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Gravitational Faraday rotation induced from interacting gravitational plane waves

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The polarization of gravitational plane waves is studied. In particular, it is found that each interacting gravitational plane wave can be split into two parts: the shock part and the impulsive part. The plane of the polarization for the plane gravitational shock waves usually gets rotated due to two kinds of interaction: one is the nonlinear interacton between the oppositely moving plane gravitational shock waves, and the other is the interaction between the gravitational shock wave and the matter fields which are present. The change of polarization due to the nonlinear interaction is exactly a gravitational analogue of Faraday rotation, but with the oppositely moving gravitational shock wave as the magnetic field and medium. The effect of the above two kinds of interaction on the impulsive gravitational plane waves is different from that on the shock waves. Only the interaction between the impulsive gravitational wave and the matter fields can change the polarization of the impulsive wave.

I. INTRODUCTION

Einstein's general relativity predicts the existence of gravitational waves. Among these waves are the simplest cylindrical and plane waves. The cylindrical gravitational waves were first studied by Einstein and Rosen [1]. Since then, a lot of effort has been devoted to this subject, and many remarkable features have been found. One of these features is the nonlinear interaction. Piran, Safier, and Stark [2] have found that, because of this interaction, a conversion occurs between different polarization modes of a cylindrical gravitational wave. Specifically, they have found that, if an outgoing (or ingoing) cylindrical wave is linearly polarized, its polarization vector rotates as it propagates. This phenomenon was interpreted by Piran, Safier, and Stark as an exact analogue of the electromagnetic Faraday rotation, but with the ingoing (or outgoing)$\times$mode wave component playing the role of both the magnetic field and medium.

The plane gravitational waves, on the other hand, were first studied by Brinkman [3], Rosen [4], Bondi [5], and Bondi, Pirani, and Robinson [6]. In 1965, Penrose [7] discussed the focusing effect, and pointed out that the focusing effect of single plane waves should cause the colliding plane waves to interact strongly and eventually develop space-time singularities. Motivated by Penrose's above conjecture, Szekeres [8,9], and Khan and Penrose [10] first studied the collision of two such gravitational plane waves, and found that the colliding plane gravitational waves indeed produced space-time singularities after the collision, although now we know that this does not inevitably occur [11–14].

In this paper we shall study another feature of interacting gravitational plane waves—the effect of the interaction between plane gravitational waves and the effect of the interaction between the gravitational plane waves and matter fields on the polarization of the plane gravitational waves [15].

The structure of the paper is as follows. To facilitate our discussion, in Sec. II we briefly review the space-time for interacting plane gravitational waves. Following it, in Sec. III the geodesic deviation of a null congruence is studied, and the relations between amplitude and polarization of a plane gravitational wave and the Weyl scalar which represents the wave is given in terms of the local basis. To compare the polarization of a plane gravitational wave at different points along the wave path, a parallel-transported basis along the wave path is found in Sec. IV. Using the Bianchi identities, which represent the interaction between the free gravitational field and matter fields [16], the change of the polarization of a gravitational plane wave is given relative to the parallel-transported basis. As an application of the results obtained in Sec. VI, the polarization of head-on colliding gravitational plane waves is considered in Sec. V. The effect of several specific matter fields on the polarization is studied. To further illustrate the feature, in Sec. VI we study some exact solutions, which represent the collision of a variably polarized gravitational plane wave and an impulsive shell of null dust (consisting of unidentified massless particles). In Sec. VII our main conclusions are derived.

II. SPACE-TIMES WITH TWO COMMUTING SPACELIKE KILLING VECTORS

We consider space-time that allows two commuting spacelike Killing vectors (with open orbits), say, ξ^{μ}_{A} ($A=2,3$; $\mu=0,1,2,3$). The coordinate system is chosen so that $\xi^{\mu}_{A}=\delta^{\mu}_{A}$, and the metric takes the form [9,17]

$$ds^2=2e^{-M}du\,dv-e^{-U}(e^{V}\cosh W\,dx^2-2\sinh W\,dx\,dy+e^{-V}\cosh W\,dy^2)\,, \tag{2.1}$$

where M, U, V, and W are functions of only the null coordinates u and v, and $\{x^{\mu}\}\equiv\{u,v,x,y\}$.

Solutions of metric (2.1) could describe cosmological models [18–20], and interacting gravitational plane waves [9,17].

If we choose the null tetrad as

$$l^\mu = B\delta^\mu_v\ ,\quad n^\mu = A\delta^\mu_u\ ,$$
$$m^\mu = \zeta^2\delta^\mu_2 + \zeta^3\delta^\mu_3\ ,\quad \bar{m}^\mu = \bar{\zeta}^2\delta^\mu_2 + \bar{\zeta}^3\delta^\mu_3\ , \tag{2.2}$$

where

$$\zeta^2 = \frac{e^{(U-V)/2}}{\sqrt{2}}\left[\cosh\frac{W}{2} + i\sinh\frac{W}{2}\right]\ ,$$
$$\zeta^3 = \frac{e^{(U+V)/2}}{\sqrt{2}}\left[\sinh\frac{W}{2} + i\cosh\frac{W}{2}\right]\ , \tag{2.3}$$
$$M = \ln(AB)\ ,$$

and an overbar denotes a complex conjugate, we will find that the only nonvanishing Weyl and Ricci scalars are, respectively, Ψ_0, Ψ_2, Ψ_4, and Φ_{00}, Φ_{02}, Φ_{11}, Φ_{22}, Λ, which have been given in Refs. [17,21].

On the other hand, it is easy to show that

$$l^\mu{}_{;\nu}l^\nu = -\frac{B(A_{,v})}{A}l^\mu,\quad n^\mu{}_{;\nu}n^\nu = \frac{A(B_{,u})}{B}n^\mu\ , \tag{2.4}$$

where a semicolon denotes covariant differentiation and a comma partial differentiation. Equation (2.4) shows that each of the null vectors l^μ and n^μ defines a null geodesic congruence. When the function A is chosen to be constant, the null geodesics defined by l^μ are affinely parametrized, while the function B is chosen to be constant, the null geodesics defined by n^μ are affinely parametrized.

Our following analysis is based on the decomposition of the Riemann tensor:

$$R_{\mu\nu\lambda\rho} = C_{\mu\nu\lambda\rho} + \tfrac{1}{2}(g_{\mu\lambda}R_{\nu\rho} + g_{\nu\rho}R_{\mu\lambda} - g_{\nu\lambda}R_{\mu\rho} - g_{\mu\rho}R_{\nu\lambda}) + \tfrac{1}{2}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho})R\ , \tag{2.5}$$

where $R_{\mu\lambda}$ is the Ricci tensor, R the Ricci scalar, and $C_{\mu\nu\lambda\rho}$ the Weyl tensor, being thought of as representing the free gravitational field, which interacts with the matter fields, $R_{\mu\lambda}$, through the Bianchi identities [16]. In terms of the nonvanishing Weyl scalars, the Weyl tensor is written as [21]

$$C_{\mu\nu\lambda\delta} = -4(\Psi_2+\bar{\Psi}_2)(l_{[\mu}n_{\nu]}l_{[\lambda}n_{\delta]} + m_{[\mu}\bar{m}_{\nu]}m_{[\lambda}\bar{m}_{\delta]}) + 4(\Psi_2-\bar{\Psi}_2)(l_{[\mu}n_{\nu]}m_{[\lambda}\bar{m}_{\delta]} + m_{[\mu}\bar{m}_{\nu]}l_{[\lambda}n_{\delta]})$$
$$-4[\Psi_0 n_{[\mu}\bar{m}_{\nu]}n_{[\lambda}\bar{m}_{\delta]} - \Psi_2(l_{[\mu}m_{\nu]}n_{[\lambda}\bar{m}_{\delta]} + n_{[\mu}\bar{m}_{\nu]}l_{[\lambda}m_{\delta]}) + \Psi_4 l_{[\mu}m_{\nu]}l_{[\lambda}m_{\delta]} + \text{c.c.}]\ , \tag{2.6}$$

where

$$A_{[\mu}B_{\nu]} \equiv \tfrac{1}{2}(A_\mu B_\nu - A_\nu B_\mu)\ . \tag{2.7}$$

The Weyl scalars Ψ_0, Ψ_2, and Ψ_4 have the following physical interpretation [16]: The Ψ_0 term represents the transverse gravitational-wave component in the n^μ direction, the Ψ_2 term a "Coulomb-like" component, and the Ψ_4 term the transverse gravitational-wave component in the l^μ direction. Since in the present case Ψ_1 and Ψ_3 vanish identically, the longitudinal components in the n^μ and l^μ directions are zero.

III. THE NULL GEODESIC DEVIATION AND THE AMPLITUDE AND POLARIZATION OF A PLANE GRAVITATIONAL WAVE

Null geodesic deviation was first studied by Pirani and Schild [22] as an attempt to give a geometrical and physical interpretation to the Weyl tensor. Later on, Szekeres [16] chose the timelike geodesics, as the latter brings out the Petrov structure more clearly. In this section we adopt the Pirani-Schild approach, since it has the following advantage in the present case: (a) null geodesics are invariant under conformal transformations of the Riemann space-time; (b) as shown above, the null vectors l^μ ad n^μ define two null geodesic congruences, which makes the task of studying the null geodesic deviation very simple; (c) gravitational plane waves in the space-time described by (2.1) propagate along these two null geodesic congruences.

