parameters O^*_0, $\hat{O}^*$, and $O^*_{1\,\text{loop}}$ get the same value at the high-mass scale. Below that scale, while $\hat{O}^*$ remains constant, $O^*_{1\,\text{loop}}$ evolves with a slope O_m, which at most is of order of the low mass. In other words, once established at the high-mass scale, the hierarchy remains stable all the way down to the light scale. In spite of the appearances, this is not a virtue of the simplicity of the model chosen in this section, but a general feature which we will see arising in other rather general models of physical relevance. This feature can be explicitly checked by letting the renormalization-group operator $\mu d/d\mu$ act directly on both sides of the formulas (2.10):

$$\mu\frac{d\Lambda^*_{1\,\text{loop}}}{d\mu} = \frac{\hbar}{32\pi^2} m^{*4}_{1\,\text{loop}} \ ,$$
$$\mu\frac{dm^{*2}_{1\,\text{loop}}}{d\mu} = \frac{\hbar}{16\pi^2} m^{*2}_{1\,\text{loop}} \lambda^*_{1\,\text{loop}} \ , \tag{2.12}$$
$$\mu\frac{d\lambda_{1\,\text{loop}}}{d\mu} = \frac{\hbar}{16\pi^2} 3\lambda^{*2}_{1\,\text{loop}} \ ,$$

which explicitly display the *stability* of the hierarchy.

III. SPONTANEOUS SYMMETRY BREAKING

We now study the previous class of models for the case when the symmetry $\pi \to -\pi$ is spontaneously broken at the high scale. The Lagrangian is the same as before, but with $M^2 \to -M^2$. In order to do perturbation theory around the tree vacuum,[7] we shift π by its vacuum expectation value (VEV) $v = (6M^2/\lambda_1)^{1/2}$ and obtain the tree-level Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\pi)^2 - \left[\Lambda - \frac{3}{2}\frac{M^4}{\lambda_1}\right] - \frac{\overline{m}^2}{2}\phi^2 - \frac{\overline{M}^2}{2}\pi^2$$
$$- \frac{\lambda_2}{4!}\phi^4 - \frac{\lambda_1}{4!}\pi^4 - \frac{\lambda_3}{4}\pi^2\phi^2 - \frac{\lambda_3 v}{2}\pi\phi^2 - \frac{\lambda_1 v}{6}\pi^3 \ ,$$

where we have introduced the defintions

$$\overline{M}^2 \equiv 2M^2, \quad \overline{m}^2 \equiv m^2 + \frac{3\lambda_3}{\lambda_1} M^2 \ .$$

To make the model a multiple scale model and to study the low-energy effective action, we will assume that at the tree level the hierarchy condition

$$\overline{m}^2 \ll \overline{M}^2$$

is satisfied. In the limit $(\Box/M^2)$, $(\overline{m}^2/\overline{M}^2) \to 0$, one can solve the equation of motion for π in a power series; to order ϕ^4 and $v/\overline{M}^4$, one gets

$$\pi(\phi) = -\frac{\lambda_3 v}{2\overline{M}^2}\phi^2 - \frac{\lambda_3^2 v}{8\overline{M}^4}\phi^4 + O\left[\frac{v}{M^6}\right] \ .$$

The tree-level effective low-energy action is then computed to give

$$\bar{\Gamma}[\phi]\equiv\lim_{\bar{M}\to\infty} S(\pi(\phi),\phi)$$
$$=\int d^4x\left[\frac{1}{2}(\partial\phi)^2-\left[\Lambda-\frac{3}{2}\frac{M^4}{\lambda_1}\right]-\frac{\bar{m}^2}{2}\phi^2-\frac{1}{4!}\left[\lambda_2-\frac{9\lambda_3^2}{\lambda_1}\right]\phi^4\right] . \quad (3.1)$$

By identification with a monoscale $\lambda\phi^4$ theory, we see that the corresponding tree-level effective parameters are given by

$$\phi_0^*\equiv\phi,\quad \Lambda_0^*\equiv\Lambda-\frac{3}{2}\frac{M^4}{\lambda_1},\quad m_0^{*2}\equiv\bar{m}^2,\quad \lambda_0^*\equiv\lambda_2-\frac{9\lambda_3^2}{\lambda_1} .$$

Computing the one-loop correction to the full theory as we did in the previous model, one gets

$$\Gamma(\phi,\pi)=\int d^4x[\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}(\partial\pi)^2-V(\phi,\pi)-V^{(1)}(\phi,\pi)+O(\hbar^2)] . \quad (3.2)$$

The first-order quantum correction to the tree-level potential is

$$V^{(1)}(\pi,\phi)=\sum_{m,n=0}^{m+n=4} a_{mn}\pi^m\phi^n ,$$

and whose nonvanishing coefficients are

$$a_{00}=-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^2}\mathrm{tr}\ln(k^2-\mathcal{M}^2) ,$$
$$a_{10}\equiv-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}(\lambda_3G+\lambda_1F)v ,$$
$$a_{02}\equiv-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{2}(\lambda_2G+\lambda_3F)-\lambda_3^2v^2FG\right] ,$$
$$a_{20}\equiv-\frac{i\hbar}{4}\int\frac{d^4k}{(2\pi)^4}[\lambda_3G+\lambda_1F-v^2(\lambda_3^2G^2+\lambda_1^2F^2)] ,$$
$$a_{04}\equiv-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{8}(\lambda_2^2G^2+\lambda_3^2F^2)+\frac{\lambda_3^2v^2}{2}(\lambda_2G^2F+\lambda_3GF^2)-\frac{\lambda_3^4v^4}{2}F^2G^2\right] ,$$
$$a_{40}\equiv-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{8}(\lambda_3^2G^2+\lambda_1^2F^2)+\frac{v^2}{2}(\lambda_3^3G^3+\lambda_1^3F^3)-\frac{v^4}{4}(\lambda_3^4G^4+\lambda_1^4F^4)\right] ,$$
$$a_{30}\equiv-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{2}(\lambda_3^2G^2+\lambda_1^2F^2)+\frac{v^2}{3}(\lambda_3^3G^3+\lambda_1^3F^3)\right]v ,$$
$$a_{22}\equiv-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{4}(\lambda_2\lambda_3G^2+\lambda_1\lambda_3F^2+4\lambda_3^2FG)-v^4(\lambda_3^4G^3F+\lambda_3^2\lambda_1^2F^3G+\lambda_1\lambda_3^3F^2G^2)+\frac{v^2}{2}(\lambda_2\lambda_3^2G^3+\lambda_3\lambda_1^2F^3+5\lambda_3^3G^2F+5\lambda_3^2\lambda_1F^2G)\right] ,$$
$$a_{12}\equiv-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{2}(\lambda_2\lambda_3G^2+\lambda_1\lambda_3F^2+4\lambda_3^2FG)+v^2(\lambda_3^3G^2F+\lambda_3^2\lambda_1F^2G)\right]v .$$

Here we have introduced the notation $F(k)\equiv-(k^2-\bar{M}^2)^{-1}$ and $G(k)\equiv-(k^2-\bar{m}^2)^{-1}$, and $\mathcal{M}^2=\mathrm{diag}(\bar{M}^2,\bar{m}^2)$ is the mass matrix of the theory already shifted to its vacuum. The equation of motion for π computed from Eq. (3.2) is

$$a_{10}+2(M^2+a_{20})\pi+\left[\frac{\lambda_1}{3!}+4a_{40}\right]\pi^3+\left[\frac{\lambda_3}{2}+2a_{22}\right]\phi^2\pi+\left[\frac{\lambda_3v}{2}+a_{12}\right]\phi^2+\left[\frac{\lambda_1v}{2}+3a_{30}\right]\pi^2=0 .$$

Again, we can compute its solution in a power series as

$$\pi=\alpha+\beta\phi^2+\gamma\phi^4+\text{higher-order powers} ,$$

and where the first coefficients are readily computed to be

$$\alpha=-\frac{a_{10}}{\bar{M}^2}+O(\hbar^2) ,$$
$$\beta=-\frac{\lambda_3v}{2\bar{M}^2}+\frac{\lambda_3va_{20}}{\bar{M}^4}-\frac{\lambda_3a_{10}}{\bar{M}^4}-\frac{a_{12}}{\bar{M}^2}+O(\hbar^2) .$$

Note that the first term in β is the tree-level contribution for $\pi=\pi(\phi)$ that we have already computed. Plugging this back into the one-loop corrected action and substituting the values of a_{mn} gives

$$\bar{\Gamma}[\phi]\equiv\lim_{\bar{M}\to\infty}\Gamma(\pi(\phi),\phi)=\int d^4x\left[\frac{1}{2}(\partial\phi)^2-(\Lambda_0^*+a_{00})-\frac{1}{2}(\bar{m}^{\,2}-2\hbar Q)\phi^2-\frac{1}{4!}(\lambda_0^*-24\hbar P)\phi^4\right], \tag{3.3}$$

where P and Q are the integrals

$$Q\equiv\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{2}\left[\lambda_2-\frac{3\lambda_3^2}{\lambda_1}\right]G-\lambda_3F-(\lambda_3 v)^2FG\right],$$

$$P\equiv\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{8}\left[\lambda_2-\frac{3\lambda_3^2}{\lambda_1}\right]^2G^2-\frac{\lambda_3^2}{2}F^2+\frac{\lambda_3^3v^2}{\bar{M}^2}FG+\frac{\lambda_3^2v^2}{2}\left[\lambda_2-\frac{3\lambda_3^2}{\lambda_1}\right]^2G^2F-\lambda_3^3v^2GF^2-\frac{\lambda_3^4v^4}{2}G^2F^2\right].$$

In the limit of the large hierarchy, these contributions reduce to

$$\lim_{\bar{m}^{\,2}/\bar{M}^2\to 0}Q=-\frac{1}{32\pi^2}\left[\frac{\bar{m}^{\,2}}{2}\left[\lambda_2-\frac{9\lambda_3^2}{\lambda_1}\right]\ln\frac{\bar{m}^{\,2}}{\nu^2}+\frac{3\lambda_3^2}{\lambda_1}\bar{m}^{\,2}\ln\frac{\bar{M}^2}{\nu^2}-\lambda_3\bar{M}^2\left[1-\frac{3\lambda_3}{\lambda_1}\right]\ln\frac{\bar{M}^2}{\nu^2}\right],$$

$$\lim_{\bar{m}^{\,2}/\bar{M}^2\to 0}P=-\frac{1}{32\pi^2}\left[\frac{1}{8}\left[\lambda_2-\frac{9\lambda_3^2}{\lambda_1}\right]^2\ln\frac{\bar{m}^{\,2}}{\nu^2}+\frac{1}{2}\left[\lambda_3^2-6\frac{\lambda_3^3}{\lambda_1}+3\frac{\lambda_2\lambda_3^2}{\lambda_1}-18\frac{\lambda_3^4}{\lambda_1^2}\right]\ln\frac{\bar{M}^2}{\nu^2}\right].$$

Following the same procedure as in our previous example, we may now define the one-loop effective parameters through

$$\begin{aligned}\Lambda^*_{1\,\text{loop}}&\equiv\Lambda_0^*+\frac{\hbar}{64\pi^2}\bar{M}^4\ln\frac{\bar{M}^2}{\nu^2},\\ m^{*2}_{1\,\text{loop}}&\equiv m_0^{*2}+\frac{\hbar}{32\pi^2}\left[\frac{6\lambda_3^2}{\lambda_1}\bar{m}^{\,2}-2\lambda_3\bar{M}^2\left[1-\frac{3\lambda_3}{\lambda_1}\right]\right]\ln\frac{\bar{M}^2}{\nu^2},\\ \lambda^*_{1\,\text{loop}}&\equiv\lambda_0^*+\frac{\hbar}{32\pi^2}\left[12\lambda_3^2-72\frac{\lambda_3^3}{\lambda_1}+36\frac{\lambda_1\lambda_3^2}{\lambda_1}-216\frac{\lambda_3^4}{\lambda_1^2}\right]\ln\frac{\bar{M}^2}{\nu^2}.\end{aligned} \tag{3.4}$$

To check for stability and decoupling, we must examine the scaling behavior of these parameters. Taking into account the renormalization-group equations for the original parameters and the expression of the O_0^* parameters, one readily shows that the set of $O^*_{1\,\text{loop}}$ parameters in Eq. (3.4) satisfy the renormalization-group equations given in Eq. (2.12). This demonstrates, explicitly, the advertised hierarchy stability. We may rewrite Eq. (3.4) in terms of a set of careted parameters, just as we did for the symmetric case of the first section. In fact, the parameters (3.4) are of the general form (2.6), and since they are decoupled, we can write them as in Eq. (2.8). As before, the careted parameters are fixed under the renormalization group and therefore independent of the scale μ. Now, once the hierarchy is fixed (at the tree level) at the scale $\bar{M}$, i.e., $\bar{m}^{\,2}=m_0^{*2}\ll\bar{M}^2$, where $\hat{O}^*=O^*_{1\,\text{loop}}$, it is fixed at the one-loop level, but not only at this scale, but at all scales below $\bar{M}$, due to the hierarchically small attained value of $\hat{m}^{*2}$, and the closure of the RGE's for the effective one-loop parameters.

