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Spherically symmetric solutions in dimensionally reduced spacetimes with a higher-dimensional cosmological constant

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Certain static solutions of D-dimensional gravity with a higher-dimensional cosmological constant Λ are studied. The solutions are taken to be spherically symmetric in the physical $(m+2)$-dimensional spacetime, where $D=m+n+2$ (or more generally the m-sphere is replaced by an arbitrary Einstein space), while the internal space is an arbitrary n-dimensional Einstein space. The global properties of all such solutions are derived by considering the equivalent dimensionally reduced system in $m+2$ dimensions, and by using techniques from the theory of dynamical systems after a judicious choice of variables. All solutions with a nonzero Λ are either found to contain naked singularities or not be asymptotically flat, as would be expected from the "no-hair" theorems. A recent "black-hole" solution derived by Kim and Cho in the context of these models is shown to be incorrect.

I. INTRODUCTION

If higher dimensions are a physical reality then many familiar physical systems in four dimensions should have higher-dimensional counterparts in which the four physical dimensions and the extra dimensions split off distinctly. In particular, if one adopts the Kaluza-Klein viewpoint, it should be possible to find regular black-hole solutions which preserve spherical symmetry in the physical spacetime, while having a compact internal space. Although many solutions have been found which have higher-dimensional spherical or axial symmetry [1], relatively little is known about higher-dimensional solutions which are spherically symmetric in four dimensions only, the extra dimensions being compact.

It is true that nontrivial black-hole solutions have been found, and extensively studied, in the five-dimensional Kaluza-Klein theory [2–4]. However, if one wishes to generalize solutions to $D>5$ then extra complications can arise, since it is now possible for the internal space to have curvature. Another possible departure is that rather than taking the higher-dimensional action to be purely Einstein gravity, it may be more natural to consider actions which arise in supergravity models or the low-energy limit of string theory.

Solutions with four-dimensional spherical symmetry for $D>5$ have been discussed by Dobiasch and Maison [2], Lee [5], and Lee and Lou [6] in the case of pure gravity (with off-diagonal terms of the higher-dimensional metric corresponding to Abelian gauge fields), by Myers [7] in Einstein-Maxwell theory, by van Baal *et al.* [8] in 11-dimensional supergravity, and by Ivanov [9] for an action appropriate to the low-energy limit of string theory (including a dilaton and electromagnetic fields). All these authors, apart from van Baal *et al.*, considered a Ricci-flat internal space only. van Baal *et al.* took the internal space to be a seven-sphere, but were unable to analytically construct any regular black-hole solutions with a compact internal space. Instead, by numerical integration, they found a class of solutions which have the novel feature of a naked lightlike singularity.

Mignemi and I [10] have considered the problem of compactified black-hole solutions using a somewhat different approach. We classified all solutions of the D-dimensional vacuum Einstein equations which have the form

$$ds_D^2=-e^{2\mu}dt^2+e^{2\nu}d\bar r^{\,2}+\bar r^{\,2}\bar g_{ij}d\bar x^{\,i}d\bar x^{\,j}+e^{2\omega}\tilde g_{ab}d\tilde y^{\,a}d\tilde y^{\,b}\ , \tag{1.1a}$$

where $\mu=\mu(\bar r)$, $\nu=\nu(\bar r)$, $\omega=\omega(\bar r)$, and $\bar g_{ij}(\bar x)$ and $\tilde g_{ab}(\tilde y)$ are metrics on arbitrary Einstein spaces of dimension m and n, respectively:

$$\bar R_{ij}=(m-1)\bar\lambda\bar g_{ij}\ , \tag{1.1b}$$

$$\tilde R_{ab}=(n-1)\tilde\lambda\tilde g_{ab}\ . \tag{1.1c}$$

The case in which $\bar g_{ij}$ represents a two-sphere and the internal space is compact is of course of most physical interest. The approach we used in Ref. [10], henceforth denoted I, was to consider the equivalent dimensionally reduced system of equations and to make a choice of variables which made it possible to write the field equations as an autonomous system of first-order differential equations. Such a choice is possible because the model is in fact equivalent to a nonlinear Toda lattice [11,12], which is known to be completely integrable. Thus all the global properties of the solutions (1.1) can be determined, even though analytic solutions cannot be written down in general. A similar dynamical systems approach has also been usefully applied to higher-dimensional cosmological models [13].