Let us first consider the null geodesics defined by l^μ. Let η^μ be the deviation vector between neighbor geodesics, and $\eta^\mu l_\mu = 0$. Then, the geodesic deviation is given by

$$\frac{D^2\eta^\mu}{D\lambda^2} = -R^\mu{}_{\nu\lambda\rho}l^\nu l^\rho\eta^\lambda\ . \tag{3.1}$$

Inserting Eqs. (2.5) and (2.6) into Eq. (3.1), finally we obtain

$$\frac{D^2\eta^\mu}{D\lambda^2} = e^M[-\Phi_{00}e_0^{\mu\nu} + 2\,\text{Re}(\Psi_0)e_+^{\mu\nu} + 2\,\text{Im}(\Psi_0)e_\times^{\mu\nu}]\eta_\nu\ , \tag{3.2}$$

where

$$e_0^{\mu\nu} \equiv e_2^\mu e_2^\nu + e_3^\mu e_3^\nu\ ,\quad e_+^{\mu\nu} \equiv e_2^\mu e_2^\nu - e_3^\mu e_3^\nu\ ,$$
$$e_\times^{\mu\nu} \equiv e_2^\mu e_3^\nu - e_3^\mu e_2^\nu\ ; \tag{3.3}$$

$$e_2^\mu \equiv \frac{1}{\sqrt{2}}(m^\mu + \bar{m}^\mu)\ ,\quad e_2^\mu \equiv -\frac{i}{\sqrt{2}}(m^\mu - \bar{m}^\mu)\ , \tag{3.4}$$

and $\text{Re}(\Psi_0)$ denotes taking the real part of Ψ_0, while $\text{Im}(\Psi_0)$ denotes taking the imaginary part of Ψ_0.

For all physically realistic matter fields we have

$$\Phi_{00} \geq 0\ . \tag{3.5}$$

Equation (3.2) allows us the following physical interpretation. Let S_O and S_P be infinitesimal 2-elements spanned by e_2 and e_3 and orthogonal to a null geodesic C at neighbor points O and P of C, and let S be an infinitesimal circle with center O, lying in S_O [see Fig.

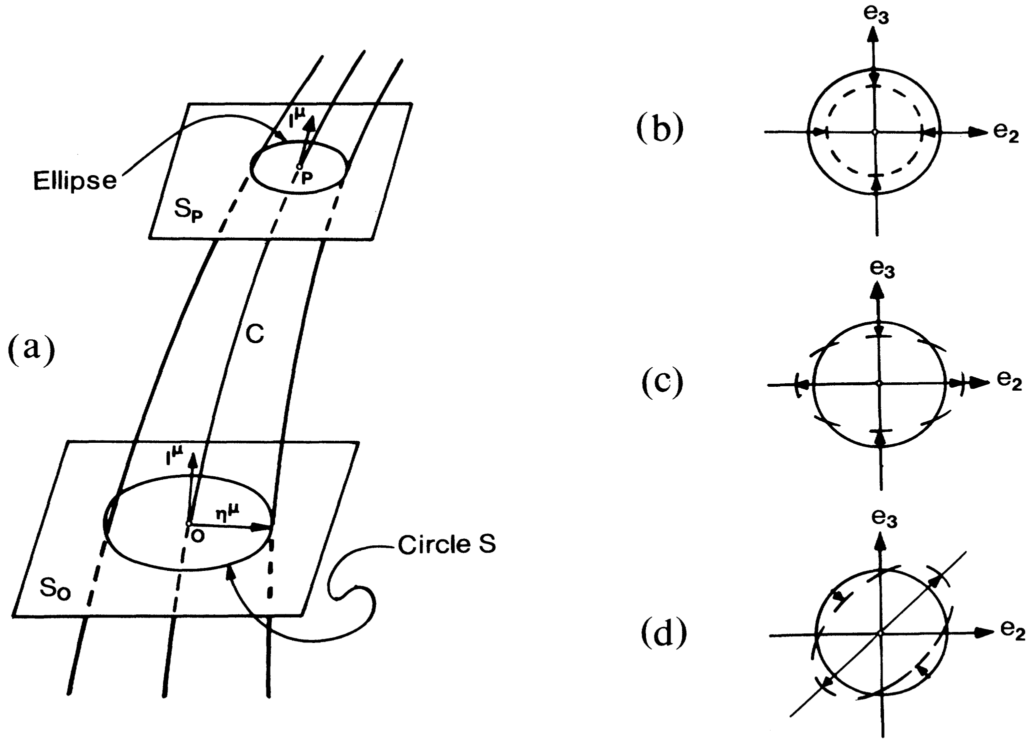

FIG. 1. A null geodesic congruence meets the S_O plane in the circle S. Because of the force generated by the Ψ_0 gravitational plane-wave component, the image of the circle S on the S_P-plane is a sheared ellipse.

1(a)]. Suppose that a beam of light rays meets S_O in the circle S, then let us observe the image of these light rays on S_P. The first term on the right-hand side of Eq. (3.2) shows that matter fields always make the circle S contracted [see Fig. 1(b)]. The second term, corresponding to the contribution of the real part of Ψ_0, makes the circle elliptic with the main major axis along e_2 [see Fig. 1(c)]. The last term on the right-hand side of Eq. (3.2), corresponding to the contribution of the imaginary part of Ψ_0, makes the circle also elliptic but with the main major axis tilted at 45° to e_2 and e_3 [see Fig. 1(d)]. Thus, the image of these light rays on S_P is a sheared ellipse. When $\mathrm{Im}(\Psi_0)=0$, it is an ellipse without shearing; i.e., its main major axis is along e_2.

If we make the following rotation in the (e_2, e_3) plane,

$$\begin{aligned} e_2^{\mu} &= \cos\varphi_0 e'^{\mu}_2 + \sin\varphi_0 e'^{\mu}_3 , \\ e_3^{\mu} &= -\sin\varphi_0 e'^{\mu}_2 + \cos\varphi_0 e'^{\mu}_3 , \end{aligned} \tag{3.6}$$

we find

$$\begin{aligned} e_0^{\mu\nu} &= e'^{\mu\nu}_0 , \\ e_+^{\mu\nu} &= \cos 2\varphi_0 e'^{\mu\nu}_+ + \sin 2\varphi_0 e'^{\mu\nu}_\times , \\ e_\times^{\mu\nu} &= -\sin 2\varphi_0 e'^{\mu\nu}_+ + \cos 2\varphi_0 e'^{\mu\nu}_\times . \end{aligned} \tag{3.7}$$

In terms of e'^{μ}_2 and e'^{μ}_3, Eq. (3.2) becomes

$$\frac{D^2\eta^\mu}{D\lambda^2} = e^M\{-\Phi_{00} e'^{\mu\nu}_0 + 2[\cos 2\varphi_0 \mathrm{Re}(\Psi_0) - \sin 2\varphi_0 \mathrm{Im}(\Psi_0)] e'^{\mu\nu}_+ + 2[\sin 2\varphi_0 \mathrm{Re}(\Psi_0) + \cos 2\varphi_0 \mathrm{Im}(\Psi_0)] e'^{\mu\nu}_\times\}\eta_\nu . \tag{3.8}$$

If the angle is chosen such that

$$\sin 2\varphi_0 \mathrm{Re}(\Psi_0) + \cos 2\varphi_0 \mathrm{Im}(\Psi_0) = 0 , \tag{3.9}$$

or equivalently

$$\tan 2\varphi_0 = -\frac{\mathrm{Im}(\Psi_0)}{\mathrm{Re}(\Psi_0)} , \tag{3.10}$$

then Eq. (3.8) becomes

$$\frac{D^2\eta^\mu}{D\lambda^2} = e^M[-\Phi_{00} e'^{\mu\nu}_0 + 2(\Psi_0\overline{\Psi}_0)^{1/2} e'^{\mu\nu}_+]\eta_\nu . \tag{3.11}$$

It follows that the main major axis of the ellipse is along e'^{μ}_2. We call e'^{μ}_2 the direction of the polarization of the Ψ_0 wave [23,15]. The angle φ_0 is the polarization angle of the plane gravitational wave with respect to the (e_2^μ, e_3^μ) basis. The relative accelerations of neighbor geodesics are proportional to $(\Psi_0\overline{\Psi}_0)^{1/2}$, which does not relate to any observer. Thus, the $(\Psi_0\bar{\Psi}_0)^{1/2}$ term represents the absolute amplitude of the relative accelerations of neighbor rays.

In a similar fashion, if we consider the geodesic deviation of the null congruence defined by n^μ, we will find

$$\frac{D^2\eta^\mu}{D\lambda^2}=e^M[-\Phi_{22}e_0^{\mu\nu}+2\,\mathrm{Re}(\Psi_4)e_+^{\mu\nu}-2\,\mathrm{Im}(\Psi_4)e_\times^{\mu\nu}]\eta_\nu \ . \tag{3.12}$$

Following the discussion given after Eq. (3.2) we can see that the above equation will be brought to the form

$$\frac{D^2\eta^\mu}{D\lambda^2}=e^M[-\Phi_{22}e'^{\mu\nu}_0+2(\Psi_4\bar{\Psi}_4)^{1/2}e'^{\mu\nu}_+]\eta_\nu \ , \tag{3.13}$$

if the rotated angle φ_4 is chosen such that

$$\tan 2\varphi_4=\frac{\mathrm{Im}(\Psi_4)}{\mathrm{Re}(\Psi_4)} \ . \tag{3.14}$$

Thus, the angle φ_4 is the polarization angle of the Ψ_4 wave with respect to the (e_2^μ,e_3^μ) basis, and the $(\Psi_4\bar{\Psi}_4)^{1/2}$ term represents the absolute amplitude of the relative accelerations of neighbor rays due to the Ψ_4 wave.