IV. MOST GENERAL SCALAR THEORY

The most general scalar potential, compatible with the requirement of renormalizability in four dimensions and involving two-scale scalar fields, is given by

$$\begin{aligned}V(\pi,\phi)=\;&\Lambda+h_1\pi+h_2\phi+\frac{M^2}{2}\pi^2+\frac{m^2}{2}\phi^2+\omega^2\pi\phi\\&+\frac{g_1}{6}\pi^3+\frac{g_2}{6}\phi^3+\frac{g_3}{2}\pi^2\phi+\frac{g_4}{2}\phi^2\pi+\frac{\lambda_1}{24}\pi^4\\&+\frac{\lambda_2}{24}\phi^4+\frac{\lambda_5}{6}\pi^3\phi+\frac{\lambda_4}{6}\phi^3\pi+\frac{\lambda_3}{4}\pi^2\phi^2.\end{aligned} \tag{4.1}$$

These interactions do not respect any of the symmetries one could impose on π or ϕ. This makes this case the most interesting to analyze because there are no *a priori* arguments to simplify the structure of quantum corrections. Hence decoupling, if possible, is a most outstanding phenomenon and cannot be adscribed to quantum corrections respecting any symmetry.

When substituting π by the solution of its equation of motion, one gets, in the limit $(\Box/M^2)\to 0$,

$$\pi=\pi(\phi)=\sum_{n=0}^{\infty}a_n\phi^n.$$

Note that now there is an a_o term. This term is a consequence of the tadpole h_1 and amounts to a shift in π to get the true ground state. The term linear in ϕ induces a contribution to the kinetic term of ϕ, of the form

$$\tfrac{1}{2}(1+a_1^2)(\partial\phi)^2,$$

which implies a finite wave-function renormalization to get a properly normalized propagator for the light degree of freedom, i.e.,

$$\phi=(1+a_1^2)^{-1/2}\phi_0^* \ .$$

From the equations of motion for the heavy field, we see at once that these parameters a_0 and a_1 are determined by the equations

$$h_1+M^2a_0+\frac{g_1}{2}a_0^2+\frac{\lambda_1}{6}a_0^3=0 \ ,$$

$$\left[M^2+g_1a_0+\frac{\lambda_1}{2}a_0^2\right]a_1+\omega^2+g_3a_0+\frac{\lambda_5}{2}a_0^2=0 \ ,$$

which will be used later on. (At this point the details of their solutions are uninteresting to us.) Finally, at the tree level, the effective action for the light degree of freedom is readily found to be

$$\overline{\Gamma}[\phi_0^*]=\int d^4x\,[\tfrac{1}{2}(\partial\phi_0^*)^2-\Lambda_0^*-h_0^*\phi_0^*-\tfrac{1}{2}m_0^{*2}\phi_0^{*2}-\tfrac{1}{6}g_0^*\phi_0^{*3}-\tfrac{1}{24}\lambda_0^*\phi_0^{*4}] \ , \tag{4.2}$$

where the effective tree-level parameters O_0^* are defined in terms of the shifted parameters $\overline{O}$ of the original Lagrangian O as follows:

$$\Lambda_0^*\equiv\overline{\Lambda} \ , \quad g_0^*\equiv\frac{\overline{g}_2+3\overline{g}_4a_1+3\overline{g}_3a_1^2+\overline{g}_1a_1^3}{(1+a_1^2)^{3/2}} \ ,$$

$$h_0^*\equiv\frac{\overline{h}_2}{(1+a_1^2)^{1/2}} \ ,$$

$$\lambda_0^*\equiv\frac{\lambda_2+4\lambda_4a_1+6\lambda_3a_1^2+4\lambda_5a_1^3+\lambda_1a_1^4}{(1+a_1^2)^2} \ ,$$

$$m_0^{*2}\equiv\frac{\overline{m}^{\,2}+\overline{\omega}^{\,2}a_1}{1+a_1^2} \ .$$

These formulas were written in terms of the shifted parameters, which we display here for the sake of completeness:

$$\overline{\Lambda}\equiv\Lambda+h_1a_0+\frac{M^2}{2}a_0^2+\frac{g_1}{6}a_0^3+\frac{\lambda_1}{24}a_0^4, \quad \overline{g}_1\equiv g_1+\lambda_1a_0 \ ,$$

$$\overline{h}_2\equiv h_2+\omega^2a_0+\frac{g_3}{2}a_0^2+\frac{\lambda_5}{6}a_0^3 \ , \quad \overline{g}_2\equiv g_2+\lambda_4a_0 \ ,$$

$$\overline{M}^{\,2}\equiv M^2+g_1a_0+\frac{\lambda_1}{2}a_0^2 \ , \quad \overline{g}_3\equiv g_3+\lambda_5a_0 \ , \tag{4.3}$$

$$\overline{m}^{\,2}\equiv m^2+g_4a_0+\frac{\lambda_3}{2}a_0^2 \ , \quad \overline{g}_4\equiv g_4+\lambda_3a_0 \ ,$$

$$\overline{\omega}^{\,2}\equiv\omega^2+g_3a_0+\frac{\lambda_5}{2}a_0^2 \ .$$

In the above treatment it was implicitly assumed that there was a hierarchy and that it was stable. In other words, if $m^2\ll M^2$, then the same applied to $\overline{m}^{\,2}$ and m_0^{*2}. From their definitions one has that m^2 and $\overline{m}^{\,2}$ are of the same order of magnitude only if

$$ga_0\le m^2, \quad \lambda a_0^2\le m^2 \ ,$$

which gives a strong condition for the VEV of the heavy field

$$|a_0|\le m \ .$$

For such a small a_0 one gets $a_0\sim -h_1/M^2$, and therefore the tadpole should also be small, $h_1\le mM^2$.

These results can also be obtained by arguments of *naturalness* applied to the parameters in the original potential. One observes that if the linear and cubic terms of the potential vanish, then the theory increases its symmetry to $\pi\to-\pi$, $\phi\to-\phi$. It is natural to expect that the corresponding parameters of these linear and cubic terms be some small quantity. In the $N=1$ supergravity models, one finds a similar feature, with the small dimensionful parameter being $m_{3/2}$. One sees that the rescaling

$$h_1=m_{3/2}M^2\tau_1, \quad h_2=m_{3/2}m^2\tau_2, \quad g_i=m_{3/2}A_i\lambda_1$$

is enough to satisfy the tree-level hierarchy conditions. (Here, and in what follows, we will symbolically represent by $m_{3/2}$ a mass scale of the order of the light-mass scale already present in the model; this terminology is clearly borrowed from softly broken supersymmetric theories.) This is achieved by writing a_0, a_1, and the shifted parameters (4.3) in terms of the new dimensionless τ's and A's, all of which are of order 1. For instance, $a_0=-m_{3/2}\tau_1+O(m_{3/2}^3/M^2)$.

Finally, and at the tree level, we require in order that we have a hierarchy the mass matrix of the model must have a heavy and a light eigenvalue, so that also ω^2 is bounded by $\omega^2\le mM$, and therefore $a_1\le m/M$. A stronger condition is, however, necessary. This is due to the fact that to keep ϕ light, one must have that its shifted parameters be of the same order as the original ones; for the tadpole this gives $\overline{h}_2\sim m_{3/2}m^2$, so that there are two possibilities: namely,

$$\text{(i)}\ \ h_1\sim m_{3/2}^2m^2, \quad a_0\sim m_{3/2}/M, \quad \omega^2\sim mM \ ,$$

$$\text{(ii)}\ \ h_1\sim m_{3/2}M^2, \quad a_0\sim m_{3/2}, \quad \omega^2\sim m_{3/2}m \ .$$

Probably the second case looks more natural because it makes the symmetry-breaking parameter ω^2 proportional to the "small" parameter $m_{3/2}$ (the other possibility, $\omega^2\sim m_{3/2}M$, is ruled out because ω^2 does not vanish with $m_{3/2}$ if there is a geometrical scale; in addition, $\overline{h}_2$ gets too large).

So far, we have only considered the conditions that the parameters of a theory of the type described by Eq. (4.1) must satisfy at the tree level so that it be a bona fide two-scale model, i.e., such that its ground state describes a heavy and a light degree of freedom. We did the same in the preceding section, where we assumed that the tree-level mass $\overline{m}^{\,2}=m^2+(3\lambda_3/\lambda_1)M^2$ was small compared with $\overline{M}^2$. In what follows we will assume that the theory is quantized at the large scale M, around the ground state of the heavy field, where its tadpole vanishes, and that the parameters $\overline{O}$ have sizes suited for the existence of these two scales at the tree level.

When taking into account quantum corrections, one must bear in mind the fact that some parameters which were small at the tree level may acquire sizable quantum corrections. For instance, a_1 was negligibly small above; however, it gets large quantum contributions through the inhomogeneous terms in its RG equation. In other words, what was negligible at the large scale, where renormalization is performed, becomes finite and large at lower scales. In this work we handle this problem by keeping track of all the parameters, or combinations of these, susceptible of receiving destabilizing quantum corrections, which could make them unnaturally large at some scale.