In a recent paper [14] Kim and Cho considered higher-dimensional Einstein gravity with a higher-dimensional cosmological constant Λ and tried to find solutions of the form (1.1) with the added restriction that $\tilde\lambda=0$; i.e., the internal space is Ricci flat, and $\bar g_{ij}$ represents the metric on a two-sphere. By making a particular ansatz for the dimensionally reduced fields, Kim and Cho claimed to find a "black-hole" solution for nonzero Λ. Unfortunately, however, they made two errors which invalidate their results. First, they made a mistake in deriving the higher-dimensional Ricci tensor and consequently one of their field equations was wrong [15]. This mistake was compounded later by a somewhat more serious error: having integrated some of their field equations to derive expressions for the metric functions, Kim and Cho neglected to substitute the functions they obtained back into the constraint equation to see what additional restrictions it placed on the integration constants. Had they done so, they would have discovered that there are *no* solutions in the context of their metric ansatz (with the exception of one unphysical solution which does not possess an asymptotic region). This remains true if the correct field equations are used, as is demonstrated in the Appendix.

We should note that in order for black-hole solutions to be consistent with the Kaluza-Klein interpretation the radius of the extra dimensions $e^{2\omega}$, should be asymptotically constant. Consequently in the present model realistic black-hole solutions would appear to be ruled out by the "no-hair" theorems [16]—it is well known that no regular black-hole solutions can be found if the four-dimensional Einstein action is coupled to a (massive or massless) asymptotically flat scalar field [17]. The field equations derived from the D-dimensional action

$$S=\int d^D x\sqrt{-g}\,\frac{1}{4\kappa_D^2}(R-2\Lambda) \tag{1.2}$$

are of course equivalent to those derived from the dimensionally reduced action [18]

$$\hat{S}=\int d^{m+2}\hat{x}\sqrt{-\hat{g}}\left[\frac{\hat{R}}{4\kappa^2}-\frac{1}{m}\hat{g}^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma-\mathcal{V}(\sigma)\right], \tag{1.3a}$$

where

$$\kappa\sigma=\tfrac{1}{2}\sqrt{n(m+n)}\,\omega\ , \tag{1.3b}$$

$$\mathcal{V}(\sigma)=\frac{1}{4\kappa^2}\exp\left[\frac{-4\sqrt{n}\,\kappa\sigma}{m\sqrt{m+n}}\right]\times\left[2\Lambda-n(n-1)\tilde{\lambda}\exp\left[\frac{-4\kappa\sigma}{\sqrt{n(m+n)}}\right]\right], \tag{1.3c}$$

$\kappa^2=4\pi G$ is the $(m+2)$-dimensional gravitational constant, $\kappa_D^2=\kappa^2\int d^n\tilde{y}(\det\tilde{g}_{ab})^{1/2}$ is the D-dimensional gravitational constant, and $\hat{g}\equiv\det(\hat{g}_{\alpha\beta})$ and $\hat{R}$ refer to the conformally rescaled $(m+2)$-dimensional metric:

$$\hat{g}_{\alpha\beta}d\hat{x}^{\alpha}d\hat{x}^{\beta}=e^{2n\omega/m}(-e^{2\mu}dt^2+e^{2\nu}d\bar{r}^2+\bar{r}^2\bar{g}_{ij}d\bar{x}^{i}d\bar{x}^{j})\ . \tag{1.3d}$$

The requirement that $e^{2\omega}$ is asymptotically constant is equivalent to the requirement that σ is asymptotically constant, or alternatively that σ is asymptotically flat if we make a trivial rescaling of the internal dimensions to set $e^{2\omega}\to 1$ at spatial infinity, and thus we expect the no-hair theorem to hold [19]. Since no-hair theorems have only been derived for scalar fields with particular potentials, it is of course not ruled out that the field equations derived from (1.3a) could have regular black-hole solutions. However, in light of the great number of no-hair results which have been derived for various forms of matter, such a result is not to be expected.