When the angle φ_0 (φ_4) is constant everywhere, we say that the corresponding plane gravitational wave is constantly polarized; otherwise it is variably polarized.

IV. A PARALLEL-TRANSPORTED BASIS ALONG A GRAVITATIONAL PLANE-WAVE PATH AND THE CHANGE OF POLARIZATION OF THE WAVE

The above definition of the polarization angle for a plane gravitational wave is local, since the (e_2^μ,e_3^μ) basis is not parallel transported along either the Ψ_0 wave path or the Ψ_4 wave path. In fact, from Eqs. (2.1) and (2.2) we find

$$e_2{}^\mu{}_{;\nu}l^\nu=\tfrac{1}{2}\sinh WV_{,\nu}l^\nu e_3^\mu \ , \quad e_3{}^\mu{}_{;\nu}l^\nu=-\tfrac{1}{2}\sinh WV_{,\nu}l^\nu e_2^\mu \ , \tag{4.1}$$
$$e_2{}^\mu{}_{;\nu}n^\nu=\tfrac{1}{2}\sinh WV_{,\nu}n^\nu e_3^\mu \ , \quad e_3{}^\mu{}_{;\nu}n^\nu=-\tfrac{1}{2}\sinh WV_{,\nu}n^\nu e_2^\mu \ .$$

In order to compare the polarization of a plane gravitational wave at two different points along the wave path, we have to find a parallel-transported basis carried by the wave, and then define the polarization angle relative to this parallel-transported basis. In this way we can see that the change of polarization angle of the wave along the wave path has absolutely physical meaning and independent of the choice of the coordinates. For example, if the change (relative to a parallel-transported basis) is zero, it means that the polarization vector e'^μ_2 defined above is parallel to the parallel-transported basis. Such defined parallelism is independent of the coordinates.

To find a parallel-transported basis along the null geodesics defined by l^μ, we make a coordinate rotation in the (e_2^μ,e_3^μ) plane but with the angle denoted by $\varphi_4^{(0)}$ and the new basis by $\lambda_{(2)}^\mu$ and $\lambda_{(3)}^\mu$, respectively. Then we find

$$\lambda_{(2)}{}^\mu{}_{;\nu}l^\nu=(\tfrac{1}{2}\sinh WV_{,\nu}-\varphi^{(0)}_{4,\nu})l^\nu\lambda^\mu_{(3)} \ ,$$
$$\lambda_{(3)}{}^\mu{}_{;\nu}l^\nu=-(\tfrac{1}{2}\sinh WV_{,\nu}-\varphi^{(0)}_{4,\nu})l^\nu\lambda^\mu_{(2)} \ . \tag{4.2}$$

Therefore, if the angle $\varphi_4^{(0)}$ is chosen so that

$$\tfrac{1}{2}\sinh WV_{,v}-\varphi_4{}^{(0)}{}_{,v}=0 \ , \tag{4.3}$$

then the spacelike orthogonal vectors $\lambda^\mu_{(2)}$ and $\lambda^\mu_{(3)}$ are parallel transported along the null geodesics (or the Ψ_4 wave path) defined by l^μ, and the difference

$$\theta_4\equiv\varphi_4-\varphi_4^{(0)} \tag{4.4}$$

defines the angle between the polarization direction of the Ψ_4 wave and the $\lambda^\mu_{(2)}$ direction.

Similarly, if the basis (e_2^μ,e_3^μ) is rotated so that the rotated angle $\varphi_0^{(0)}$ satisfies

$$\tfrac{1}{2}\sinh WV_{,u}-\varphi_0{}^{(0)}{}_{,u}=0 \ , \tag{4.5}$$

the vectors $\lambda^\mu_{(2)}$ and $\lambda^\mu_{(3)}$ are parallel transported along the Ψ_0 wave path, and the angle

$$\theta_0\equiv\varphi_0-\varphi_0^{(0)} \tag{4.6}$$

determines the polarization direction of the Ψ_0 wave relative to the $(\lambda^\mu_{(2)},\lambda^\mu_{(3)})$ basis.

On the other hand, from the Bianchi identities, we find

$$\begin{aligned}A\Psi_{0,u}=\tfrac{1}{2}\{&A[4(\ln B)_{,u}+U_{,u}-i2\sinh WV_{,u}]\Psi_0-3B(\cosh WV_{,v}-iW_{,v})\Psi_2-2B\Phi_{02,v}\\&+B(U_{,v}-i2\sinh WV_{,v})\Phi_{02}-2B(\cosh WV_{,v}-iW_{,v})\Phi_{11}-A(\cosh WV_{,u}-iW_{,u})\Phi_{00}\} \ ,\\B\Psi_{4,v}=\tfrac{1}{2}\{&B[4(\ln A)_{,v}+U_{,v}+i2\sinh WV_{,v}]\Psi_4-3A(\cosh WV_{,u}+iW_{,u})\Psi_2-2A\Phi_{20,u}\\&+A(U_{,u}+i2\sinh WV_{,u})\Phi_{20}-2A(\cosh WV_{,u}+iW_{,u})\Phi_{11}-B(\cosh WV_{,v}+iW_{,v})\Phi_{22}\} \ .\end{aligned} \tag{4.7}$$

It is now convenient to introduce the following "scale-invariant" quantities via the relations [9,17]

$$\begin{aligned}&\Psi_0^{(0)}=B^{-2}\Psi_0 \ , \quad \Psi_2^{(0)}=(AB)^{-1}\Psi_2 \ , \quad \Psi_4^{(0)}=A^{-2}\Psi_4 \ , \quad \Phi_{00}^{(0)}=B^{-2}\Phi_{00} \ , \quad \Phi_{11}^{(0)}=(AB)^{-1}\Phi_{11} \ ,\\&\Phi_{02}^{(0)}=(AB)^{-1}\Phi_{02} \ , \quad \Phi_{22}^{(0)}=A^{-2}\Phi_{22} \ , \quad \Lambda^{(0)}=(AB)^{-1}\Lambda \ .\end{aligned} \tag{4.8}$$

Since from now on only the "scale-invariant" terms are used, we shall drop all of the superscript zeros without causing any confusion.

Inserting Eq. (4.8) into Eq. (4.7) we obtain

$$\begin{aligned}\Psi_{0,u}=&\tfrac{1}{2}\{[U_{,u}-i2\sinh WV_{,u}]\Psi_0-3(\cosh WV_{,v}-iW_{,v})\Psi_2-2\Phi_{02,v}+(U_{,v}-2M_{,v}-i2\sinh WV_{,v})\Phi_{02}\\&-2(\cosh WV_{,v}-iW_{,v})\Phi_{11}-(\cosh WV_{,u}-iW_{,u})\Phi_{00}\}\ ,\\\Psi_{4,v}=&\tfrac{1}{2}\{[U_{,v}+i2\sinh WV_{,v}]\Psi_4-3(\cosh WV_{,u}+iW_{,u})\Psi_2-2\Phi_{20,u}+(U_{,u}-2M_{,u}+i2\sinh WV_{,u})\Phi_{20}\\&-2(\cosh WV_{,u}+iW_{,u})\Phi_{11}-(\cosh WV_{,v}+iW_{,v})\Phi_{22}\}\ .\end{aligned}\tag{4.9}$$

From the above equation we can see that the two plane gravitational waves Ψ_0 and Ψ_4 interact with each other through the Coulomb-like field Ψ_2. The components Φ_{02} and Φ_{11} of the matter field interact with both Ψ_0 and Ψ_4. In other words, Φ_{02} and Φ_{11} are "gravitationally active" to both of them [16]. Φ_{00} is "gravitational active" only to Ψ_0, and Φ_{22} only to Ψ_4, while Λ is "gravitationally inert" to both Ψ_0 and Ψ_4.

Combining Eqs. (3.10), (4.5), (4.6), and (4.9) we find that the change of polarization angle of the Ψ_0 wave along the wave path is given by

$$\begin{aligned}\theta_{0,u}=&-\frac{1}{4\Psi_0\overline{\Psi}_0}\{3[\cosh WV_{,v}\mathrm{Im}(\Psi_0\overline{\Psi}_2)+W_{,v}\mathrm{Re}(\Psi_0\overline{\Psi}_2)]+2\,\mathrm{Im}(\Psi_0\Phi_{20,v})+(2M_{,v}-U_{,v})\mathrm{Im}(\Psi_0\Phi_{20})\\&+2\Phi_{11}[\cosh WV_{,v}\mathrm{Im}(\Psi_0)+W_{,v}\mathrm{Re}(\Psi_0)]+\Phi_{00}[\cosh WV_{,u}\mathrm{Im}(\Psi_0)+W_{,u}\mathrm{Re}(\Psi_0)]\\&-2\sinh WV_{,v}\mathrm{Re}(\Psi_0\Phi_{20})\}\ .\end{aligned}\tag{4.10}$$

Similarly, for the Ψ_4 wave, the change of the polarization angle along the wave path is given by

$$\begin{aligned}\theta_{4,v}=&\frac{1}{4\Psi_4\overline{\Psi}_4}\{3[\cosh WV_{,u}\mathrm{Im}(\Psi_4\overline{\Psi}_2)-W_{,u}\mathrm{Re}(\Psi_4\overline{\Psi}_2)]+2\,\mathrm{Im}(\Psi_4\Phi_{02,u})+(2M_{,u}-U_{,u})\mathrm{Im}(\Psi_4\Phi_{02})\\&+2\Phi_{11}[\cosh WV_{,u}\mathrm{Im}(\Psi_4)-W_{,u}\mathrm{Re}(\Psi_4)]+\Phi_{22}[\cosh WV_{,v}\mathrm{Im}(\Psi_4)-W_{,v}\mathrm{Re}(\Psi_4)]\\&+2\sinh WV_{,u}\mathrm{Re}(\Psi_4\Phi_{02})\}\ .\end{aligned}\tag{4.11}$$

When $W=0$, all the Weyl and Ricci scalars are real. Consequently, Eqs. (4.10) and (4.11) yield

$$\theta_{0,u}=0=\theta_{4,v}\quad (W=0)\ .\tag{4.12}$$

That is, in the collinear case the polarization of plane gravitational waves does not change. In the following we shall consider only the cases where $W\neq 0$.