In order to carry out the one-loop computation, it is not necessary to keep all the terms in the action: it is enough to construct the classical renormalized action $S^*[\phi]$ of Eq. (2.7). Taking into account (2.6), one sees that only the quantum corrections proportional to $L=\ln(\overline{M}^2/\mu^2)$ are necessary. To get these one only needs to retain the terms proportional to L in

$$\Gamma^{(1)}(\pi,\phi)=\frac{i\hbar}{2}\int d^4x\int\frac{d^4k}{(2\pi)^2}\mathrm{tr}\ln(k^2-\overline{\mathcal{M}}^2)+\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^2}\mathrm{tr}\ln[k^2-G(k)U(\pi,\phi)]\ ,$$

where we have introduced the matrices

$$\overline{\mathcal{M}}^2\equiv\begin{bmatrix}\overline{M}^2 & \overline{\omega}^2\\ \overline{\omega}^2 & \overline{m}^2\end{bmatrix},\quad \overline{G}_1\equiv\begin{bmatrix}\overline{g}_1 & \overline{g}_3\\ \overline{g}_3 & \overline{g}_4\end{bmatrix},\quad \overline{G}_2\equiv\begin{bmatrix}\overline{g}_3 & \overline{g}_4\\ \overline{g}_4 & \overline{g}_2\end{bmatrix},\quad \Lambda_1\equiv\begin{bmatrix}\lambda_1 & \lambda_3\\ \lambda_3 & \lambda_5\end{bmatrix},\quad \Lambda_2\equiv\begin{bmatrix}\lambda_5 & \lambda_4\\ \lambda_4 & \lambda_2\end{bmatrix},\quad \Lambda_3\equiv\begin{bmatrix}\lambda_3 & \lambda_5\\ \lambda_5 & \lambda_4\end{bmatrix},$$

with $G(k)\equiv(k^2-\overline{\mathcal{M}}^2)^{-1}$. The function $U(\pi,\phi)$ is the 2×2 matrix of the second-order derivatives of the *shifted* potential energy:

$$U(\pi,\phi)=\overline{G}_1\pi+\overline{G}_2\phi+\tfrac{1}{2}\Lambda_1\pi^2+\Lambda_3\pi\phi+\tfrac{1}{2}\Lambda_2\phi^2\ .$$

Finally, the terms proportional to L can be collected into

$$\lim_{m_0^*/\overline{M}\to0}\Gamma^{(1)}[\pi,\phi,L]=\frac{\hbar}{32\pi^2}\int d^4x\left[\sum_{m,n=0}^{m+n=4}a_{mn}\pi^m\phi^n\right]\ln\frac{c_1}{\mu^2}\ .$$

Here c_1 is the heavy eigenvalue of the square of the shifted mass matrix $\overline{\mathcal{M}}^2$, and the coefficients in the expansion a_{mn} are explicitly given by

$$a_{00}=-\tfrac{1}{2}c_1^2\ ,$$

$$a_{10}=-\mathrm{tr}(\overline{G}_1\overline{\mathcal{M}}^2)+\frac{m_0^{*2}}{1+a_1^2}\mathrm{tr}(\overline{G}_1A)\ ,$$

$$a_{01}=-\mathrm{tr}(\overline{G}_2\overline{\mathcal{M}}^2)+\frac{m_0^{*2}}{1+a_1^2}\mathrm{tr}(\overline{G}_2A)\ ,$$

$$a_{20}=-\frac{1}{2}\mathrm{tr}(\overline{G}_1^2+\Lambda_1\overline{\mathcal{M}}^2)+\frac{1}{2}\left[\frac{m_0^{*2}}{1+a_1^2}\mathrm{tr}(\Lambda_1A)+\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\overline{G}_1A)^2\right],$$

$$a_{02}=-\frac{1}{2}\mathrm{tr}(\overline{G}_2^2+\Lambda_2\overline{\mathcal{M}}^2)+\frac{1}{2}\left[\frac{m_0^{*2}}{1+a_1^2}\mathrm{tr}(\Lambda_2A)+\frac{1}{(1+\alpha_1^2)^2}\mathrm{tr}(\overline{G}_2A)^2)\right],$$

$$a_{11}=-\mathrm{tr}(\overline{G}_1\overline{G}_2+\Lambda_3\overline{\mathcal{M}}^2)+\left[\frac{m_0^{*2}}{1+a_1^2}\mathrm{tr}(\Lambda_3A)+\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\overline{G}_1A\overline{G}_2A)\right],$$

$$a_{30}=-\frac{1}{2}\mathrm{tr}(\overline{G}_1\Lambda_1)+\frac{1}{2}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\overline{G}_1A\Lambda_1A)\ ,$$

$$a_{03}=-\frac{1}{2}\mathrm{tr}(\overline{G}_2\Lambda_2)+\frac{1}{2}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\overline{G}_2A\Lambda_2A)\ ,$$

$$a_{21}=-\frac{1}{2}\mathrm{tr}(\overline{G}_2\Lambda_1+2\overline{G}_1\Lambda_3)+\frac{1}{2}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\overline{G}_2A\Lambda_1A+\overline{G}_1A\Lambda_3A)\ ,$$

$$a_{12}=-\frac{1}{2}\mathrm{tr}(\overline{G}_1\Lambda_2+2\overline{G}_2\Lambda_3)+\frac{1}{2}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\overline{G}_1A\Lambda_2A+\overline{G}_2A\Lambda_3A)\ ,$$

$$a_{40}=-\frac{1}{8}\mathrm{tr}(\Lambda_1^2)+\frac{1}{8}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\Lambda_1A)^2\ ,$$

$$a_{04}=-\frac{1}{8}\mathrm{tr}(\Lambda_2^2)+\frac{1}{8}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\Lambda_2 A)^2 ,$$

$$a_{31}=-\frac{1}{2}\mathrm{tr}(\Lambda_1\Lambda_3)+\frac{1}{2}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\Lambda_1 A\Lambda_3 A) ,$$

$$a_{13}=-\frac{1}{2}\mathrm{tr}(\Lambda_2\Lambda_3)+\frac{1}{2}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\Lambda_2 A\Lambda_3 A) ,$$

$$a_{22}=-\frac{1}{4}\mathrm{tr}(\Lambda_1\Lambda_2+2\Lambda_3^2)+\frac{1}{4}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\Lambda_1 A\Lambda_2 A+2\Lambda_3 A\Lambda_3 A) .$$

(Note that these expressions differ by a normalizing factor of $\hbar/32\pi^2$ from the ones in the previous section. This redefinition has been done for the sake of keeping the typography as streamlined as possible. The factor has, of course, been taken into account at all times.)

The matrix A is defined by

$$A\equiv\begin{bmatrix} a_1^2 & a_1 \\ a_1 & 1 \end{bmatrix} .$$

Armed with the above results, it is now straightforward to compute the classical renormalized action $S^*[\phi]$ of Eq. (2.8), given by

$$S^*[\phi^*]=\lim_{m_0^*/\overline{M}\to 0}\{S[\overline{\pi}(\phi),\phi]+\Gamma^{(1)}[\overline{\pi}(\phi),\phi,L]+O(\hbar^2)\} .$$

In performing this computation we have been careful and included the radiative correction to the heavy tadpole a_0 and proportional to a_{10}. As we saw in the previous example, this induces a shift in the VEV of π by

$$\delta a_0=\frac{\hbar}{32\pi^2}\frac{a_{10}}{\overline{M}^2}+O(\hbar^2) .$$

This must be taken into account when computing $\overline{\pi}(\phi)$.

The renormalized action for the light degree of freedom can, as in the previous cases, be written solely in terms of the effective one-loop parameters, whose values are given below in terms of *careted* parameters:

$$A^*_{1\,\mathrm{loop}}=\hat{\Lambda}^*-\frac{\hbar}{64\pi^2}m_0^{*4}\ln\frac{c_1}{\mu^2} ,$$

$$h^*_{1\,\mathrm{loop}}=\hat{h}^*-\frac{\hbar}{32\pi^2}m_0^{*2}g_0^*\ln\frac{c_1}{\mu^2} ,$$

$$m^{*2}_{1\,\mathrm{loop}}=\hat{m}^{*2}-\frac{\hbar}{32\pi^2}(g_0^{*2}+m_0^{*2}\lambda_0^*)\ln\frac{c_1}{\mu^2} ,$$

$$g^*_{1\,\mathrm{loop}}=\hat{g}^*-\frac{\hbar}{32\pi^2}3g_0^*\lambda_0^*\ln\frac{c_1}{\mu^2} ,$$

$$\lambda^*_{1\,\mathrm{loop}}=\hat{\lambda}^*-\frac{\hbar}{32\pi^2}3\lambda_0^{*2}\ln\frac{c_1}{\mu^2} .$$

For brevity's sake we only give the value of the careted cosmological constant:

$$\hat{\Lambda}^*\equiv\Lambda^*+\frac{\hbar}{64\pi^2}(c_1^2+c_2^2)\ln\frac{c_1^2}{\mu^2}$$
$$=\Lambda^*+\frac{\hbar}{64\pi^2}(\overline{M}^4+2\varpi^4+\overline{m}^4)\ln\frac{c_1^2}{\mu^2} .$$

In general, one finds that *all* the *careted* parameters are of the form shown in Eq. (2.9), i.e.,

$$\hat{O}^*=O_0^*+\frac{1}{2}\left[\mu\frac{dO_0^*}{d\mu}\right]\ln\frac{c_1}{\mu^2} .$$

To check that the $\hat{O}^*$ are small, one only need check that at the scale c_1 the tree-level parameters O_0^* are small, in other words, that at the tree level there was a hierarchy. Observe that the quantum corrections are functions of the light scale, depending on M (the heavy scale) only through the logarithm. Thus our calculation compels one to conclude that if the *careted* parameters are small, the *one-loop* parameters will remain small *at all scales.* In other words, the hierarchy remains stable.

V. TWO-SCALE WESS-ZUMINO MODEL

In this section we will extend our generalization of Weisberger's method to a supersymmetric theory. We will not use a component formulation; instead, we use the superspace-superfield formulation of globally $N=1$ supersymmetric theories because it simplifies the algebra and gives compact and transparent results. Our notation and conventions are those of Refs. 8 and 9.

Let us consider the simplest, renormalizable, globally supersymmetric and two-scale model in four spacetime dimensions:

$$S(B,L)=\int d^4x\, d^4\theta(\overline{B}B+\overline{L}L)+\int d^4x[d^2\theta W(B,L)+\mathrm{H.c.}] , \tag{5.1}$$

where the superpotential function is

$$W(B,L)=\frac{M}{2}B^2+\frac{m}{2}L^2+m_{BL}BL+\frac{\lambda_1}{6}B^3+\frac{\lambda_2}{6}L^3+\frac{\lambda_3}{2}B^2L+\frac{\lambda_4}{2}L^2B .$$

Taking B and L to be chiral superfields, this is the Wess-Zumino (WZ) action. We have not included a kinetic term $B\overline{L}+\mathrm{H.c.}$ in order to avoid irrelevant complications. This term is generated in perturbation theory, but we will eliminate it by an appropriate wave-function renormalization. We are also free to add the linear terms $\rho_B B+\rho_L L$ to the superpotential. Nevertheless, we can always perform a redefinition of fields, sources, and parameters such that these tadpoles disappear from the action. Furthermore, this redefinition is stable because zero tadpoles are fixed points of the renormalization group.

The mass matrix is

$$\mathcal{M}=\begin{bmatrix} M & m_{BL} \\ m_{BL} & m \end{bmatrix} .$$

We compute its eigenvalues and find the tree-level conditions for dealing with two widely different mass scales: $m \ll M$ and $m_{BL}^2 \leq Mm$. In any physically relevant theory, m may be typically $m \sim 10^2$ GeV, while M may be taken of the order of the grand unification scale, $M \sim 10^{16}$ GeV. This allows us to have $m_{BL} \leq 10^{10}$ GeV. This size of the masses leads to a hierarchy problem. This can be seen by computing the RGE of the light mass:

$$\mu\frac{dm}{d\mu}=\frac{\hbar}{32\pi^2}[2m(\mathrm{tr}\Lambda_2^2)+2m_{BL}(\mathrm{tr}\Lambda_1\Lambda_2)] ,$$

where

$$\Lambda_1=\begin{bmatrix} \lambda_1 & \lambda_3 \\ \lambda_3 & \lambda_4 \end{bmatrix}, \quad \Lambda_2=\begin{bmatrix} \lambda_3 & \lambda_4 \\ \lambda_4 & \lambda_2 \end{bmatrix} .$$

Therefore, the RGE's show that the tree-level hierarchy is lost, since $m(\mu)$ has contributions proportional to m_{BL}. We can also compute the RGE of the mass m_{BL} and note that it has a contribution proportional to M.