In the case of the nontrivial Kaluza-Klein black-hole solutions mentioned above, the scalar charge is a dependent function of the other charges (electric and magnetic) of the theory [4], a result consistent with the no-hair theorems. However, there are no extra charges in the model considered here.

Since we are (in principle) seeking solutions which represent regular black holes in the dimensionally reduced theory, we require that the dimensionally reduced fields should be regular on a regular horizon. In particular, the scalar field σ and curvature invariants constructed from the $(m+2)$-dimensional metric $\hat{g}_{\alpha\beta}$ should be finite on the horizon. This condition is of course stronger than simply requiring that quantities be regular according to the higher-dimensional metric: particular combinations of the higher-dimensional metric components, which are not invariant in the higher-dimensional spacetime, are to be interpreted as the physical fields of the theory. For the models which we are considering here regularity in the dimensionally reduced spacetime implies regularity in the higher-dimensional spacetime, but the converse is not true, as is illustrated by the well-known example of the Euclidean Schwarzschild metric,

$$ds_D^2=-dt^2+\frac{d\bar{r}^2}{1-\left[\frac{\bar{c}}{\bar{r}}\right]^{m-1}}+\bar{r}^2d\Omega_m^2+\left[1-\left[\frac{\bar{c}}{\bar{r}}\right]^{m-1}\right]d\tilde{y}^2\ , \tag{1.4}$$

where $\bar{c}$ is a constant, which solves the field equations derived from (1.2) if $\Lambda=0$ and $n=1$. The metric (1.4) has an apparent singularity at $\bar{r}=\bar{c}$, a "bolt" [20] corresponding to a fixed point of the $\partial/\partial\tilde{y}$ Killing vector, which can be removed if the $\tilde{y}$ coordinate is identified with period $4\pi\bar{c}/(m-1)$. Thus the manifold is completely regular from the D-dimensional viewpoint. However, the dimensionally reduced fields, which are given by

$$\hat{g}_{\alpha\beta}d\hat{x}^{\alpha}d\hat{x}^{\beta}=\left[1-\left[\frac{\bar{c}}{\bar{r}}\right]^{m-1}\right]^{1/m}dt^2+\frac{d\bar{r}^2}{\left[1-\left[\frac{\bar{c}}{\bar{r}}\right]^{m-1}\right]^{(m-1)/m}}+\left[1-\left[\frac{\bar{c}}{\bar{r}}\right]^{m-1}\right]^{1/m}\bar{r}^2d\Omega_m^2 \tag{1.5a}$$

and

$$\kappa\sigma=\tfrac{1}{2}(m+1)^{1/2}\ln\left[1-\left[\frac{\bar{c}}{\bar{r}}\right]^{m-1}\right], \qquad (1.5b)$$

are singular at $\bar{r}=\bar{c}$. This situation is very similar to that of the Kaluza-Klein monopole, which is completely regular in five dimensions but not from the viewpoint of the equivalent dimensionally reduced theory [21].

In using a definition of regularity based in the dimensionally reduced spacetime we may of course overlook other special solutions which similarly to the Euclidean Schwarzschild solution are regular in higher dimensions. However, since our primary aim is to investigate the properties of spacetimes obtained by dimensional reduction, assuming a Kaluza-Klein interpretation, this will not concern us here.