For a perfect fluid, the energy-stress tensor reads

$$\begin{aligned}&T_{(\alpha)(\beta)}=(\mu+p)u_{(\alpha)}u_{(\beta)}-\eta_{(\alpha)(\beta)}p\ ,\\&u_{(\alpha)}u_{(\beta)}\eta^{(\alpha)(\beta)}=1\ ,\end{aligned}\tag{4.13}$$

where $u_{(\alpha)}$ denotes the null tetrad components of the four-velocity of the fluid, μ the energy density, p the pressure, and

$$[\eta_{(\alpha)(\beta)}]=[\eta^{(\alpha)(\beta)}]=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{bmatrix}\ .\tag{4.14}$$

Since all of the metric coefficients are the functions of only u and v, without loss of generality we assume that $u_{(\alpha)}$, μ, and p are only the functions of u and v, too, and that $u_{(\alpha)}$ have the form

$$u_{(\alpha)}=\frac{1}{\sqrt{2}}[u_{(0)},u_{(0)}^{-1},0,0]\ ,\tag{4.15}$$

where $u_{(0)}=u_{(0)}(u,v)$.

From the Einstein field equations we find that the nonvanishing Ricci scalars are given by

$$\begin{aligned}&\Phi_{00}=\frac{B^{-2}}{4}(\mu+p)u_0^2\ ,\quad \Phi_{22}=\frac{A^{-2}}{4}(\mu+p)u_0^{-2}\ ,\\&\Phi_{11}=\frac{(AB)^{-1}}{8}(\mu+p)\ ,\quad \Lambda=\frac{(AB)^{-1}}{24}(\mu-3p)\ .\end{aligned}\tag{4.16}$$

Before closing this section we note that in the present case Petrov type-N solutions cannot exist in a perfect fluid [16], since, as mentioned previously, the gravitational waves propagate along the null geodesics defined either by l^μ or n^μ.

V. THE POLARIZATION OF COLLIDING GRAVITATIONAL PLANE WAVES

As an application, in this section, let us consider the collision and interaction of plane gravitational waves coupled or without coupled matter shells. In the following we shall consider only the head-on collision, since it is always possible to make a Lorentzian transformation to a coordinate system in which the two waves approach each other from exactly opposite spatial directions [9].

The space-time for the collision and interaction of such two plane gravitational waves can be arranged as follows. Prior to the collision, there are two plane waves moving toward each other in opposite directions: one is incident in region II where $u<0$ and $v>0$, and the other is incident in region III where $u>0$ and $v<0$. In region II all the functions M, U, V, and W depend only on v, the only nonvanishing Weyl scalar is Ψ_0, while in region III they depend only on u and the only nonvanishing Weyl scalar is Ψ_4. After they collide at the surface $u=0=v$, the two incoming waves enter the interaction region IV

$(u,v>0)$, in which the functions M, U, V, and W depend on both u and v. Region I $(u,v<0)$ is the region in which the two incoming waves do not arrive yet. Thus, the space-time in this region is flat and all the above functions vanish [21].

To consider the polarization of colliding plane gravitational waves, we first notice that the Weyl and Ricci scalars in such space-times, in general, consist of two parts [21], the shock part and the impulsive part, although the former essentially includes three different cases: smooth wave, shock wave, and the wave with unbounded wave front [14] (the treatment for these three cases, however, is the same for the present problem, so in the present case we do not distinguish them). As shown in Ref. [21], all of them can be written in the form

$$\begin{aligned}
&\Psi_0(u,v)=\Psi_0^{\rm IV}(uH(u),v)H(v)+\Psi_0^{\rm im}(uH(u))\delta(v)\ ,\\
&\Psi_2(u,v)=\Psi_2^{\rm IV}(u,v)H(u)H(v)\ ,\\
&\Psi_4(u,v)=\Psi_4^{\rm IV}(u,vH(v))H(u)+\Psi_4^{\rm im}(vH(v))\delta(u)\ ,
\end{aligned}\tag{5.1}$$

and

$$\begin{aligned}
&\Phi_{00}(u,v)=\Phi_{00}^{\rm IV}(uH(u),v)H(v)+\Phi_{00}^{\rm im}(uH(u))\delta(v)\ ,\\
&\Phi_{22}(u,v)=\Phi_{22}^{\rm IV}(u,vH(v))H(u)+\Phi_{22}^{\rm im}(vH(v))\delta(u)\ ,\\
&\Phi_{02}(u,v)=\bar{\Phi}_{20}(u,v)=\Phi_{02}^{\rm IV}(u,v)H(u)H(v)\ ,\\
&\Phi_{11}(u,v)=\Phi_{11}^{\rm IV}(u,v)H(u)H(v)\ ,\\
&\Lambda(u,v)=\Lambda^{\rm IV}(u,v)H(u)H(v)\ ,
\end{aligned}\tag{5.2}$$

where

$$\begin{aligned}
\Psi_0^{\rm im}(uH(u))&\equiv-\tfrac{1}{2}[\cosh W(uH(u),0)V_{,v}(uH(u),0)\\
&\qquad -iW_{,v}(uH(u),0)]\ ,\\
\Psi_4^{\rm im}(vH(v))&\equiv-\tfrac{1}{2}[\cosh W(0,vH(v))V_{,u}(0,vH(v))\\
&\qquad +iW_{,u}(0,vH(v))]\ ,
\end{aligned}\tag{5.3}$$

$$\begin{aligned}
&\Phi_{00}^{\rm im}(uH(u))\equiv\tfrac{1}{2}U_{,v}(uH(u),0)\ ,\\
&\Phi_{22}^{\rm im}(vH(v))\equiv\tfrac{1}{2}U_{,u}(0,vH(v))\ ,
\end{aligned}\tag{5.4}$$

and

$$\begin{aligned}
&U(0,v)\equiv\lim_{u\to0^+}U(u,v)\ ,\quad U(u,0)\equiv\lim_{v\to0^+}U(u,v)\ ,\\
&U_{,u}(0,v)\equiv\lim_{u\to0^+}U_{,u}(u,v)\ ,\quad U_{,v}(u,0)\equiv\lim_{v\to0^+}U_{,v}(u,v)\ ,
\end{aligned}\tag{5.5}$$

etc. The quantities with the superscript IV denote the ones calculated in region IV, $H(x)$ the Heaviside step function, which is unity for the non-negative arguments and otherwise zero, and $\delta(x)$ the Dirac delta function.

The terms $\Phi_{00}^{\rm im}$ and $\Phi_{22}^{\rm im}$ are, in general, thought of as representing impulsive shells of null dust with support, respectively, on the hypersurfaces $u=0$ and $v=0$ [24–26] while the terms $\Psi_0^{\rm im}$ and $\Psi_4^{\rm im}$ are thought of as representing the gravitational impulsive wave part with support on the above two hypersurfaces [10,27,28]. For each part of Ψ_0 and Ψ_4 we can assign a polarization angle. In the following we use $\theta_0^{\rm sh}$ and $\theta_0^{\rm im}$ to denote the polarization angles, respectively, for the shock part and impulsive part of Ψ_0, and $\theta_4^{\rm sh}$, $\theta_4^{\rm im}$ the shock part and the impulsive part of Ψ_4.

In region II, all the metric coefficients are functions of only v, consequently, the nonvanishing Weyl scalar Ψ_0 is a function of only v, too. So we have $\theta_0^{\rm sh}=\theta_0^{\rm sh}(v)$ and $\theta_0^{\rm im}=$const. That is, the shock part, in general, is variably polarized while the impulsive part is always constantly polarized in region II. Along the wave path, $\theta_0^{\rm sh}$ and $\theta_0^{\rm im}$ do not change in this region

$$\theta_0{}^{\rm sh}{}_{,u}=0=\theta_0{}^{\rm im}{}_{,u}\ .\tag{5.6}$$

Of course, in regions I and III, $\theta_0^{\rm sh}$ and $\theta_0^{\rm im}$ vanish, since Ψ_0 vanishes there.

Similarly, for the Ψ_4 wave, we have

$$\theta_4{}^{\rm sh}{}_{,v}=0=\theta_4{}^{\rm im}{}_{,v}\ ,\tag{5.7}$$

in region III, and $\theta_4^{\rm sh}$, $\theta_4^{\rm im}$ vanish in regions I and II.