We now proceed to apply our method to this problem. As before, we obtain the solution to the equation of motion of the heavy superfield

$$-\frac{\bar{D}^2}{4}\bar{B}+\frac{\partial W}{\partial B}=0 .$$

The solution $B=B(L)$ is written in terms of an infinite set of proper functions, which have to be supersymmetric and Lorentz invariant and, furthermore, have to conserve the chiral character of B; i.e., they must satisfy the constraint that $\bar{D}_{\dot{\alpha}}B=0$. These conditions immediately limit the form of the expansion to

$$B(L)=a_1L+a_2L^2+a_3L^3+a_4\left[\frac{\bar{D}^2}{4}\bar{L}\right] +a_5\left[\frac{\bar{D}^2}{4}\bar{L}^2\right]+a_6\left[\frac{\bar{D}^2}{4}\bar{L}L\right]+B_4(L) .$$

$B_4(L)$ contains proper functions of mass dimension ≥ 4, and indeed they are being suppressed by powers of $(M)^{-n}$ with $n\geq 3$. The coefficients a_i are obtained by substituting $B(L)$ in the equation of motion and matching factors of the same proper function. This calculation is done by using the well-known properties of the D algebra.[9] A vapid calculation leads to the following set of coupled algebraic equations:

$$a_1=-\frac{m_{BL}}{M} , \quad a_2=-\frac{1}{2M^2}(\lambda_1a_1^2+2\lambda_3a_1+\lambda_4) ,$$
$$a_3=-\frac{1}{M}(\lambda_1a_1+\lambda_3)a_2 , \quad a_4=\frac{1}{M}a_1 ,$$
$$a_5=\frac{1}{M}a_2 , \quad a_6=-\frac{1}{M}(\lambda_1a_1+\lambda_3)a_4 .$$

The low-energy effective action is obtained by eliminating the HDF from the full action (5.1). This is achieved by computing $S(B(L),L)$ in the large hierarchy limit. We find again the same type of results that were found before. The effects of nonrenormalizable interactions on $S(B(L),L)$ are suppressed by powers of the small-mass scale or external momenta over the large masses. In other words, when the heavy-particle masses are very large compared to the small-mass scale or external momenta, the effective theory is renormalizable. Explicitly, the final result is

$$\lim_{E/M\to 0} S[B(L),L]=\int d^4x\, d^4\theta(1+a_1^2)|L|^2+\int d^4x\{d^2\theta[\tfrac{1}{2}(m+m_{BL}a_1)L^2 +\tfrac{1}{6}(\lambda_2+3\lambda_4a_1+3\lambda_3a_1^2+\lambda_1a_1^3)L^3]+\text{H.c.}\} . \tag{5.2}$$

Hence the effective low-energy theory is a one-scale Wess-Zumino model with effective field and parameters given by

$$L_0^*=(1+a_1^2)^{1/2}L ,\quad m_0^*=\frac{m+m_{BL}a_1}{1+a_1^2} ,\quad \lambda_0^*=\frac{\lambda_2+3\lambda_4a_1+3\lambda_3a_1^2+\lambda_1a_1^3}{(1+a_1^2)^{3/2}} .$$

The effective mass is, in fact, small at $\mu=M$. Using the renormalization-group equations of the full theory, one computes

$$\mu\frac{dm_0^*}{d\mu}=\frac{\hbar}{32\pi^2}2m_0^*\left[\frac{a_1^2(\mathrm{tr}\Lambda_1^2)+2a_1(\mathrm{tr}\Lambda_1\Lambda_2)+(\mathrm{tr}\Lambda_2^2)}{1+a_1^2}+a_1\frac{m_0^*}{M}(\mathrm{tr}\Lambda_1\Lambda_2)\right] ,$$
$$\mu\frac{d\lambda_0^*}{d\mu}=\frac{\hbar}{32\pi^2}3\lambda_0^*\left[\frac{a_1^2(\mathrm{tr}\Lambda_1^2)+2a_1(\mathrm{tr}\Lambda_1\Lambda_2)+(\mathrm{tr}\Lambda_2^2)}{1+a_1^2}+a_1\frac{m_0^*}{M}(\mathrm{tr}\Lambda_1\Lambda_2)\right]-\frac{\hbar}{32\pi^2}3\frac{m_0^*}{M}\frac{a_1^2\lambda_1+2a_1\lambda_3+\lambda_2}{(1+a_1^2)^{1/2}}(\mathrm{tr}\Lambda_1\Lambda_2) . \tag{5.3}$$

The reader will note that these renormalization-group equations contain some very small terms (they are suppressed by the factor m_0^*/M). We will keep them in our calculations and will find that decoupling takes place *order by order in powers of* m_0^*/M. [These results have also been obtained and checked, using the component-field formalism. In this case one has to solve three equations of motion: one for a heavy complex scalar field, another for its Weyl fermion superpartner, and, finally, one for the auxiliary component. We solve this set of coupled differential equations in the kinematic region $p \ll M$. We obtain B, B_α, and F_B as functions of L, L_α, and F_L. These functions are power-series expansions of Lorentz-invariant proper functions constructed from the basic blocks L, L_α, and F_L. They may also be obtained directly from the superfield expansion by using the projection technique. The heavy component fields as functions of the light ones have to be substituted into the full component action. Then, after taking its low-energy limit, one obtains the component version of the superspace action (5.2).]

The next step is the computation of the first quantum correction to the full theory, $\Gamma^{(1)}(B,L)$, and eliminate the HDF from there. The result can be compactly given by[8]

$$\Gamma^{(1)}(B,L)=-\frac{\hbar}{2}\mathrm{Tr}\ln\left[\delta^8(x-x')+\int\frac{d^4k}{(2\pi)^4}X(k,x,x')\times e^{ik\cdot(x-x')}\right], \tag{5.4}$$

where $X(k,x,x')$ is a 4×4 matrix which may be written as

$$X(k,x,x')=\begin{bmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{bmatrix}\delta^4(\theta-\theta') .$$

Here we have introduced the following set of 2×2 matrices:

$$X_{11}\equiv V(x)\mathcal{M}G(k)\frac{\bar D^2D^2}{16k^2}, \quad X_{12}\equiv -V(x)G(k)\frac{\bar D^2}{4},$$

$$X_{21}\equiv\overline{X_{12}}, \quad X_{22}\equiv\overline{X_{11}},$$

$$G(k)\equiv(-1)(k^2+\mathcal{M}^2)^{-1}, \quad V(x)\equiv\Lambda_1B(x)+\Lambda_2L(x) .$$

The overbar means a Hermitian conjugate. The free momentum running along the loop is represented by k. By Tr we indicate a functional trace; i.e., the trace is performed on internal and Lorentz indices and on superspace variables.

We now expand the logarithm in $\Gamma^{(1)}(B,L)$. In this process we will find a set of multidimensional θ integrals which are done by using the properties of the D algebra.[10] At the end one checks explicitly that $\Gamma^{(1)}(B,L)$ does not contain any superpotential proper functions (nonrenormalization theorem). The final result is

$$\Gamma^{(1)}(B,L)=\frac{\hbar}{2}\int d^4x\,d^4\theta\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}[V(x)G(k)\bar V(x)G(k)]+\text{higher-dimensional proper functions} .$$

The same arguments of Sec. II also work for this case; i.e., these higher-dimensional proper functions either vanish in the low-energy limit or are the one-particle irreducible 1PI n-point functions of the effective theory we will give below. We write all of the mass parameters of the action $\Gamma^{(1)}(B,L)$ in terms of m_0^* and a_1 and expand its 1PI proper functions in powers of the small parameter $\epsilon\equiv m_0^*/M$. After all the dust settles, we find that only the following D terms survive in this procedure:

$$\lim_{\epsilon\to0}\Gamma^{(1)}(B,L)=\int d^4x\,d^4\theta[(a_{11}+\alpha_{11})|B|^2+(a_{22}+\alpha_{22})|L|^2+(a_{12}+\alpha_{12})(B\bar L+\text{H.c.})] .$$

[We assume that the divergent part of $\Gamma^{(1)}(B,L)$ was canceled by an appropriate counterterm Lagrangian.] As in the pure scalar case, the terms containing the logarithm of the light mass reproduce the n-point functions of the low-energy effective theory. We have divided the coefficients of the D terms into two parts. The reasons for this splitting will become clear later. These coefficients are given by

$$a_{11}=-\frac{\hbar}{32\pi^2}\mathrm{tr}(\Lambda_1^2)\ln\frac{c_1}{\mu^2},$$

$$\alpha_{11}=\frac{\hbar}{32\pi^2}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\Lambda_1A\Lambda_1A)\ln\frac{c_1}{\mu^2},$$

$$a_{22}=-\frac{\hbar}{32\pi^2}\mathrm{tr}(\Lambda_2^2)\ln\frac{c_1}{\mu^2},$$

$$\alpha_{22}=\frac{\hbar}{32\pi^2}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\Lambda_2A\Lambda_2A)\ln\frac{c_1}{\mu^2},$$

$$a_{12}=-\frac{\hbar}{32\pi^2}\mathrm{tr}(\Lambda_1\Lambda_2)\ln\frac{c_1}{\mu^2},$$

$$\alpha_{12}=\frac{\hbar}{32\pi^2}\frac{1}{(1+a_1^2)^2}\mathrm{tr}(\Lambda_1A\Lambda_2A)\ln\frac{c_1}{\mu^2},$$

where A is the 2×2 matrix:

$$A\equiv\begin{bmatrix}a_1^2 & a_1\\ a_1 & 1\end{bmatrix}.$$

Recall that $a_1\equiv-m_{BL}/M$, and c_1 represents the heavy scale.