Our definition of asymptotic flatness shall similarly apply to quantities defined in the dimensionally reduced spacetime: for an "asymptotically flat" spacetime $\hat{R}_{\alpha\beta\gamma\delta}$ must vanish as $r\rightarrow\infty$, where r is a radial coordinate based in the dimensionally reduced spacetime. Since the relation between the lower- and higher-dimensional radial coordinates is

$$\bar{r}=e^{-n\omega/m}r=e^{-2\sqrt{n}\,\kappa\sigma/(m\sqrt{m+n}\,)}r, \qquad (1.6)$$

the asymptotic region defined by $r\rightarrow\infty$ will also correspond to an asymptotic region in terms of $\bar{r}$ if $e^{2\kappa\sigma}$ is asymptotically constant, or if $e^{2\kappa\sigma}\sim r^{a}$, where $a<m\sqrt{m+n}\,/\sqrt{n}$, but not otherwise. If $e^{2\kappa\sigma}$ is asymptotically constant then asymptotic flatness of the $(m+2)$-dimensional metric implies asymptotic flatness of the higher-dimensional metric, but of course differences are possible in other circumstances.

The purpose of this paper is to extend the results of I to include a higher-dimensional cosmological constant, thereby classifying all the spherically symmetric solutions which Kim and Cho sought to investigate. In fact, the model we will investigate is somewhat more general than that of Kim and Cho because we will allow the internal space to possibly have curvature ($\tilde{\lambda}\neq 0$). Also, although we are primarily interested in the case in which $\bar{g}_{ij}$ represents a two-sphere, we shall continue to take it to represent an arbitrary m-dimensional Einstein space, as this leads to a symmetry in the equations which will become apparent. This symmetry is a relic of the symmetry between the two Einstein spaces in (1.1): we could equally well choose the m-dimensional space to be the internal space, while the n-dimensional space corresponded to the spatial sections at spatial infinity. Instead of the coordinate r defined by (1.6) the appropriate radial coordinate to use would be

$$\tilde{r}=e^{m\omega/n}\bar{r}=r^{m/n}, \qquad (1.7)$$

based in an $(n+2)$-dimensional spacetime with a conformally rescaled metric

$$ds^2_{(n+2)}=\tilde{r}^{\,2m/n}(-e^{2\mu}dt^2+e^{2\nu}d\tilde{r}^{\,2}+e^{2\omega}\tilde{g}_{ab}d\tilde{y}^{\,a}d\tilde{y}^{\,b}), \qquad (1.8)$$

instead of (1.3d).

II. THE DYNAMICAL SYSTEM

Rather than working directly with the higher-dimensional action, as Kim and Cho did, we will use the equivalent field equations derived from the dimensionally reduced action (1.3): namely,

$$\hat{R}_{\alpha\beta}=\frac{4\kappa^2}{m}(\partial_\alpha\sigma\partial_\beta\sigma+\hat{g}_{\alpha\beta}\mathcal{V}), \qquad (2.1a)$$

$$\frac{1}{\sqrt{-\hat{g}}}\partial_\alpha(\sqrt{-\hat{g}}\;\partial^\alpha\sigma)=\frac{m}{2}\frac{\partial\mathcal{V}}{\partial\sigma}. \qquad (2.1b)$$

We will follow Refs. [10,12] by choosing coordinates

$$\hat{g}_{\alpha\beta}d\hat{x}^{\,\alpha}d\hat{x}^{\,\beta}=e^{2u}(-dt^2+r^{2m}d\xi^2)+r^2\bar{g}_{ij}d\bar{x}^{\,i}d\bar{x}^{\,j}, \qquad (2.2)$$

where $u=u(\xi)$ and $r=r(\xi)$. In addition, we will define the functions ζ, η, and χ by

$$\zeta=u+(m-1)\ln r, \qquad (2.3)$$

$$\eta=u+m\ln r-\frac{2\sqrt{m+n}\;\kappa\sigma}{m\sqrt{n}}, \qquad (2.4)$$

$$\chi=u+m\ln r-\frac{2\sqrt{n}\;\kappa\sigma}{m\sqrt{m+n}}. \qquad (2.5)$$

With these choices the field equations become

$$\zeta''=(m-1)^2\bar{\lambda}e^{2\zeta}+n(n-1)\tilde{\lambda}e^{2\eta}-2\Lambda e^{2\chi}, \qquad (2.6a)$$