Note that the angle $\varphi_4^{(0)}$ $(\varphi_0^{(0)})$ is constant in region III (region II) [see Eqs. (4.3) and (4.5)]. Without loss of generality we can choose $\varphi_4^{(0)}$ $(\varphi_0^{(0)})$ to be zero in region III (region II) so that the $(\lambda^\mu_{(2)},\lambda^\mu_{(3)})$ basis coincides with the (e_2^μ,e_3^μ) basis in these regions.

In the interaction region, IV, the situation is quite different. From the generalized Bianchi identities [21] we find

$$\begin{aligned}
\Psi_0{}^{\rm IV}{}_{,u}=\tfrac{1}{2}\{&[U_{,u}-i2\sinh WV_{,u}]\Psi_0^{\rm IV}-3(\cosh WV_{,v}-iW_{,v})\Psi_2^{\rm IV}-2\Phi_{02}{}^{\rm IV}{}_{,v}+(U_{,v}-2M_{,v}-i2\sinh WV_{,v})\Phi_{02}^{\rm IV}\\
&-2(\cosh WV_{,v}-iW_{,v})\Phi_{11}^{\rm IV}-(\cosh WV_{,u}-iW_{,u})\Phi_{00}^{\rm IV}\}\ ,\\
\Psi_4{}^{\rm IV}{}_{,v}=\tfrac{1}{2}\{&[U_{,v}+i2\sinh WV_{,v}]\Psi_4^{\rm IV}-3(\cosh WV_{,u}+iW_{,u})\Psi_2^{\rm IV}-2\Phi_{20}{}^{\rm IV}{}_{,u}+(U_{,u}-2M_{,u}+i2\sinh WV_{,u})\Phi_{20}^{\rm IV}\\
&-2(\cosh WV_{,u}+iW_{,u})\Phi_{11}^{\rm IV}-(\cosh WV_{,v}+iW_{,v})\Phi_{22}^{\rm IV}\}\quad(u,v>0)
\end{aligned}\tag{5.8}$$

and

$$\begin{aligned}
&\Psi_0{}^{\rm im}{}_{,u}=\tfrac{1}{2}\{[U_{,u}-i2\sinh WV_{,u}]\Psi_0^{\rm im}-2\Phi_{02}^{\rm IV}-(\cosh WV_{,u}-iW_{,u})\Phi_{00}^{\rm im}\}\quad(u>0,v=0)\\
&\Psi_4{}^{\rm im}{}_{,v}=\tfrac{1}{2}\{[U_{,v}+i2\sinh WV_{,v}]\Psi_4^{\rm im}-2\Phi_{20}^{\rm IV}-(\cosh WV_{,v}+iW_{,v})\Phi_{22}^{\rm im}\}\quad(u=0,v>0)\ .
\end{aligned}\tag{5.9}$$

Equation (5.9) shows that the impulsive gravitational wave $\Psi_0^{\rm im}$ interacts only with the matter components $\Phi_{02}^{\rm IV}$ and $\Phi_{00}^{\rm im}$, and $\Psi_4^{\rm im}$ interacts only with $\Phi_{20}^{\rm IV}$ and $\Phi_{22}^{\rm im}$.

Using Eqs. (3.10), (3.14), (5.8), and (5.9), we find

$$\theta_0{}^{sh}{}_{,u} = -\frac{1}{4\Psi_0^{IV}\bar{\Psi}_0^{IV}}\{3[\cosh W V_{,v}\,\mathrm{Im}(\Psi_0^{IV}\bar{\Psi}_2^{IV}) + W_{,v}\mathrm{Re}(\Psi_0^{IV}\bar{\Psi}_2^{IV})] + 2\,\mathrm{Im}(\Psi_0^{IV}\Phi_{20}{}^{IV}{}_{,v}) + (2M_{,v} - U_{,v})\mathrm{Im}(\Psi_0^{IV}\Phi_{20}^{IV})$$
$$+ 2\Phi_{11}^{IV}[\cosh W V_{,v}\,\mathrm{Im}(\Psi_0^{IV}) + W_{,v}\mathrm{Re}(\Psi_0^{IV})] + \Phi_{00}^{IV}[\cosh W V_{,u}\,\mathrm{Im}(\Psi_0^{IV}) + W_{,u}\mathrm{Re}(\Psi_0^{IV})]$$
$$- 2\sinh W V_{,v}\mathrm{Re}(\Psi_0^{IV}\Phi_{20}^{IV})\}\ ,$$
$$\theta_4{}^{sh}{}_{,v} = \frac{1}{4\Psi_4^{IV}\bar{\Psi}_4^{IV}}\{3[\cosh W V_{,u}\,\mathrm{Im}(\Psi_4^{IV}\bar{\Psi}_2^{IV}) - W_{,u}\mathrm{Re}(\Psi_4^{IV}\bar{\Psi}_2^{IV})] + 2\,\mathrm{Im}(\Psi_4^{IV}\Phi_{02}{}^{IV}{}_{,u}) + (2M_{,u} - U_{,u})\mathrm{Im}(\Psi_4^{IV}\Phi_{02}^{IV})$$
$$+ 2\Phi_{11}^{IV}[\cosh W V_{,u}\,\mathrm{Im}(\Psi_4^{IV}) - W_{,u}\mathrm{Re}(\Psi_4^{IV})] + \Phi_{22}^{IV}[\cosh W V_{,v}\,\mathrm{Im}(\Psi_4^{IV}) - W_{,v}\mathrm{Re}(\Psi_4^{IV})]$$
$$+ 2\sinh W V_{,u}\mathrm{Re}(\Psi_4^{IV}\Phi_{02}^{IV})\}\quad (u,v>0), \tag{5.10}$$

and

$$\theta_0{}^{im}{}_{,u} = -\frac{1}{4\Psi_0^{im}\bar{\Psi}_0^{im}}\{[\cosh W V_{,u}\,\mathrm{Im}(\Psi_0^{im}) + W_{,u}\mathrm{Re}(\Psi_0^{im})]\Phi_{00}^{im} + 2\,\mathrm{Im}(\Psi_0^{im}\Phi_{20}^{IV})\}\quad (u>0, v=0),$$
$$\theta_4{}^{im}{}_{,v} = \frac{1}{4\Psi_4^{im}\bar{\Psi}_4^{im}}\{[\cosh W V_{,v}\,\mathrm{Im}(\Psi_4^{im}) - W_{,v}\mathrm{Re}(\Psi_4^{im})]\Phi_{22}^{im} + 2\,\mathrm{Im}(\Psi_4^{im}\Phi_{02}^{IV})\}\quad (u=0, v>0)\ . \tag{5.11}$$

Equation (5.10) shows that, due to the interaction between the two plane gravitational shock waves and the interaction with the matter fields, the polarization directions of the Ψ_0^{IV} and Ψ_4^{IV} waves get changed relative to the parallel-transported basis along the wave paths. In other words, the polarization direction of a plane gravitational shock wave is no longer parallel transported along the wave path, because of the above two kinds of interaction (note that for a single gravitational plane wave the polarization is always constant along the wave path). The change of polarization of a plane gravitational shock wave caused by the interaction with the other plane gravitational shock wave is exactly the analogue of the well-known electromagnetic Faraday rotation, but having the other plane gravitational shock wave as the medium and the magnetic field. We call the effect caused by the interaction between the plane gravitational waves and matter fields the deflection effect, and the effect caused by the interaction between the plane gravitational waves the gravitational Faraday effect (or gravitational Faraday rotation).

Equations (5.11), on the other hand, show that the case for impulsive gravitational waves is different from that for the shock waves. In particular, only the interaction between the impulsive waves and matter fields can change the polarization of the impulse gravitational plane waves.

To further illustrate the properties of the polarization of colliding plane gravitational waves, in the rest of this section we restrict ourselves to several specific cases which are most interesting from the point of view of physics.

Because of the symmetry shared by the two plane gravitational waves, it is sufficient to consider only one of them, say, the Ψ_0 wave. In addition, since we are now working in region IV, we do not make any more specific statements about it in the following, and understand all the following results valid only in that region (plus its two boundaries $u=0$, $v>0$ and $u>0$, $v=0$).

A. The space-time being vacuous

When the space-time is vacuous, the corresponding collision is a purely gravitational one, and the Ricci scalars are zero

$$\Phi_{ij}=0\ ,\quad \Lambda=0\ . \tag{5.12}$$

Equations (5.10) and (5.12) yield

$$\theta_0{}^{sh}{}_{,u} = -\frac{3}{4\Psi_0^{IV}\bar{\Psi}_0^{IV}}[\cosh W V_{,v}\,\mathrm{Im}(\Psi_0^{IV}\bar{\Psi}_2^{IV}) + W_{,v}\mathrm{Re}(\Psi_0^{IV}\bar{\Psi}_2^{IV})]\quad (u,v>0)\ . \tag{5.13}$$

It follows that, if $\Psi_2^{IV}=0$, we have $\theta_0{}^{sh}{}_{,u}=0$. However, it has been shown [9] that due to the nonlinear interaction between the Ψ_0^{IV} and Ψ_4^{IV} waves, the Coulomb-like gravitational field Ψ_2^{IV} necessarily appears in the interaction region, IV. Thus, in the vacuum, the change of polarization of a plane gravitational shock wave is totally due to the nonlinear interaction with the other plane gravitational shock wave.