As always, we compute the full theory to one-loop order, $\Gamma(B,L)$, by adding the tree-level action and its corresponding quantum correction computed in full, and *then* taking the limit to the low-energy regime. In this model we find an additional complication due to wave-function renormalization effects. We find that they can be taken into account in several ways. For example, (a) we might construct the effective low-energy action directly from $\Gamma(B,L)$ without any change in the field basis, and (b) by a *finite* wave-function redefinition, we can eliminate from $\Gamma(B,L)$ the a_{ij} or the α_{ij} or both or an arbitrary combination of them. We will proceed according to (b). Our motivation to do this resides in the fact that this allows one to *explicitly* check how decoupling takes place, without first disregarding the terms of $O(m_0^*/M)$ in Eq. (5.3). One can then go ahead reabsorb the a_{ij} by a finite redefinition of the superfields into

$$B=\hat{B}+\frac{\hbar}{64\pi^2}[\hat{B}(\mathrm{tr}\Lambda_1^2)+\hat{L}(\mathrm{tr}\Lambda_1\Lambda_2)]\ln\frac{c_1}{\mu^2}\ ,$$
$$L=\hat{L}+\frac{\hbar}{64\pi^2}[\hat{L}(\mathrm{tr}\Lambda_2^2)+\hat{B}(\mathrm{tr}\Lambda_1\Lambda_2)]\ln\frac{c_1}{\mu^2}\ . \tag{5.5}$$

The action $\Gamma(B,L)$ is then written as a functional of the field basis (5.5). Note that this field redefinition induces a change of the coefficients in the superpotential. From now on everything proceeds as before. We solve the equation of motion for the heavy field and plug it into the full action in order to eliminate the HDF. We omit those proper functions which enter suppressed by powers of $1/M$ or m_0^*/M. Finally, we perform a finite redefinition of the wave function which allows us to recover the standard normalization of the kinetic term. We get an effective theory which is a one-scale Wess-Zumino model:

$$S^*(L^*_{1\,\mathrm{loop}})=\int d^4x\, d^4\theta |L^*_{1\,\mathrm{loop}}|^2+\int d^4x[d^2\theta\, W^*(L^*_{1\,\mathrm{loop}})+\mathrm{H.c.}]\ ,$$

with the superpotential function being

$$W^*(L^*_{1\,\mathrm{loop}})\equiv\tfrac{1}{2}m^*_{1\,\mathrm{loop}}L^{*2}_{1\,\mathrm{loop}}+\tfrac{1}{6}\lambda^*_{1\,\mathrm{loop}}L^{*3}_{1\,\mathrm{loop}}\ .$$

The effective one-loop parameters are very complicated functions of the original ones:

$$m^*_{1\,\mathrm{loop}}\equiv m_0^*\left\{1+\frac{\hbar}{32\pi^2}\,\frac{a_1^2(\mathrm{tr}\Lambda_1^2)+2a_1(\mathrm{tr}\Lambda_1\Lambda_2)+(\mathrm{tr}\Lambda_2^2)}{1+a_1^2}-\frac{\hbar}{32\pi^2}\lambda_0^{*2}+\frac{\hbar}{32\pi^2}a_1\frac{m_0^*}{M}(\mathrm{tr}\Lambda_1\Lambda_2)+O(\hbar^2)\right\}\ln\frac{c_1}{\mu^2}\ ,$$
$$\lambda^*_{1\,\mathrm{loop}}=\lambda_0^*\left\{1+\frac{\hbar}{32\pi^2}\,\frac{3}{2}\,\frac{a_1^2(\mathrm{tr}\Lambda_1^2)+2a_1(\mathrm{tr}\Lambda_1\Lambda_2)+(\mathrm{tr}\Lambda_2^2)}{1+a_1^2}-\frac{\hbar}{32\pi^2}\,\frac{3}{2}\lambda_0^{*2}+\frac{\hbar}{32\pi^2}\,\frac{3}{2}a_1\frac{m_0^*}{M}(\mathrm{tr}\Lambda_1\Lambda_2)+O(\hbar^2)\right\}\ln\frac{c_1}{\mu^2}$$
$$-\frac{\hbar}{32\pi^2}\,\frac{3}{2}\,\frac{m_0^*}{M}\,\frac{a_1^2\lambda_1+2a_1\lambda_3+\lambda_2}{(1+a_1^2)^{1/2}}(\mathrm{tr}\Lambda_1\Lambda_2)\ln\frac{c_1}{\mu^2}+O(\hbar^2)\ . \tag{5.6}$$

The tree-level effective parameters are those of (5.2). These formulas give us the one-loop effective parameters as functions of the original ones. With them we can now explicitly display their size and, simultaneously, prove that the decoupling theorem is satisfied even for m_0^*/M-order contributions. Indeed, from the definitions (5.6) it follows that

$$\mu\frac{dm^*_{1\,\mathrm{loop}}}{d\mu}=\frac{\hbar}{32\pi^2}2m^*_{1\,\mathrm{loop}}\lambda^{*2}_{1\,\mathrm{loop}}\ ,$$
$$\mu\frac{d\lambda^*_{1\,\mathrm{loop}}}{d\mu}=\frac{\hbar}{32\pi^2}3\lambda^{*3}_{1\,\mathrm{loop}}\ .$$

Therefore, the effective parameters are, in fact, decoupled. When these renormalization-group equations are integrated together with the boundary conditions

$$m^*_{1\,\mathrm{loop}}(\mu=M)=m_0^*(\mu=M)\ ,$$
$$\lambda^*_{1\,\mathrm{loop}}(\mu=M)=\lambda_0^*(\mu=M)\ ,$$

the initial hierarchy problem disappears since $m_0^*(\mu=M)$ is really small. This result allows us to reorganize the expressions (5.6) just as we did in the pure scalar case:

$$m^*_{1\,\mathrm{loop}}\equiv\hat{m}_0^*-\frac{\hbar}{32\pi^2}m_0^*\lambda_0^{*2}\ln\frac{c_1}{\mu^2}\ ,$$
$$\lambda^*_{1\,\mathrm{loop}}\equiv\hat{\lambda}_0^*-\frac{\hbar}{32\pi^2}\,\frac{3}{2}\lambda_0^{*3}\ln\frac{c_1}{\mu^2}\ . \tag{5.7}$$

These careted parameters are such that the last two expressions for the effective parameters, (5.6) and (5.7) coincide. Their dependence in $\hbar\ln(c_1/\mu^2)$ is such that they are renormalization-group invariant.

VI. GENERAL SOFTLY BROKEN WZ MODEL

The supersymmetric version of the standard model of particle physics may be a softly broken supersymmetric theory, since the "soft" breaking terms only generate logarithmic ultraviolet divergences, and hence the gauge hierarchy problem of ordinary grand unified theories (GUT's) is not present (absence of quadratic divergences). However, it is known that the soft breaking of supersymmetry may destabilize the gauge hierarchy. In principle, this may happen in the case where the light and heavy sectors are gauge singlets, without any gauge symmetry to protect the low-energy scales from the quantum corrections produced by the heavy masses.[11,12]

In this section we will apply our procedure to a softly broken Wess-Zumino model. The description of this theory through the superfield-superspace formalism requires the introduction of explicit, θ-dependent terms (spurions) in the Lagrangian.[11] The fundamental ideas of the procedure remain the same as in the previous cases, but it will be necessary to introduce some new mathematical tools for dealing with the calculations. In particular, we develop a technique for solving superfield equations of motion, and computing one-loop effective actions, which involves spurions (explicit θ factors) and supersymmetric covariant derivatives. The final result of this procedure is a compact formula which contains *all* the quantum information of the theory.

We write down the most general, two-scale, renormalizable, and softly broken Wess-Zumino model in four dimensions. This is described by the action

$$S(B,L)=\int d^4x\, d^4\theta\{|B|^2+|L|^2-\theta^2\bar{\theta}^2[\mu_B^2|B|^2+\mu_L^2|L|^2+\mu_{BL}^2(B\bar{L}+\text{H.c.})]\}$$
$$+\int d^4x\{d^2\theta[W(B,L)-\theta^2 f(B,L)]+\text{H.c.}\} . \tag{6.1}$$

In this expression $W(B,L)$ and $f(B,L)$ are the superpotential functions, which for the most general case are given by

$$W(B,L)=\rho_B B+\rho_L L+\frac{M}{2}B^2+\frac{m}{2}L^2+m_{BL}BL+\frac{\lambda_1}{6}B^3+\frac{\lambda_2}{6}L^3+\frac{\lambda_3}{2}B^2L+\frac{\lambda_4}{2}L^2B ,$$
$$f(B,L)=b+h_B B+h_L L+\frac{f_B}{2}B^2+\frac{f_L}{2}L^2+f_{BL}BL+\frac{g_1}{6}B^3+\frac{g_2}{6}L^3+\frac{g_3}{2}B^2L+\frac{g_4}{2}L^2B .$$

$f(B,L)$ parametrizes the soft breaking terms. This model contains every feature needed for building $N=1$ supergravity extensions of the Weinberg-Salam model and grand unified models. In order to be consistent with the phenomenological requirements of these $N=1$ scenarios, we have to impose [11] the tree-level mass hierarchy

$$\frac{m}{M}\ll 1 , \quad m_{BL}^2\le Mm .$$

Concomitantly, the soft breaking terms must be constrained by the naturalness of the spontaneous symmetry breaking of local supersymmetry:

$$\begin{aligned}
&\mu_B^2=m_{3/2}^2\sigma_B,\quad \mu_L^2=m_{3/2}^2\sigma_L,\quad \mu_{BL}^2=m_{3/2}^2\sigma_{BL} ,\\
&b=m_{3/2}\Lambda ,\\
&h_B=m_{3/2}M^2\tau_B,\quad h_L=m_{3/2}m^2\tau_L ,\\
&f_B=m_{3/2}MB_B,\quad f_L=m_{3/2}mB_L,\quad f_{BL}=m_{3/2}m_{BL}B_{BL} ,\\
&g_1=m_{3/2}\lambda_1A_1,\quad g_2=m_{3/2}\lambda_2A_2,\quad g_3=m_{3/2}\lambda_3A_3 ,\\
&g_4=m_{3/2}\lambda_4A_4,\quad m_{3/2}\sim m .
\end{aligned}\tag{6.2}$$

All σ's, τ's, B's, A's, and λ's are of order 1. This choice of parameters summarizes the essential features of models having phenomenological applications. By the same token, we can consider (6.2) as the hypothesis fixing the size of the tree-level parameters at $\mu=M$.

Finally, we make some considerations relative to the linear terms in B and L. We know that the tadpole terms can be eliminated from the action by an appropriate redefinition of the fields, sources, and parameters. This is a good strategy for the supersymmetric tadpoles ρ_B and ρ_L, since it is stable under perturbation theory. ($\rho_B=\rho_L=0$ are fixed points of the renormalization group; cf., e.g., Ref 9). It is therefore safe to set $\rho_B=\rho_L=0$. However, this choice is impossible for the non-supersymmetric tadpoles h_B and h_L since they receive quantum corrections, and hence the redefinition must be redone order by order in perturbation theory.

After these considerations we can now go ahead with our strategy. We start by eliminating the HDF from the tree-level action. We solve the tree-level heavy superfield equation of motion in the appropriate kinematic region:

$$-\frac{\bar{D}^2}{4}(\bar{B}-m_{3/2}^2\sigma_B\theta^2\bar{\theta}^2\bar{B}-m_{3/2}^2\sigma_{BL}\theta^2\bar{\theta}^2\bar{L})$$
$$+\frac{\partial W}{\partial B}-\theta^2\frac{\partial f}{\partial B}=0 . \tag{6.3}$$

In order to eliminate the inhomogeneous term $-m_{3/2}M^2\tau_B\theta^2$ from the equation, we *necessarily* have to perform the shift

$$B=B'+a_0+b_0\theta^2 . \tag{6.4}$$

After substituting (6.4) and (6.3) and demanding the vanishing of the field-independent terms, one obtains a couple of algebraic equations; these determine a_0 and b_0 as functions of the original parameters:

$$b_0+Ma_0+\frac{\lambda_1}{2}a_0^2=0 ,$$
$$(M+\lambda_1a_0)b_0-m_{3/2}(M^2\tau_B+MB_Ba_0$$
$$+m_{3/2}\sigma_Ba_0+\tfrac{1}{2}\lambda_1A_1a_0^2)=0 .$$

Both of these equations are exact. The remaining equation for B' is also exact. Obviously, the equation for B' can be obtained from the action $S(B,L)=S(B'+a_0+b_0\theta^2,L)$. In this action we search for dangerous terms which could spoil the tree-level mass hierarchy. One such term is

$$-\tfrac{1}{2}\int d^4x\, d^2\theta\, \theta^2(m_{3/2}mB_L+m_{3/2}\lambda_4 A_4 a_0-\lambda_4 b_0)L^2 \ . \tag{6.5}$$

Expanding a_0 and b_0 in a power series in $m_{3/2}/M$, we find the dominant terms to be

$$a_0=-m_{3/2}\tau_B \ , \quad b_0=m_{3/2}M\tau_B \ ,$$

which substituted into (6.5) give

$$m_{3/2}mB_L-m_{3/2}^2\lambda_4 A_4\tau_B-m_{3/2}M\lambda_4\tau_B\sim -m_{3/2}M\lambda_4\tau_B \ .$$

Therefore, the nonsupersymmetric heavy tadpole τ_B spoils the tree-level hierarchy of the theory. The coupling constant λ_4 is not the culprit: In fact, the shifted action $S(B',L)$ also has the superpotential term

$$\int d^4x\, d^2\theta \left[m_{BL}a_0+\frac{\lambda_3}{2}a_0^2\right] L \ .$$

Its dominant term $-m_{3/2}m_{BL}\tau_B$ also destroys the hierarchy, showing that τ_B is clearly the responsible parameter.