$$\eta''=m(m-1)\bar{\lambda}e^{2\zeta}+(n-1)^2\tilde{\lambda}e^{2\eta}-2\Lambda e^{2\chi}, \qquad (2.6b)$$

$$\chi''=m(m-1)\bar{\lambda}e^{2\zeta}+n(n-1)\tilde{\lambda}e^{2\eta}-\frac{(m+n+1)2\Lambda e^{2\chi}}{m+n}, \qquad (2.6c)$$

with the constraint

$$m(m+1)\zeta'^2+2mn\zeta'\eta'+n(n+1)\eta'^2-2(m+n)(m\zeta'+n\eta')\chi'+(m+n)(m+n-1)\chi'^2+m(m-1)\bar{\lambda}e^{2\zeta}+n(n-1)\tilde{\lambda}e^{2\eta}-2\Lambda e^{2\chi}=0. \qquad (2.6d)$$

These equations can be recast in the form of a five-dimensional autonomous system of first-order differential equations. If we define variables V, W, X, Y, and Z by

$$V=\chi',\quad W=|2\Lambda|^{1/2}e^{\chi},\quad X=\zeta',\quad Y=\eta',\quad Z=|\tilde{\lambda}|^{1/2}e^{\eta}, \qquad (2.7)$$

then the constraint (2.6d) can be regarded as a definition of $e^{2\zeta}$. Eliminating the $e^{2\zeta}$ terms from (2.6a)–(2.6c) we therefore obtain the system

$$V'=\frac{-\epsilon_2}{m+n}W^2-P\ , \tag{2.8a}$$

$$W'=VW\ , \tag{2.8b}$$

$$X'=\frac{n}{m}(n-1)\epsilon_1 Z^2-\frac{\epsilon_2}{m}W^2-\frac{m-1}{m}P\ , \tag{2.8c}$$

$$Y'=-(n-1)\epsilon_1 Z^2-P\ , \tag{2.8d}$$

$$Z'=YZ\ , \tag{2.8e}$$

where

$$\begin{aligned}P\equiv{}& m(m+1)X^2+2mnXY+n(n+1)Y^2\\&-2(m+n)(mX+nY)V\\&+(m+n)(m+n-1)V^2\ ,\end{aligned} \tag{2.8f}$$

$$\epsilon_1=\begin{cases}1, & \tilde{\lambda}>0,\\ 0, & \tilde{\lambda}=0,\\ -1, & \tilde{\lambda}<0\ ,\end{cases} \tag{2.8g}$$

and

$$\epsilon_2=\begin{cases}1, & \Lambda>0,\\ 0, & \Lambda=0,\\ -1, & \Lambda<0\ .\end{cases} \tag{2.8h}$$

Although the phase space is five-dimensional it is nonetheless amenable to analysis because of various symmetries. Equations (2.8b) and (2.8e) ensure that trajectories cannot cross either the $W=0$ or the $Z=0$ subspaces. These two subspaces correspond physically to the cases in which $\Lambda=0$ and $\tilde{\lambda}=0$, respectively. The $\Lambda=0$ ($W=0$) system was studied in I. There is a trivial symmetry between trajectories in the $W>0$ and $W<0$ portions of the phase space, and between trajectories in the $Z<0$ and $Z>0$ portions of the phase space. This merely reflects the fact that the equations are invariant under $e^{\chi}\to -e^{\chi}$, and under $e^{\eta}\to -e^{\eta}$.

If $W=0$ then

$$V=\frac{1}{m+n-1}(mX+nY+c_1)\ , \tag{2.9}$$

while if $Z=0$ then

$$Y=\frac{1}{n+1}[-mX+(m+n)V+c_2]\ , \tag{2.10}$$

where c_1 and c_2 are arbitrary constants. Thus in each case a further degree of freedom can be integrated out, giving rise to a three-dimensional autonomous system. Another special surface in the phase space is the hyperboloid defined by $\bar{\lambda}=0$, or

$$m(m+1)X^2+2mnXY+n(n+1)Y^2-2(m+n)(mX+nY)V+(m+n)(m+n-1)V^2+n(n-1)\epsilon_1 Z^2-\epsilon_2 W^2=0\ . \tag{2.11}$$

This surface once again partitions the phase space into distinct regions which trajectories cannot cross.