On the other hand, from Eqs. (5.11) and (5.12) we find

$$\theta_0{}^{im}{}_{,u}=0\ . \tag{5.14}$$

That is, the impulsive plane gravitational wave does not change its polarization after the collision, when the space-time is empty. From Eq. (5.9) we can see that in the present case the Ψ_0^{im} wave component does not interact with the others.

B. The space-time filled with null dust

When the space-time is filled with null dust, the energy-stress tensor can be written in the form [29,25,30]

$$T_{\mu\nu}=\varepsilon_1 l_\mu l_\nu + \varepsilon_2 n_\mu n_\nu\ , \tag{5.15}$$

which is the superposition of two pure radiation fields, where ε_1 and ε_2 are nonnegative. Equation (5.15) represents a pair of oppositely moving null dust clouds with the energy density ε_1 and ε_2, respectively, each of which is separately conserved [25]. The corresponding nonvanishing Ricci scalars are given by

$$\Phi_{00}=\tfrac{1}{2}B^{-2}\varepsilon_2 \ , \quad \Phi_{22}=\tfrac{1}{2}A^{-2}\varepsilon_1 \ . \tag{5.16}$$

On the other hand, Eq. (5.2) shows that, like Ψ_0 and Ψ_4, the components Φ_{00} and Φ_{22}, in general, consist of two parts: the H-function part and the δ-function part. When attention is restricted to the interaction region, only the H-function part remains, and from Eq. (5.10) we find

$$\theta_0{}^{\mathrm{sh}}{}_{,u}=-\frac{1}{4\Psi_0^{\mathrm{IV}}\bar{\Psi}_0^{\mathrm{IV}}}\{3[\cosh WV_{,v}\mathrm{Im}(\Psi_0^{\mathrm{IV}}\bar{\Psi}_2^{\mathrm{IV}})+W_{,v}\mathrm{Re}(\Psi_0^{\mathrm{IV}}\bar{\Psi}_2^{\mathrm{IV}})]+\Phi_{00}^{\mathrm{IV}}[\cosh WV_{,u}\mathrm{Im}(\Psi_0^{\mathrm{IV}})+W_{,u}\mathrm{Re}(\Psi_0^{\mathrm{IV}})]\} \ . \tag{5.17}$$

Thus, unlike the vacuum case, $\theta_0{}^{\mathrm{sh}}{}_{,u}$ can now be different from zero even when $\Psi_2^{\mathrm{IV}}=0$, because of the presence of the last term on the right-hand side of Eq. (5.17), which represents the interaction between Ψ_0^{IV} and Φ_{00}^{IV} [see Eq. (5.8)].

It was shown [26] that, when null dust is present, the collision of two plane gravitational waves does not require the Coulomb-like gravitational field Ψ_2^{IV} to appear necessarily in the interaction region. Thus, a plane gravitational shock wave can change its polarization due to the deflection effect.

On the other hand, Eq. (5.11) now becomes

$$\theta_0{}^{\mathrm{im}}{}_{,u}=-\frac{1}{4\Psi_0^{\mathrm{im}}\bar{\Psi}_0^{\mathrm{im}}}\{[\cosh WV_{,u}\mathrm{Im}(\Psi_0^{\mathrm{im}})+W_{,u}\mathrm{Re}(\Psi_0^{\mathrm{im}})]\Phi_{00}^{\mathrm{im}}\} \quad (u>0, v=0) \ . \tag{5.18}$$

Obviously, when Φ_{00}^{im} is different from zero the polarization of the impulsive part of Ψ_0, in general, changes after the collision because of the interaction between Ψ_0^{im} and Φ_{00}^{im}.

C. The space-time filled with a massless scalar field

For a massless scalar field, the energy-stress tensor is given by

$$T_{\mu\nu}=\phi_{;\mu}\phi_{;\nu}-\tfrac{1}{2}g_{\mu\nu}\phi_{;\lambda}\phi^{;\lambda} \ , \tag{5.19}$$

where ϕ satisfies the massless Klein-Gordon equation

$$\phi_{;\mu;\nu}g^{\mu\nu}=0 \ . \tag{5.20}$$

Since in the present case all of the metric coefficients are functions of u and v only, without loss of generality, we assume that $\phi=\phi\,(u,v)$. Then Eq. (5.20) becomes

$$2\phi_{,uv}-U_{,u}\phi_{,v}-U_{,v}\phi_{,u}=0 \ . \tag{5.21}$$

The nonvanishing Ricci scalars are given by

$$\begin{aligned}&\Phi_{00}=\tfrac{1}{2}\phi_{,v}{}^2 \ , \quad \Phi_{22}=\tfrac{1}{2}\phi_{,u}{}^2 \ ,\\ &\Phi_{11}=\tfrac{1}{4}\phi_{,u}\phi_{,v} \ , \quad \Lambda=-\tfrac{1}{12}\phi_{,u}\phi_{,v} \ .\end{aligned} \tag{5.22}$$

Inserting Eq. (5.22) into Eq. (5.10) we find

$$\begin{aligned}\theta_0{}^{\mathrm{sh}}{}_{,u}=-\frac{1}{8\Psi_0^{\mathrm{IV}}\bar{\Psi}_0^{\mathrm{IV}}}\{&6[\cosh WV_{,v}\mathrm{Im}(\Psi_0^{\mathrm{IV}}\bar{\Psi}_2^{\mathrm{IV}})+W_{,v}\mathrm{Re}(\Psi_0^{\mathrm{IV}}\bar{\Psi}_2^{\mathrm{IV}})]\\ &+\phi_{,v}[\cosh W(\phi_{,u}V_{,v}+\phi_{,v}V_{,u})\mathrm{Im}(\Psi_0^{\mathrm{IV}})+(\phi_{,u}W_{,v}+\phi_{,v}W_{,u})\mathrm{Re}(\Psi_0^{\mathrm{IV}})]\} \ .\end{aligned} \tag{5.23}$$

Equation (5.23) shows that a plane gravitational shock wave can change its polarization due to both the nonlinear interaction between the two plane gravitational shock waves and the interaction with the massless scalar field ϕ.

When the space-time is filled only with a massless scalar field we have

$$\Phi_{00}^{\mathrm{im}}=0=\Phi_{22}^{\mathrm{im}} \ , \tag{5.24}$$

or, equivalently,

$$U_{,u}(u=0,v)=0=U_{,v}(u,v=0) \ . \tag{5.25}$$

Combining Eqs. (5.24) and (5.11) we find

$$\theta_0{}^{\mathrm{im}}{}_{,u}=0 \ , \tag{5.26}$$

which means that an impulsive plane gravitational wave does not change its polarization when it passes through a massless scalar field, since in this case there is no interaction between the gravitational impulsive wave and the massless scalar field [see Eqs. (5.9)].

Note that Eq. (5.25) is also the condition under which the hypersurfaces $u=0$ and $v=0$ are free of matter [14].

D. The space-time filled with a non-null electromagnetic field

When an electromagnetic field is null, its energy-stress tensor takes the form of a pure radiation field, which has been already discussed in Sec. V B. Thus, in this subsection, we consider only the non-null case.

For an electromagnetic field $F_{\mu\nu}$, the energy-stress tensor takes the form

$$T_{\mu\nu}=F_{\mu\lambda}F^{\lambda}{}_{\nu}-\tfrac{1}{2}g_{\mu\nu}F_{\rho\lambda}F^{\lambda\rho} \ , \tag{5.27}$$

where the antisymmetric tensor $F_{\mu\nu}$ satisfies the Maxwell equations

$$F_{[\mu\nu;\lambda]}=0\ ,\quad F_{\mu\nu;\lambda}g^{\nu\lambda}=0\ . \tag{5.28}$$

Introducing the notation

$$\begin{aligned}\Phi_0&\equiv F_{(0)(2)}=F_{\mu\nu}l^{\mu}m^{\nu}\ ,\\ \Phi_2&\equiv -F_{(1)(3)}=-F_{\mu\nu}n^{\mu}\overline{m}^{\nu}\ ,\\ \Phi_1&\equiv\tfrac{1}{2}[F_{(0)(2)}-F_{(2)(3)}]\\ &=\tfrac{1}{2}(F_{\mu\nu}l^{\mu}n^{\nu}-F_{\mu\nu}m^{\mu}\overline{m}^{\nu})\ ,\end{aligned} \tag{5.29}$$

or inversely

$$\begin{aligned}F_{\mu\nu}=2\{&-\Phi_0 n_{[\mu}\overline{m}_{\nu]}-\overline{\Phi}_0 n_{[\mu}m_{\nu]}+\Phi_2 l_{[\mu}m_{\nu]}+\overline{\Phi}_2 l_{[\mu}\overline{m}_{\nu]}\}\\ &-4\,\mathrm{Re}(\Phi_1)l_{[\mu}n_{\nu]}+i4\,\mathrm{Im}(\Phi_1)m_{[\mu}\overline{m}_{\nu]}\ ,\end{aligned} \tag{5.30}$$

we find that the Ricci scalars are given by

$$\Phi_{mn}=\Phi_m\overline{\Phi}_n\ ,\quad \Lambda=0\quad (m,n=0,1,2)\ . \tag{5.31}$$

Since the component Φ_1 defined by Eq. (5.29) is zero in regions II and III, using the Maxwell equations (5.28) it can be easily shown that it must be also zero in region IV [31,17]. If we introduce the "scale invariant" quantities

$$\Phi_0^{(0)}=B^{-1}\Phi_0\ ,\quad \Phi_2^{(0)}=A^{-1}\Phi_2\ , \tag{5.32}$$

and drop the superscript zero, then the nonvanishing Ricci ("scale invariant") scalars are

$$\Phi_{00}^{\mathrm{IV}}=\Phi_0\overline{\Phi}_0\ ,\quad \Phi_{02}^{\mathrm{IV}}=\Phi_0\overline{\Phi}_2=\overline{\Phi}_{20}^{\mathrm{IV}}\ ,\quad \Phi_{22}^{\mathrm{IV}}=\Phi_2\overline{\Phi}_2\ , \tag{5.33}$$

and the Maxwell field equations (5.28) read

$$\begin{aligned}2\Phi_{0,u}&=(U_{,u}-i\sinh WV_{,u})\Phi_0-(\cosh WV_{,v}-iW_{,v})\Phi_2\ ,\\ 2\Phi_{2,v}&=(U_{,v}+i\sinh WV_{,v})\Phi_2-(\cosh WV_{,u}+iW_{,u})\Phi_0\ .\end{aligned} \tag{5.34}$$

Note that in the present case all of the Ricci scalars have only the shock part; otherwise, the Maxwell potentials Φ_k (or equivalently, the electromagnetic field tensor $F_{\mu\nu}$) will contain the square roots of δ function, which is not acceptable physically [32].