It is easy to convince oneself that the light character of the L sector is preserved if, and only if, in the original action the condition

$$\tau_B \le \frac{m}{M} \quad \text{at } \mu=M \ , \tag{6.6}$$

is respected. In other words, if $h_B=m_{3/2}M^2\tau_B$ with $\tau_B\sim 1$ at $\mu=M$, then the shifted masses do not have a tree-level mass hierarchy. It should be clear that the choice of value for τ_B is done at the tree level. Later, we will have to check that higher-order radiative corrections do not spoil this hierarchy.

We now consider the remaining equation for B'. Its solution may be given as a power-series expansion in the supersymmetric and Lorentz invariants of the theory. They form a minimal and complete set of proper functions which solve the equation of motion. It is useful to introduce the operator $R\equiv D^2\theta^2=-4+4\theta D+\theta^2 D^2$. Some of its properties can be found in Ref. 14. Some useful additional properties are given in the Appendix. As in the exactly supersymmetric (SUSY) case, the expansion has to be consistent with the chirality constraint of B. These requirements reduce the form of the expansion to

$$B'=B'(L)=B_1+B_2+B_3+\theta^2(b_1+b_2+b_3) \ ,$$

where, explicitly,

$$B_1(L)=a_1L+a_2\frac{\bar R}{4}\bar L+a_3\frac{\bar R R}{16}L \ ,$$

$$B_2(L)=a_4L^2+a_5\frac{\bar R}{4}\bar L^2+a_6\frac{\bar R}{4}\bar L L+a_7\frac{\bar R R}{16}L^2$$
$$+a_8\frac{\bar R R}{16}\bar L L+a_9 L\frac{\bar R R}{16}L+a_{10}\frac{\bar D^2}{4}\bar L$$
$$+a_{11}\frac{\bar R}{4}\frac{D^2}{4}L+a_{12}\frac{\bar R R}{16}\frac{\bar D^2}{4}\bar L \ ,$$

$$b_1(L)=A_2L+A_3\frac{\bar R}{4}\bar L \ ,$$

$$b_2(L)=A_5L^2+A_7\frac{\bar R}{4}\bar L^2+A_8\frac{\bar R}{4}\bar L L+A_{11}\frac{\bar D^2}{4}\bar L$$
$$+A_{12}\frac{\bar R}{4}\frac{D^2}{4}L \ .$$

The functions $B_3(L)$ and $b_3(L)$ contain proper functions that go like M^{-n} with $n\ge 2$ and $n\ge 1$, respectively. The coefficients a_i, a_{ij}, A_i, and A_{ij} are computed in the usual fashion: One substitutes $B'=B'(L)$ into the heavy superfield equation of motion and sets the coefficient of each one of the proper functions equal to zero. The result of this procedure is an infinite set of coupled algebraic equations which determine, by iterative procedure, the a's and A's. For instance, the equation determining a_1 is $(M+\lambda_1 a_0)a_1+m_{BL}+\lambda_3 a_0=0$.

After having obtained $B=B(L)$, we may eliminate the HDF from the tree-level action by computing $S(B(L),L)$. We did this calculation without imposing any condition on the parameters of the softly broken sector (6.2). This straightforward, albeit long, calculation shows several interesting properties.

(i) The proper functions contained in $B_3(L)$ and $b_3(l)$ *do not contribute* to the low-energy limit of $S(B(L),L)$ because their coefficients are suppressed by powers of the heavy mass. It turns out that the dangerous terms disappear on shell.

(ii) The parameters of the supersymmetric sector get contributions from the softly broken sector and vice versa. The simplest example of this situation comes out when we substitute B by $a_0+b_0\theta^2$ into the full action. A direct consequence of this property is that the final supersymmetric effective parameters have contributions which are proportional to the soft-breaking terms.

(iii) The low-energy effective action coming from $S(B(L),L)$ can, in principle, not be a softly broken SUSY theory. The action $S(B(L),L)$ has a set of hard-breaking terms, such as, for example,

$$\int d^4\theta\, L^2\bar L, \quad \int d^4\theta\, \theta^2\bar\theta^2 L^2\frac{D^2}{4}L, \quad \int d^4\theta\, \theta^2\bar\theta^2 L\Box L \ .$$

The effective theory is renormalizable, but these interaction terms spoil its *soft* character.

It is easy to check that the hard-breaking terms disappear *if and only if* the parameters of the broken sector in the full theory satisfy (6.2) and (6.6). This may be rephrased as follows: *The naturalness condition on the breaking of local supersymmetry leads to a low-energy effective theory which is a softly broken, one-scale Wess-Zumino model. This it should be noted is only a consequence of the smallness of the gravitino mass.*

Within these conditions both the large hierarchy limit and $m_{3/2}/M\to 0$ lead (after a finite redefinition of L necessary for obtaining a standard kinetic term) to the following one-scale, tree-level effective theory:

$$S^*(L_0^*)\equiv \lim_{m_{3/2}/M\to 0} S(B(L_0^*),L_0^*)$$
$$=\int d^4x\, d^4\theta(|L_0^*|^2-m_{3/2}^2\theta^2\bar\theta^2\sigma_0^*|L_0^*|^2)+\int d^4x[d^2\theta(W^*(L_0^*)-m_{3/2}\theta^2 f^*(L_0^*))+\text{H.c.}] . \tag{6.7}$$

In order to cast this result in the simplest form, we have introduced the effective potential functions [cf. Eq. (6.1)]

$$W^*\equiv\rho_0^*L_0^*+\tfrac{1}{2}m_0^*L_0^{*2}+\tfrac{1}{6}\lambda_0^*L_0^{*3} ,$$
$$f^*\equiv b_0^*+m_0^{*2}\tau_0^*L_0^*+\tfrac{1}{2}m_0^*B_0^*L_0^{*2}+\tfrac{1}{6}\lambda_0^*A_0^*L_0^{*3} .$$

Here the quantities denoted by asterisks are the effective tree-level parameters. They are related to the original parameters through the definitions

$$\rho_0^*=\frac{1}{(1+a_1^2)^{1/2}}\left[m_{BL}a_0+\frac{\lambda_3}{2}a_0^2\right] ,$$

$$m_0^*=\frac{1}{(1+a_1^2)}[m+\lambda_4 a_0+(m_{BL}+\lambda_3 a_0)a_1] ,$$

$$\lambda_0^*=\frac{1}{(1+a_1^2)^{3/2}}(\lambda_2+3\lambda_4 a_1+3\lambda_3 a_1^2+\lambda_1 a_1^3) ,$$

$$\sigma_0^*=\frac{1}{(1+a_1^2)}(\sigma_L+2a_1\sigma_{BL}+a_1^2\sigma_B) ,$$

$$m_{3/2}b_0^*=\tfrac{1}{2}b_0^2+m_{3/2}(b+M^2\tau_B a_0+\tfrac{1}{2}m_{3/2}\sigma_B a_0^2+\tfrac{1}{2}MB_B a_0^2+\tfrac{1}{6}\lambda_1 A_1 a_0^3) ,$$

$$m_{3/2}m_0^{*2}\tau_0^*=\frac{1}{(1+a_1^2)^{1/2}}\left[-(m_{BL}+\lambda_3 a_0)b_0+m_{3/2}(m^2\tau_L+m_{BL}B_{BL}a_0+m_{3/2}\sigma_{BL}a_0+\frac{1}{2}\lambda_3 A_3 a_0^2)\right] ,$$

$$m_{3/2}m_0^*B_0^*=\frac{1}{1+a_1^2}[m_{3/2}mB_L+m_{3/2}\lambda_4 A_4 a_0-\lambda_4 b_0+a_1^2(m_{3/2}MB_B+m_{3/2}\lambda_1 A_1 a_0-\lambda_1 b_0)$$
$$+2a_1(m_{3/2}m_{BL}B_{BL}+m_{3/2}\lambda_3 A_3 a_0-\lambda_3 b_0)] ,$$

$$\lambda_0^*A_0^*=\frac{1}{(1+a_1^2)^{3/2}}(\lambda_2 A_2+3\lambda_4 A_4 a_1+3\lambda_3 A_3 a_1^2+\lambda_1 A_1 a_1^3) .$$

From these explicit expressions one checks immediately the previous statements about the size of the heavy tadpole τ_B. Only for those values (6.2) and (6.6) are the effective tree-level parameters really light.

As in the scalar case, it is convenient to write the one-loop correction for the theory in terms of the shifted heavy superfield B'. (The shift implies that we are building up the radiative correction around the classical minimum of the heavy sector of the theory.) In terms of the superfield basis B' and L, the action $\Gamma^{(1)}(B',L)$ has the same functional form as the corresponding quantum correction for the exactly supersymmetric case (5.3), but with the X_{ij} matrices

$$X_{11}=U(x)\mathcal{M}'G(k)\frac{\bar D^2D^2}{16k^2}-\left[\mathcal{F}'\mathcal{M}'G(k)\theta^2\frac{\bar D^2D^2}{16k^2}+\mu^2G(k)\theta^2\frac{\bar D^2}{4}\bar\theta^2\frac{D^2}{4}\right] ,$$
$$X_{12}=-U(x)G(k)\frac{\bar D^2}{4}+\left[\mathcal{F}'G(k)\theta^2\frac{\bar D^2}{4}+\mu^2\mathcal{M}'G(k)\theta^2\frac{\bar D^2}{4}\bar\theta^2\frac{D^2\bar D^2}{16k^2}\right] ,$$
$$X_{21}=\overline{X_{12}} ,$$
$$X_{22}=\overline{X_{11}} , \tag{6.8}$$

where

$$U(x)\equiv\Lambda_1B(x)+\Lambda_2L(x)-\theta^2m_{3/2}G_1B(x)-\theta^2m_{3/2}G_2L(x) .$$

The matrices Λ_1 and Λ_2 are the ones that were introduced for the supersymmetric case; the remaining are

$$G_1 \equiv \begin{bmatrix} \lambda_1 A_1 & \lambda_3 A_3 \\ \lambda_3 A_3 & \lambda_4 A_4 \end{bmatrix},$$

$$G_2 \equiv \begin{bmatrix} \lambda_3 A_3 & \lambda_4 A_4 \\ \lambda_4 A_4 & \lambda_2 A_2 \end{bmatrix},$$

$$\mu^2 \equiv m_{3/2}^2 \begin{bmatrix} \sigma_B & \sigma_{BL} \\ \sigma_{BL} & \sigma_L \end{bmatrix},$$

$$\mathcal{M}' \equiv \begin{bmatrix} M+\lambda_1 a_0 & M_{BL}+\lambda_3 a_0 \\ m_{BL}+\lambda_3 a_0 & m+\lambda_4 a_0 \end{bmatrix},$$

$$\mathcal{F}' \equiv \begin{bmatrix} m_{3/2} M B_B + m_{3/2}\lambda_1 A_1 a_0 - \lambda_1 b_0 & m_{3/2} m_{BL} B_{BL} + m_{3/2}\lambda_3 A_3 a_0 - \lambda_3 b_0 \\ m_{3/2} m_{BL} B_{BL} + m_{3/2}\lambda_3 A_3 a_0 - \lambda_3 b_0 & m_{3/2} m B_L + m_{3/2}\lambda_4 A_4 a_0 - \lambda_4 b_0 \end{bmatrix}.$$

The function $G(k)$ is constructed with the shifted mass matrix $G(k) = -(k^2+\mathcal{M}'^2)^{-1}$. We point out that the softly broken terms in the original action have been considered as interaction terms.