Further simplifications arise if any two of the constants Λ, $\tilde{\lambda}$, and $\bar{\lambda}$ are simultaneously zero. In each of these cases it is in fact possible to integrate the field equations exactly. This was done in I for cases (i) $\Lambda=0$, $\tilde{\lambda}=0$ [22] and (ii) $\Lambda=0$, $\bar{\lambda}=0$. In both cases all solutions were found to have naked singularities, except for some particular choices of the integration constants. In the $\tilde{\lambda}=0$ case this combination merely gave the trivial case of the Schwarzschild solution with an everywhere constant scalar field. In the $\bar{\lambda}=0$ case a new solution with a regular horizon was obtained, but it was not asymptotically flat. By symmetry this latter solution of course corresponds to the Schwarzschild solution that we would have obtained if we had chosen to dimensionally reduce to $n+2$ instead of $m+2$ dimensions, as may be readily seen by transforming to the coordinates defined by (1.7) and (1.8).

The new case which arises here is (iii) $\tilde{\lambda}=0$, $\bar{\lambda}=0$. This case is actually very similar to cases (i) and (ii), especially when we observe that if we define $l=-(m+n)$ and

$$\hat{\Lambda}=\frac{-2\Lambda}{(m+n)(m+n+1)}\ , \tag{2.12}$$

then Eqs. (2.6) become

$$\zeta''=(m-1)^2\bar{\lambda}e^{2\zeta}+l(l-1)\hat{\Lambda}e^{2\chi}\ , \tag{2.13a}$$

$$\chi''=m(m-1)\bar{\lambda}e^{2\zeta}+(l-1)^2\hat{\Lambda}e^{2\chi}\ , \tag{2.13b}$$

$$\begin{aligned}&m(l-1)\zeta'^2-2ml\zeta'\chi'+l(m-1)\chi'^2+(l+m)c_2^2\\&\quad+m(m-1)(m+l-1)\bar{\lambda}e^{2\zeta}\\&\quad+l(l-1)(l+m-1)\hat{\Lambda}e^{2\chi}=0\ ,\end{aligned} \tag{2.13c}$$

if $\tilde{\lambda}=0$, where c_2 is the integration constant defined by (2.10). This system is in fact identical to the $\Lambda=0$ system studied in I if we make the replacements $\chi\to\eta$, $l\to n$, $\hat{\Lambda}\to\tilde{\lambda}$, and $c_2\to c_1$. Thus if $\bar{\lambda}=0$ also, then the solutions for e^{ζ} and e^{χ} can be read off from Eqs. (4.1) of I, and the other metric functions derived. We find

$$e^{\chi}=\frac{CA_1}{\hat{\Delta}}\exp(\tfrac{1}{2}C\xi)\ , \tag{2.14a}$$

$$e^{\xi}=A_0\exp\left[\frac{m+n}{m+n+1}(\chi+c_3\xi)\right]\ , \tag{2.14b}$$

$$e^{u}=A_0^m A_2^{n(m-1)/m}\left\{\frac{CA_1}{\hat{\Delta}}\exp\left[\left[\frac{1}{2}C+mc_3+(m-1)c_0\right]\xi\right]\right\}^{(m+n)/[m(m+n+1)]}\ , \tag{2.14c}$$

$$r=\frac{1}{A_0A_2^{n/m}}\left\{\frac{CA_1}{\hat{\Delta}}\exp\left[\left[\frac{1}{2}C-c_0\right]\xi\right]\right\}^{(m+n)/[m(m+n+1)]}\ , \tag{2.14d}$$