Equations (5.10), (5.33), and (5.34) give

$$\begin{aligned}\theta_0{}^{\mathrm{sh}}{}_{,u}=-\frac{1}{4\Psi_0^{\mathrm{IV}}\overline{\Psi}{}_0^{\mathrm{IV}}}\{&3[\cosh WV_{,v}\mathrm{Im}(\Psi_0^{\mathrm{IV}}\overline{\Psi}{}_2^{\mathrm{IV}})+W_{,v}\mathrm{Re}(\Psi_0^{\mathrm{IV}}\overline{\Psi}{}_2^{\mathrm{IV}})]+2\,\mathrm{Im}(\Psi_0^{\mathrm{IV}}\overline{\Phi}_{0,v}\Phi_2)\\ &-\sinh WV_{,v}\mathrm{Re}(\Psi_0^{\mathrm{IV}}\overline{\Phi}_0\Phi_2)+2M_{,v}\mathrm{Im}(\Psi_0^{\mathrm{IV}}\overline{\Phi}_0\Phi_2)\}\ .\end{aligned} \tag{5.35}$$

Thus, similar to the last two cases, the polarization of a plane gravitational shock wave can be changed, when it interacts with a non-null electromagnetic field.

On the other hand, for the impulsive part of Ψ_0, Eq. (5.11) becomes

$$\theta_0{}^{\mathrm{im}}{}_{,u}=-\frac{\mathrm{Im}(\Psi_0^{\mathrm{im}}\overline{\Phi}_0\Phi_2)}{2\Psi_0^{\mathrm{im}}\overline{\Psi}{}_0^{\mathrm{im}}}\ . \tag{5.36}$$

It follows that a gravitational impulsive wave can change its polarization due to the interaction between the impulsive wave Ψ_0^{im} and the electromagnetic field component Φ_{02}^{IV}.

VI. EXAMPLES FOR COLLIDING GRAVITATIONAL PLANE WAVES

As examples, we consider the colliding gravitational plane wave solutions obtained recently by Tsoubelis and the present author [33] using the Belinsky-Zakharov soliton technique [34]. A subclass of those solutions is given by

$$\begin{aligned}&e^{-M}=\frac{A}{Y}t^{2a(a-1)}\ ,\quad e^{-U}=t=1-uH(u)-v^2H(v)\ ,\\ &e^{V}=\frac{C}{B}t^{2a+1}\ ,\quad \sinh W=-\frac{4qvYt^{2a-1}}{A}H(v)\ ,\end{aligned} \tag{6.1}$$

where

$$\begin{aligned}A&\equiv(Y-vH(v))^{4a}+q^2(Y+vH(v))^{4a}\ ,\\ B&\equiv(Y-vH(v))^{4a}(Y+vH(v))^2\\ &\quad+q^2(Y+vH(v))^{4a}(Y-vH(v))^2\ ,\\ C&\equiv[16q^2v^2Y^2t^{4a-2}H(v)+B^2]^{1/2}\ ,\\ Y&\equiv(1-uH(u))^{1/2}\ ,\end{aligned} \tag{6.2}$$

and q,a are arbitrary constants.

It has been shown in Ref. [33] (see also Ref. [35]) that this subclass of solutions represents the collision of a plane gravitational wave and an impulsive shell of null dust. The latter may be accompanied by a constantly polarized plane gravitational wave.

From Eqs. (5.2), (6.1), and (6.2) we find that the only nonvanishing Ricci scalar is

$$\Phi_{22}=\frac{\delta(u)}{2At^{2a(a-1)+1}}\ , \tag{6.3}$$

which represents an impulsive shell of null dust with support on the hypersurface $u=0$. To study this subclass of solutions as a whole is too complicated. In the following we consider only several subcases [36].

Case α: $a=0$. In this case we find

$$\Psi_k^{\mathrm{IV}}=0\quad (k=0,2,4)\ , \tag{6.4}$$

which means that the space-time is flat in the interaction region, IV. Hence, from Eqs. (5.1) and (5.3) we find that the nonvanishing Weyl scalars are given by

$$\Psi_0(u,v)=\frac{1}{(1+q^2)Y}[(1-q^2)-i2q]\delta(v)\ ,$$

$$\Psi_2(u,v)=0\ , \tag{6.5}$$

$$\Psi_4(u,v)=\frac{vH(v)}{2(1-v^2)C}[(1-q^2)(1-v^2)+i2q(1+v^2)]\delta(u)\ .$$

Thus, in this case, the solution represents the collision of an impulsive gravitational plane wave and an impulsive shell of null dust. Since the space-time is flat in region IV, the hypersurface $t=0$ is free of curvature singularity. After the collision, the gravitational radiation (Ψ_4) is stimulated.

Since on the hypersurface $v=0$, we have $W=0$. Then, from Eq. (4.5) we find

$$\varphi_0^{(0)}=0\ . \tag{6.6}$$

Hence, Eqs. (3.10), (4.6), (6.5), and (6.6) yield

$$\theta_0{}^{\mathrm{im}}=\frac{1}{2}\arctan\left[\frac{2q}{1-q^2}\right]\ . \tag{6.7}$$

Obviously, $\theta_0{}^{\mathrm{im}}{}_{,u}=0$. That is, the polarization angle of the impulsive gravitational wave, Ψ_0^{im}, does not change. As we can see from Eq. (5.9), this is because Ψ_0^{im} does not interact with the others.

On the other hand, Eqs. (4.3) and (6.1) give

$$\varphi_4^{(0)}=\frac{1}{2}\arctan\left[\frac{4q(1-q^2)v^2H(v)}{(1+q^2)^2-(1-6q^2+q^4)v^2}\right]\ . \tag{6.8}$$

Inserting Eqs. (6.5) and (6.8) into Eq. (4.4) we finally obtain

$$\theta_4^{\mathrm{im}}=\frac{1}{2}\arctan\left[\frac{2q}{1-q^2}\right]H(v)\ . \tag{6.9}$$

It follows that, after it is created, the impulsive gravitational wave Ψ_4^{im} does not change its polarization.

Equations (6.7) and (6.9) show that the stimulated gravitational impulsive wave Ψ_4^{im} has the same polarization direction as the Ψ_0^{im} wave. Thus, if we make a coordinate rotation in the (e_2,e_3) plane, the metric should be brought into diagonal form. In fact, it is indeed the case. After rotation with the angle given by Eq. (6.7), the metric takes the form

$$ds^2=\frac{2(1+q^2)}{Y}du\,dv-(Y-vH(v))^2dx^2$$
$$-(Y+vH(v))^2dy^2\ . \tag{6.10}$$

This solution was first found by Babala [37]. Actually, Eq. (6.10) will reduce to the exact form used by Babala, after u is replaced by u'

$$u'=\begin{cases}1-2\sqrt{1-u}\ , & u\geq 0\ ,\\ u-1\ , & u\leq 0\ .\end{cases} \tag{6.11}$$

Case β: $a=\frac{1}{2}$. In this case, if we define the functions $F(x,y)$ and $G(x,y)$ as

$$\begin{aligned}F(x,y)&\equiv(x-y)^5-6q^2x(x^2-y^2)^2+q^4(x+y)^5+i2q(x^2-y^2)[(1+q^2)(5x^2+y^2)y-2(1-q^2)(x^2+2y^2)x]\ ,\\ G(x,y)&\equiv(x-y)^4(x^2+xy+y^2)-6q^2xy(x^2-y^2)^2-q^4(x+y)^4(x^2-xy+y^2)-i2q(1+q^2)(x^2-y^2)^3\ ,\end{aligned} \tag{6.12}$$

we find that the nonvanishing Weyl scalars in region IV are given by

$$\Psi_0^{\mathrm{IV}}(u,v)=\frac{3(1+q^2)Y}{tCA^2}F(Y,v)\ ,$$

$$\Psi_2^{\mathrm{IV}}(u,v)=-\frac{1}{2t^2A^2Y}G(Y,v)\ , \tag{6.13}$$

$$\Psi_4^{\mathrm{IV}}(u,v)=-\frac{3(1+q^2)v}{4tCA^2Y^2}\bar{F}(v,Y)\ .$$

Then, Eqs. (5.1) and (6.13) give

$$\Psi_0(u,v)=\Psi_0^{\mathrm{IV}}(uH(u),v)H(v)$$
$$+\frac{1}{(1+q^2)Y}\{(1-q^2)-i2q\}\delta(v)\ ,$$

$$\Psi_2(u,v)=\Psi_2^{\mathrm{IV}}(u,v)H(u)H(v)\ , \tag{6.14}$$

$$\Psi_4(u,v)=\Psi_4^{\mathrm{IV}}(u,v)H(u)H(v)$$
$$+\frac{1+q^2}{2AC}\{(1-vH(v))^3+q^2(1+vH(v))^3$$
$$+i2qv(1-v^2)H(v)\}\delta(u)\ .$$

Thus, this model represents the collision of a variably polarized gravitational shock+impulsive wave and an impulsive shell of null dust which is accompanied by a constantly polarized gravitational impulsive wave. In this model, region III is flat, while region II is curved due to the presence of the variably polarized gravitational wave.