The expansion of the functional trace for this broken theory can now be calculated in complete analogy with the unbroken case. Any term in the expansion is reduced by using D algebra until one is left with a single θ^2 integration. (We recall that only a finite set of integrands do not vanish inside the last θ integral.[10,14])

Next, we compute the limit m_0^*/M, $m_{3/2}/M \to 0$ on $\Gamma^{(1)}(B',L)$. The next step consists in rewriting the action $\Gamma^{(1)}(b',L)$ in terms of the old field B. It turns out that for technical reasons, having to do with algebraic details, it is more convenient to write $\Gamma^{(1)}(B',L)$ in terms of B.

At this point we are left with the usual task of computing the effective low-energy action to one-loop order from the full action including its radiative corrections:

$$\Gamma(B,L) = S(B,L) + \lim_{m_0^*/M,\, m_{3/2}/M \to 0} \Gamma^{(1)}(B,L) + O(\hbar^2).$$

From now on everything proceeds as in the supersymmetric case. In the end we find that the effective action is given by

$$\Gamma^*(L^*_{1\,\text{loop}}) = \int d^4x\, d^4\theta (|L^*_{1\,\text{loop}}|^2 - m_{3/2}^2 \theta^2\bar\theta^2 \sigma^*_{1\,\text{loop}} |L^*_{1\,\text{loop}}|^2) + \int d^4x \{ d^2\theta [W^*_{1\,\text{loop}}(L^*_{1\,\text{loop}}) - m_{3/2}\theta^2 f^*_{1\,\text{loop}}(L^*_{1\,\text{loop}})] + \text{H.c.} \}.$$

The superpotentials $W^*_{1\,\text{loop}}$ and $f^*_{1\,\text{loop}}$ are functions of the same form as those of the tree-level case, but written in terms of the one-loop effective field and parameters. These one-loop effective parameters are very complicated functions of the original parameters of the initial action. However, one discovers again that they can be recast as

$$\begin{aligned}
\rho^*_{1\,\text{loop}} &= \hat\rho_0^* \left[1 - \frac{\hbar}{32\pi^2}\frac{1}{2}\lambda_0^{*2} \ln\frac{c_1}{\mu^2} \right], \\
m^*_{1\,\text{loop}} &= \hat m_0^* \left[1 - \frac{\hbar}{32\pi^2}\lambda_0^{*2} \ln\frac{c_1}{\mu^2} \right], \\
\lambda^*_{1\,\text{loop}} &= \hat\lambda_0^* \left[1 - \frac{\hbar}{32\pi^2}\frac{3}{2}\lambda_0^{*2} \ln\frac{c_1}{\mu^2} \right], \\
\sigma^*_{1\,\text{loop}} &= \hat\sigma_0^* - \frac{\hbar}{32\pi^2}[(\lambda_0^* A_0^*)^2 + 3\sigma_0^*\lambda_0^{*2}] \ln\frac{c_1}{\mu^2}, \\
b^*_{1\,\text{loop}} &= \hat b_0^* - \frac{\hbar}{64\pi^2}(m_{3/2}^3\sigma_0^{*2} + 2m_{3/2} m_0^{*2}\sigma_0^* + m_{3/2} m_0^{*2} B_0^{*2} + 2\rho_0^*\lambda_0^* m_0^* B_0^*) \ln\frac{c_1}{\mu^2}, \\
m^{*2}_{1\,\text{loop}}\tau^*_{1\,\text{loop}} &= \widehat{m_0^{*2}\tau_0^*} - \frac{\hbar}{32\pi^2}\left[m_{3/2} A_0^* B_0^* \lambda_0^* m_0^* + 2m_{3/2}\lambda_0^*\sigma_0^* m_0^* + \rho_0^* A_0^* \lambda_0^{*2} + B_0^*\lambda_0^* m_0^{*2} + \frac{1}{2} m_0^{*2}\lambda_0^{*2}\tau_0^* \right] \ln\frac{c_1}{\mu^2}, \\
m^*_{1\,\text{loop}} B^*_{1\,\text{loop}} &= \widehat{m_0^* B_0^*} - \frac{\hbar}{32\pi^2}(2m_0^*\lambda_0^{*2} B_0^* + 2A_0^* m_0^* \lambda_0^{*2}) \ln\frac{c_1}{\mu^2}, \\
\lambda^*_{1\,\text{loop}} A^*_{1\,\text{loop}} &= \widehat{\lambda_0^* A_0^*} - \frac{\hbar}{32\pi^2}\frac{9}{2} A_0^* \lambda_0^{*3} \ln\frac{c_1}{\mu^2}.
\end{aligned} \tag{6.9}$$

In these formulas c_1 represents the heavy eigenvalue of the square of the shifted mass matrix $\mathcal{M}'^2$. Remarkably, these parameters have the properties we mentioned in Sec. I. To test for decoupling we must check that the one-loop effective parameters defined above satisfy the renormalization-group equations of a one-scale, softly broken Wess-Zumino model in four dimensions. That this indeed is the case may be seen by applying the scaling operator $d/d\ln\mu$ to both sides of (6.9). Up to $\hbar^2$ terms we obtain

$$
\begin{aligned}
(\rho^*_{1\,\mathrm{loop}}) &= \frac{\hbar}{32\pi^2}\rho^*_{1\,\mathrm{loop}}\lambda^{*2}_{1\,\mathrm{loop}}\,,\\
(m^*_{1\,\mathrm{loop}}) &= \frac{\hbar}{32\pi^2}2m^*_{1\,\mathrm{loop}}\lambda^{*2}_{1\,\mathrm{loop}}\,,\\
(\lambda^*_{1\,\mathrm{loop}}) &= \frac{\hbar}{32\pi^2}3\lambda^{*2}_{1\,\mathrm{loop}}\,,\\
(\sigma^*_{1\,\mathrm{loop}}) &= \frac{\hbar}{32\pi^2}[2(A^*_{1\,\mathrm{loop}}\lambda^*_{1\,\mathrm{loop}})^2+6\sigma^{*2}_{1\,\mathrm{loop}}\lambda^{*2}_{1\,\mathrm{loop}}]\,,\\
(b^*_{1\,\mathrm{loop}}) &= \frac{\hbar}{32\pi^2}(m^3_{3/2}\sigma^{*2}_{1\,\mathrm{loop}}+2m_{3/2}m^{*2}_{1\,\mathrm{loop}}\sigma^*_{1\,\mathrm{loop}}+m_{3/2}m^{*2}_{1\,\mathrm{loop}}B^{*2}_{1\,\mathrm{loop}}+2\rho^*_{1\,\mathrm{loop}}\lambda^*_{1\,\mathrm{loop}}m^*_{1\,\mathrm{loop}}B^*_{1\,\mathrm{loop}})\,,\\
(m^{*2}_{1\,\mathrm{loop}}\tau^*_{1\,\mathrm{loop}}) &= \frac{\hbar}{32\pi^2}(2m_{3/2}A^*_{1\,\mathrm{loop}}B^*_{1\,\mathrm{loop}}m^*_{1\,\mathrm{loop}}\lambda^*_{1\,\mathrm{loop}}+4m_{3/2}m^*_{1\,\mathrm{loop}}\lambda^*_{1\,\mathrm{loop}}\sigma^{*2}_{1\,\mathrm{loop}}\\
&\qquad +2\rho^*_{1\,\mathrm{loop}}\lambda^{*2}_{1\,\mathrm{loop}}A^*_{1\,\mathrm{loop}}+2m^{*2}_{1\,\mathrm{loop}}\lambda^*_{1\,\mathrm{loop}}B^*_{1\,\mathrm{loop}}+m^{*2}_{1\,\mathrm{loop}}\lambda^{*2}_{1\,\mathrm{loop}})\,,\\
(m^*_{1\,\mathrm{loop}}B^*_{1\,\mathrm{loop}}) &= \frac{\hbar}{32\pi^2}(4m^*_{1\,\mathrm{loop}}\lambda^{*2}_{1\,\mathrm{loop}}B^*_{1\,\mathrm{loop}}+4m^*_{1\,\mathrm{loop}}\lambda^{*2}_{1\,\mathrm{loop}}A^*_{1\,\mathrm{loop}})\,,\\
(\lambda^*_{1\,\mathrm{loop}}A^*_{1\,\mathrm{loop}}) &= \frac{\hbar}{32\pi^2}9A^*_{1\,\mathrm{loop}}\lambda^{*3}_{1\,\mathrm{loop}}\,.
\end{aligned}
\tag{6.10}
$$

This proves that the RGE's are closed and the ones corresponding to a fully decoupled theory. They may be integrated together with the boundary conditions

$$O^*_{1\,\mathrm{loop}}(\mu=M)=O^*_0(\mu=M)\,.$$

We observe that the one-loop effective parameters will remain light if the original parameters of the fully theory satisfy the boundary conditions (6.2) and (6.6). This proves the stability of the hierarchy of the one-loop effective parameters under renormalization.

VII. SUMMARY AND CONCLUSIONS

In this paper we have considered the low-energy limit of two-scale quantum field theories. We have studied several model theories, each with its own special field theory behavior. In all the models the heavy sector couples to the light sector via some coupling that we introduce in the tree-level potential; because of this, one has to check on the effects that the heavy sector has on the light sector due to quantum corrections. In principle, if we do not appropriately separate both sectors, there will be dangerous contributions from the heavy-light mixing whose final effect will be to make heavy the light sector.

Here we have shown that *there is* a field-theoretic solution to this problem and have illustrated its application in each of the models that we have considered. This solution is based on a straightforward application of Weisberger's procedure for multiple-scale field theories, to which we add a new, renormalization-group-based, interpretation for the effective parameters. Assuming the existence of a tree-level hierarchy, we show how it is maintained through radiative corrections without any need for either fine-tunings or having to impose global symmetries to this end. The dimensional, effective tree-level parameters receive quantum contributions which are proportional to heavy logarithms of the heavy mass, and this happens in such a way that the dangerous terms destroying the original hierarchy cancel each other. The dimensionful proportionality factors for the dimensional parameters, and the dimensionless factors for nondimensional couplings, turn out to be such that the effective low-energy parameters satisfy a set of closed renormalization-group equations. That is, the low-energy effective theory is consistent and stable under renormalization. This is a consequence of the following two properties, which are common to all of the models considered and highly nontrivial and unexpected: (i) when computing the low-energy limit of the full quantum correction to the original, two-scale theory, one obtains as the coefficients of the heavy logs the precise combinations of the tree-level effective parameters which later on display this decoupling, and (ii) the one-loop effective parameters can *always* be written in terms of a set of parameters which are fixed points of the renormalization group, plus $\hbar$ times the necessary combinations of the tree-level parameters from which follow a decoupled set of renormalization-group equations.