$$\exp\left[\frac{2\kappa\sigma}{\sqrt{n(m+n)}}\right]=\frac{1}{A_2}\left\{\frac{CA_1}{\hat{\Delta}}\exp\left[\left[\frac{1}{2}C+\frac{1}{n}(m+n)(mc_3-c_0)\right]\xi\right]\right\}^{1/(m+n+1)}\ , \tag{2.14e}$$

where

$$\hat{\Delta}\equiv(m+n+1)[\hat{\Lambda}-A_1^2\exp(C\xi)]\ , \tag{2.14f}$$

while A_0, A_1, A_2, and c_3 are arbitrary constants, C is a nonzero constant given by

$$\tfrac{1}{4}(n+1)(m+n)C^2=n(m+n+1)c_2^2+m(m+n)^2c_3^2\ , \tag{2.14g}$$

and the constant c_0 is defined by

$$c_0=\frac{1}{n+1}\left[\frac{(m+n+1)nc_2}{m+n}+mc_3\right]\ . \tag{2.14h}$$

Some restrictions on the signs of the various constants, and on the range of ξ will be imposed by the requirement of reality when roots are taken in the expressions (2.14c)–(2.14e) [23]. If we replace the coordinates (2.2) by the more standard Schwarzschild-type coordinates

$$\hat{g}_{\alpha\beta}d\hat{x}^{\alpha}d\hat{x}^{\beta}=-e^{2u}dt^2+e^{2v}dr^2+r^2\bar{g}_{ij}d\bar{x}^{i}d\bar{x}^{j}\ , \tag{2.15}$$

then e^u is given by (2.14c) while

$$e^{v}=\frac{mA_0(\hat{\Delta}\{CA_1\exp[(\tfrac{1}{2}C+c_3)\xi]\}^{m+n})^{1/(m+n+1)}}{(m+n)[CA_1^2\exp[C\xi]+(\tfrac{1}{2}C-c_0)(m+n+1)^{-1}\hat{\Delta}]}\ , \tag{2.16}$$

and ξ is defined implicitly in terms of r by (2.14d).

Similarly to cases (i) and (ii) [10] we find that the limit $\xi\to-\infty$ corresponds to $r\to 0$ except in the special instances when $c_2=(m+n)c_3=\frac{1}{2}|C|$ (and hence $c_0=\frac{1}{2}|C|$), for which $r\to$const, suggesting the possible presence of an horizon. This indeed turns out to be true: (2.14d) can be inverted and if we make the choice

$$A_1=A_0^{m+1}A_2^n,\quad \left[\frac{A_2}{A_0}\right]^{n/(m+n)}=\frac{m}{m+n}\ , \tag{2.17}$$

then we find the solution

$$ds^2=-r^2\Delta\,dt^2+\frac{dr^2}{\Delta r^{2m/(m+n)}}+r^2\bar{g}_{ij}d\bar{x}^{i}d\bar{x}^{j}\ , \tag{2.18a}$$

where

$$\Delta=\hat{\Lambda}-\frac{(m+n)^2C}{m(m+n+1)r^{m(m+n+1)/(m+n)}}\ , \tag{2.18b}$$

while the scalar field is given by

$$\exp\left[\frac{2\sqrt{m+n}\,\kappa\sigma}{m\sqrt{n}}\right]=\left[\frac{m+n}{m}\right]^{(m+n)/n}r\ . \tag{2.18c}$$

Similarly the limit $\xi\to+\infty$ also corresponds to $r\to 0$ except for the special cases when $c_2=(m+n)c_3=-\frac{1}{2}|C|$ (and hence $c_0=-\frac{1}{2}|C|$), for which $r\to$const. Equation (2.14d) can once again be inverted, and we retrieve the solution (2.18) if we now make the choice

$$A_1=\frac{\hat{\Lambda}}{A_0^{m+1}A_2^n},\quad \left[\frac{A_2}{A_0}\right]^{n/(m+n)}=\frac{m}{m+n}\ . \tag{2.19}$$

The spacetimes thus have naked singularities except in the special cases above.