As $t\rightarrow 0^+$, all the nonvanishing Weyl scalars become unbounded. Thus, the space-time now is singular on the hypersurface $t=0$.

On the other hand, Eq. (6.14) shows that Ψ_0 and Ψ_4 consist of two parts: the shock part and the impulsive part. In the following we consider them separately.

Let us first consider the impulsive part. Following the discussion carried out in case α we find

$$\theta_0^{\mathrm{im}}=\frac{1}{2}\arctan\left[\frac{2q}{1-q^2}\right]\ ,$$

$$\theta_4^{\mathrm{im}}=\frac{1}{2}\arctan\left[\frac{2qv}{(1-v)+q^2(1+v)}\right]H(v)\ . \tag{6.15}$$

Thus, the polarization angle for the impulsive part of Ψ_0 remains constant even after the collision. The reason is that, similar to the last case, $\Psi_0^{\rm im}$ does not interact with any of the others. However, for the Ψ_4 wave the situation is different. The interaction between the impulsive part of Ψ_4 and the impulsive shell of null dust $\Phi_{22}^{\rm im}$ is such as to make the polarization angle $\theta_4^{\rm im}$ change along the v axis according to Eq. (6.15).

In a similar fashion we find that for the shock part of Ψ_0 and Ψ_4 the polarization angles $\theta_0^{\rm sh}$ and $\theta_4^{\rm sh}$ are given, respectively, by

$$\theta_0^{\rm sh}=-\frac{1}{2}\arctan\left[\frac{I(v,Y)}{J(v,Y)}\right]H(v)\ ,$$
$$\theta_4^{\rm sh}=\frac{1}{2}\arctan\left[\frac{I(Y,v)}{J(Y,v)}\right]H(u)H(v)\ , \tag{6.16}$$

where the functions $I(x,y)$ and $J(x,y)$ are defined by

$$\begin{aligned} I(x,y)\equiv 2qy\{&(x^2-y^2)(x-y)(x+5y)+q^2[(4+6q^2+4q^4+q^6)x^4+4(2-2q^4-q^6)x^3y+2(4+14q^2+4q^4-3q^6)x^2y^2\\ &+4(6-6q^4+q^6)xy^3-(60-126q^2+60q^4-5q^6)y^4]\}\ ,\\ J(x,y)\equiv(x^2-y^2)^2&(x-y)+q^2[(5+10q^2+10q^4+5q^6+q^8)x^5-(3+2q^2-2q^4-3q^6-q^8)x^4y\\ &+2(3+14q^2+14q^4+3q^6-q^8)x^3y^2-2(13+14q^2-14q^4-13q^6+q^8)x^2y^3\\ &-(27-42q^2-42q^4+27q^6-q^8)xy^4+(45-210q^2+210q^4-45q^6+q^8)y^5]\ . \end{aligned} \tag{6.17}$$

Equation (6.16) shows that the plane gravitational shock waves of Ψ_0 and Ψ_4 change their polarization angles along each of their own paths. The change is due to the presence of the Coulomb-like field Ψ_2, which is the result of the nonlinear interaction between Ψ_0 and Ψ_4.

VII. CONCLUSION

In the previous sections, the polarization of interacting gravitational plane waves have been studied. It has been found that the polarization of a plane gravitational shock wave can be changed due to two kinds of interaction. One is the nonlinear interaction between two oppositely moving gravitational shock waves, and the other is the interaction between the shock wave and matter fields, which are present. The former is exactly an analogue of the well-known electromagnetic Faraday rotation, but with the other gravitational shock wave as the magnetic field and the medium. The effect of the above two kinds of interaction on the polarization of a plane gravitational impulse wave is different. In particular, the polarization of an impulsive plane gravitational wave is affected only by the interaction between the impulse wave and matter fields.

The above study of polarization of plane gravitational waves may give us a possible way to detect gravitational plane waves. In this direction, let us consider a gravitational wave moving in the direction of the z axis, emitted, say, by a remote star. If we let such a wave pass through a medium, then, we can see that, due to the interaction between the wave and the medium, the polarization of the wave will be changed. So, if we equip two rings in the (x,y) plane, one is in front of the medium, and the other is right after the medium, then we will find that, after the gravitational wave passes through them, the two rings are deformed into ellipses but with different shearing angles. Thus, by measuring the difference between these two angles we can determine whether or not a gravitational plane wave passes through the medium.

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[1] A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).
[2] T. Piran and P. N. Safier, Nature (London) **318**, 217 (1985); T. Piran, P. N. Safier, and R. F. Stark, Phys. Rev. D **32**, 3101 (1985).
[3] H. W. Brinkman, Proc. Natl. Acad. Sci. (U.S.A.) **9**, 1 (1923).
[4] N. Rosen, Phys. Z. **12**, 366 (1937).
[5] H. Bondi, Nature (London) **179**, 1072 (1957).
[6] H. Bondi, F. A. E. Pirani, and I. Robinson, Proc. R. Soc. London **A251**, 519 (1959).
[7] R. Penrose, Rev. Mod. Phys. **37**, 215 (1965).
[8] P. Szekeres, Nature (London) **228**, 1183 (1970).
[9] P. Szekeres, J. Math. Phys. **13**, 286 (1972).
[10] K. A. Khan and R. Penrose, Nature (London) **229**, 185 (1971).
[11] S. Chandrasekhar and B. C. Xanthopoulos, Proc. R. Soc. London **A408**, 175 (1986); V. Ferrari and J. Ibanez, Gen. Relativ. Gravit. **19**, 405 (1987).
[12] A. V. Feinstein and J. Ibanez, Phys. Rev. D **39**, 470 (1989).
[13] U. Yurtsever, Phys. Rev. D **38**, 1706 (1988); **40**, 329 (1989).
[14] D. Tsoubelis and A. Z. Wang, Gen. Relativ. Gravit. **21**, 807 (1989).
[15] A. Z. Wang (unpublished).
[16] P. Szekeres, J. Math. Phys. **6**, 1387 (1965); **7**, 751 (1966).
[17] J. B. Griffiths, Ann. Phys. (N.Y.) **102**, 388 (1976).

[18] P. J. Adams, R. W. Hellings, R. L. Zimmerman, H. Farhoosh, D. I. Levine, and S. Zeldich, Astrophys. J. **253**, 1 (1982).
[19] B. J. Carr and E. Verdaguer, Phys. Rev. D **28**, 2995 (1983).
[20] E. Verdaguer, in *Observational and Theoretical Aspects of Relativistic Astrophysics and Cosmology,* Proceedings of the International Course, Santander, Spain, 1984, edited by J. L. Sanz and L. J. Goicoechea (World Scientific, Singapore, 1985), p. 311.
[21] A. Z. Wang, report, 1991 (unpublished).
[22] F. A. E. Pirani and A. Schild, Bull. Acad. Polon. Sci. **9**, 543 (1961).
[23] J. Ehlers and W. Kundt, in *Gravitation: An Introduction on Current Research,* edited by L. Witten (Wiley, New York, 1962), p. 63.
[24] T. Dray and G. 't Hooft, Class. Quantum Grav. **3**, 825 (1986).
[25] A. H. Taub, J. Math. Phys. **29**, 690 (1988).
[26] D. Tsoubelis and A. Z. Wang, Gen. Relativ. Gravit. **22**, 1091 (1980).
[27] Y. Nutku and M. Halil, Phys. Rev. Lett. **39**, 1379 (1977).
[28] S. Chandrasekhar and V. Ferrari, Proc. R. Soc. London **A396**, 55 (1984).
[29] S. Chandrasekhar and B. C. Xanthopoulos, Proc. R. Soc. London **A403**, 189 (1986).
[30] D. Tsoubelis and A. Z. Wang, J. Math. Phys. **32**, 1017 (1991).
[31] P. Bell and P. Szekeres, Gen. Relativ. Gravit. **5**, 275 (1974).
[32] S. Chandrasekhar and B. C. Xanthopoulos, Proc. R. Soc. London **A398**, 223 (1985).
[33] D. Tsoubelis and A. Z. Wang, report, 1991 (unpublished).
[34] V. A. Belinsky and V. E. Zakharov, Zh. Eksp. Teor. Fiz. **75**, 1955 (1978) [Sov. Phys. JETP **48**, 985 (1978)].
[35] A. Z. Wang, Int. J. Mod. Phys. A **6**, 2273 (1991).
[36] A. Z. Wang, Ph.D. dissertation, submitted to University of Ioannina, 1990.
[37] D. Babala, Class. Quantum Grav. **4**, L89 (1987).