As far as we can tell, these results are independent of the model under consideration and very general. They hold uniformly for supersymmetric and nonsupersymmetric theories, and one is led to conclude that they must be a general property of local quantum field theories with two mass scales, for they always come out by taking the low-energy limit of the appropriate Feynman momentum integrals.[15] It is quite surprising that, for such different theories, the limit $m/M\to 0$ of the quantum correction to the full theory may always be written in terms of effective parameters defined at the tree level and without any fine-tuning. Furthermore, there are indications that this feature also holds for higher-order perturbative corrections, because the renormalization-group-invariant parameters that show up in the calculation can be considered as bare parameters with respect to the heavy-mass scale.

It is interesting to note that even for scalar theories with quartic and quadratic divergences, the procedure goes through and allows the construction of a stable, light field theory. The same holds true for softly broken supersymmetric theories. The contributions from the softly broken sector to the supersymmetric sector do not necessarily destabilize the original tree-level hierarchy. Furthermore, the effective low-energy theory is a softly broken theory if and only if the scale of the breaking of local supersymmetry $m_{3/2}$ is of the same order of magnitude as the low-energy scale m.

Finally, the discussion of the magnitude of the effective parameters and their scaling behavior leads to the same conclusions: One can have a tree-level hierarchy (when appropriate boundary conditions are imposed), and this hierarchy is maintained through radiative corrections.

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APPENDIX: USEFUL PROPERTIES

For the model described by (6.1), the equation of motion is given by (6.3). In order to solve it, we rewrite it in terms of the operator $R\equiv D^2\theta^2=-4+4\theta^\alpha D_\alpha+\theta^2 D^2$ and its complex conjugate. They have a set of properties which will be useful in the calculations. Some of them can be found in Ref. 14, but we need some extra properties which we demonstrate in this appendix. With $F(x)$ and $G(x)$ two arbitrary and general superfields, and using the properties of the D and R algebras, we can show that

$$\left[\frac{R}{4}F\right]\left[\frac{R}{4}G\right]=-\left[\frac{R}{4}FG\right],$$

$$\left[\frac{R}{4}F\right]^2=-\left[\frac{R}{4}F^2\right],$$

$$\left[\frac{R\bar{R}}{16}F\right]\left[\frac{R\bar{R}}{16}G\right]=\left[\frac{R\bar{R}}{16}FG\right],$$

$$\frac{R}{4}F\frac{D^2}{4}G=\left[\frac{R}{4}F\right]\left[\frac{D^2}{4}G\right].$$

We also have to check that the basis for proper functions that we have used in the expansion of $B=B(L)$ is complete. Let Φ be a chiral superfield and A, ψ_α, and F its component fields. Using the explicit θ expansion of the superfield and the representation by differential operators of the supersymmetric covariant derivative,[10] one finds the identities

$$\bar{\theta}^2\Phi=\bar{\theta}^2(A+\sqrt{2}\theta^\alpha\psi_\alpha+\theta^2F),$$

$$\theta^2\Phi=\theta^2 A,$$

$$\bar{\theta}^2\frac{D^2}{4}\,(\text{any})=\frac{1}{4}\varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}\frac{\partial}{\partial\theta^\alpha}\bar{\theta}^2\,(\text{any}),$$

$$\bar{\theta}^2\frac{D^2}{4}\Phi=-\bar{\theta}^2F,$$

$$\frac{D^2}{4}\Phi=-\bar{\theta}^2\Box A+i\sqrt{2}(\partial_m\psi^\alpha)\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}-\frac{1}{2}\bar{\theta}^2\theta^\alpha(\Box\psi_\alpha)-F-\frac{1}{4}\theta^2\bar{\theta}^2(\Box F)+i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}(\partial_m F),$$

$$\frac{R}{4}\Phi=-A+i\theta\sigma^m\bar{\theta}\partial_m A-\frac{1}{4}\theta^2\bar{\theta}^2\Box A,$$

$$\frac{\bar{R}R}{16}\Phi=A+i\theta\sigma^m\bar{\theta}\partial_m A+\frac{1}{4}\theta^2\bar{\theta}^2\Box A.$$

The last two properties prove that $(R/4)\Phi$ is an antichiral superfield and that $(\bar{R}R/16)\Phi$ is a chiral superfield. Furthermore, these properties show that the proper functions we have used in the expansion $B=B(L)$ are, all of them, independent. Finally, we see that the power-series expansion $B=B(L)$ does not generate any other proper function which cannot be constructed using this basis of proper functions.

From these properties we also derive the projection formulas we used in the last two sections:

$$\left.\frac{R}{4}\Phi\right|=-A,\quad \left.\frac{\bar{R}R}{16}\Phi\right|=A,$$

$$\left.\frac{\bar{R}}{4}\frac{D^2}{4}\Phi\right|=F,\quad \left.\frac{R\bar{R}}{16}\frac{D^2}{4}\Phi\right|=-F,$$

$$\left.\frac{\bar{D}^2}{4}\bar{\Phi}\frac{R}{4}\Phi\right|=A\bar{F},\quad \left.\frac{\bar{D}^2}{4}\bar{\Phi}\frac{R\bar{R}}{16}\bar{\Phi}\right|=-\bar{A}\bar{F},$$

$$\left.\frac{\bar{D}^2}{4}\bar{\Phi}^2\right|=-2\bar{A}\bar{F}+\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}},\quad \left.\frac{\bar{R}}{4}\frac{D^2}{4}\Phi^2\right|=2AF-\psi^\alpha\psi_\alpha,$$

$$\Phi|=A,\quad \left.\frac{D^2}{4}\Phi\right|=-F,\quad \left.\frac{1}{\sqrt{2}}D_\alpha\Phi\right|=\psi_\alpha.$$

Here we have used the notation anything $|\equiv$ anything $|_{\theta=\bar{\theta}=0}$.

The superfield and component formalism for constructing the effective action corresponding to a supersymmetric theory are equivalent. Obviously, we have to compare the results produced by both procedures and check that they indeed coincide. Here we need a set of identities which relate the superspace actions with the usual four-dimensional Minkowski space. In the exactly supersymmetric case, we only need the identity which gives the component form of the Wess-Zumino action.[9] When supersymmetry is softly broken, we need many other new identities, viz,

$$\int d^4\theta\,\theta^2\bar\theta^2|\Phi|^2=|A|^2\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\Phi\Box\bar\Phi=A\Box\bar A\,,$$

$$\int d^4\theta\,\theta^2 f(\Phi)=f(A)\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\Phi\frac{D^2}{4}\Phi=-AF\,,$$

$$\int d^4\theta\,\theta^2\Phi\bar\Phi=A\bar F\,,$$

$$\int d^4\theta\,\theta^2\Phi^2\bar\Phi=A^2\bar F\,,$$

$$\int d^4\theta\,\theta^2\Phi\bar\Phi^2=A(2\bar A\,\bar F-\bar\psi_{\dot\alpha}\bar\psi^{\dot\alpha})\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\Phi^2\frac{D^2}{4}\Phi=-A^2F\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\Phi\bar\Phi\frac{D^2}{4}\Phi=-A\bar AF\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\Phi\frac{D^2}{4}\Phi^2=-A(2AF-\psi^\alpha\psi_\alpha)\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\left|\frac{D^2}{4}\Phi\right|^2=|F|^2\,,$$

$$\int d^4\theta\,\bar\theta^2\Phi\frac{D^2}{4}\Phi=-F^2\,.$$

In both sides of these equations, an integral over the four-dimensional spacetime is understood.

After solving the equation of motion, we have to eliminate the HDF from the tree-level action by computing $S(B(L),L)$. This action contains nonrenormalizable terms, but we omit them since they are suppressed by powers of the heavy mass. For the renormalizable interactions we use two kinds of identities:

$$\int d^2\theta\left[\frac{\bar R}{4}\bar\Phi\right]=-\int d^4\theta\,\bar\theta^2\bar\Phi=0\,,$$

$$\int d^2\theta\left[\frac{\bar RR}{16}\,(\text{any})\right]=-\int d^4\theta\,\bar\theta^2\left[\frac{R}{4}\,(\text{any})\right]=0\,.$$

The other type of interactions do not vanish. However, we can recast them by using identities such as

$$\int d^2\theta\,L\left[\frac{\bar RR}{16}L^2\right]=-\int d^4\theta\,\theta^2\bar\theta^2L^2\frac{D^2}{4}L\,,$$

$$\int d^2\theta\,L\left[\frac{\bar R}{4}\bar L^2\right]=-\int d^4\theta\,\bar\theta^2\bar L^2L\,,$$

$$\int d^2\theta\,L^2\left[\frac{\bar RR}{16}L\right]=-\int d^4\theta\,\theta^2\bar\theta^2L\frac{D^2}{4}L^2\,,$$

$$\int d^2\theta\,L^2\left[\frac{\bar R}{4}\bar L\right]=-\int d^4\theta\,\bar\theta^2\bar LL^2\,,$$

$$\int d^2\theta\,L\left[\frac{\bar RR}{16}\bar LL\right]=-\int d^4\theta\,\theta^2\bar\theta^2\bar LL\frac{D^2}{4}L\,,$$

$$\int d^2\theta\,\theta^2\left[\frac{\bar R}{4}\bar L\right]=-\int d^4\theta\,\theta^2\bar\theta^2\bar L\,,$$

$$\int d^2\theta\,\theta^2L^2\left[\frac{\bar R}{4}\bar L\right]=-\int d^4\theta\,\theta^2\bar\theta^2L^2\bar L\,,$$

$$\int d^2\theta\,\theta^2L\left[\frac{\bar R}{4}\bar L\right]=-\int d^4\theta\,\theta^2\bar\theta^2\bar LL\,,$$

$$\int d^4\theta\,\theta^2\bar L\left[\frac{\bar R}{4}\bar L\right]=\int d^4\theta\,\theta^2\bar\theta^2\bar L\frac{\bar D^2}{4}\bar L\,,$$

$$\int d^4\theta\,L\left[\frac{R}{4}L\right]=-\int d^4\theta\,\theta^2\bar\theta^2L\Box L\,,$$

$$\int d^4\theta\,\theta^2\bar L\left[\frac{\bar R}{4}\frac{D^2}{4}L\right]=\int d^4\theta\,\theta^2\bar\theta^2\left|\frac{D^2}{4}L\right|^2\,,$$

$$\int d^4\theta\left|\frac{R}{4}L\right|^2=\int d^4\theta\,\theta^2\bar\theta^2\bar L\Box L\,.$$

After regrouping the contributions to each interaction term, using the equation of motion, and taking the low-energy limit, we find that $S(B(L),L)$ is modified by the *hard*-breaking terms

$$\int d^4\theta\,\theta^2\bar\theta^2\bar L\Box L\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2L\Box L\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\left[L\frac{D^2}{4}L+\text{H.c.}\right]\,,$$

$$\int d^4\theta(\theta^2L^2\bar L+\text{H.c.})\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\left[L\frac{D^2}{4}L^2+\text{H.c.}\right]\,,$$

$$\int d^4\theta(\theta^2L\bar L^2+\text{H.c.})\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\left[L^2\frac{D^2}{4}L+\text{H.c.}\right]\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\bar LL\left[\frac{D^2}{4}L+\text{H.c.}\right]\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2\left|\frac{D^2}{4}L\right|^2\,,$$

$$\int d^4\theta\left[\bar\theta^2L\frac{D^2}{4}L+\text{H.c.}\right]\,,$$

$$\int d^4\theta\,\theta^2\bar\theta^2(L^2\bar L+\text{H.c.})\,.$$

As we have already mentioned, these hard terms *vanish if the parameters which break supersymmetry* are proportional to $m_{3/2}$.

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