An asymptotic region is defined only for $\Lambda<0$ (i.e., $\hat{\Lambda}>0$): $r\to\infty$ when $C\xi=\ln|\hat{\Lambda}/A_1^2|$. The asymptotic form of all the $\Lambda<0$ solutions (2.14) is given by

$$e^{2u}\sim r^2,\quad e^{2v}\sim r^{-2m/(m+n)},\quad e^{2\kappa\sigma}\sim r^{m\sqrt{n}/\sqrt{m+n}}\ . \tag{2.20}$$

Thus none of the solutions is asymptotically flat. The general solution (2.14) holds for $C\neq 0$. If one sets $C=0$ while integrating the differential equations one finds a solution which corresponds to the $C=0$ limit of (2.18).

III. GLOBAL PROPERTIES OF SOLUTIONS

The analysis of the full five-dimensional phase space is greatly simplified by the fact that the only critical points at a finite distance from the origin are those for which $W=0$, $Z=0$, and $P=0$, as can be quickly seen from Eqs. (2.8). This is equivalent to saying that the only critical points are those for which all three constants Λ, $\tilde{\lambda}$, and $\bar{\lambda}$ are identically zero. Since $\Lambda=0$ these critical points are of course precisely those which have already been discussed in I. For each c_1 and for each X_0 such that

$$|X_0| \geq \left[\frac{(m-1)(m+n)}{m(m+n-1)}\right]^{1/2} |c_1| \tag{3.1}$$

we take Y_0 to be given by the solution of the quadratic equation

$$n(m-1)Y_0^2 - 2mnX_0Y_0 + m(n-1)X_0^2 + (m+n)c_1^2 = 0 \; . \tag{3.2}$$

The critical points are then given by

$$X=X_0, \quad Y=Y_0, \quad Z=0, \quad W=0 \; ,$$
$$V=\frac{1}{m+n-1}(mX_0+nY_0+c_1) \; . \tag{3.3}$$

We have of course chosen the parametrization so that c_1 coincides with the integration constant defined by (2.9). The surface defined by (3.1)–(3.3) is a cone within the $W=0$, $Z=0$ subspace.

The pattern of trajectories in the $W=0$, $Z=0$ subspace is extremely simple because this is just the special case (i) discussed above: the trajectories are the straight lines [10]

$$Y=\frac{m}{m-1}(X+k) \; ,$$
$$V=\frac{mX}{m-1}+\frac{1}{m+n-1}\left[\frac{mnk}{m-1}+c_1\right] \; , \tag{3.4}$$

where k is an arbitrary constant. In Fig. 1 we sketch the projection of these trajectories onto the X,Y plane. The bold lines correspond to sections through the cone of critical points defined by (3.2) and (3.3). As discussed in I, the critical points correspond to the limit $r \to 0$ except for the special trajectories with

$$c_1 = mk \neq 0 \; . \tag{3.5}$$

In these cases one critical point corresponds to the horizon of the trivial Schwarzschild solution, and c_1 is in fact proportional to the Schwarzschild mass. If $c_1<0$ the critical point corresponding to the horizon occurs in the first quadrant, as in Fig. 1(b), while the continuation of the line $Y=(mX+c_1)/(m-1)$ to the third quadrant gives the negative-mass Schwarzschild solution of the same absolute mass. If $c_1>0$ then the roles are reversed and the critical point corresponding to the horizon is in the third quadrant.

The eigenvalue equation for small perturbations about the critical points yields three positive (negative) eigenvalues for trajectories in the first (third) quadrant, as well as two zero eigenvalues. The zero eigenvalues correspond to degeneracies in the W and Z directions which arise because at the linearized level the relations (3.4) hold approximately. Thus each critical point in the first quadrant is a repellor for a three-dimensional set of trajectories, while each point in the third quadrant is similarly an attractor for a three-dimensional set. Thus most trajectories end at $r=0$. However, for each $c_1 \neq 0$ there is a three-dimensional set of trajectories which approach the same point H [cf. Fig. 1(b)], corresponding to a regular horizon. Hence there is a four-dimensional set of tra-

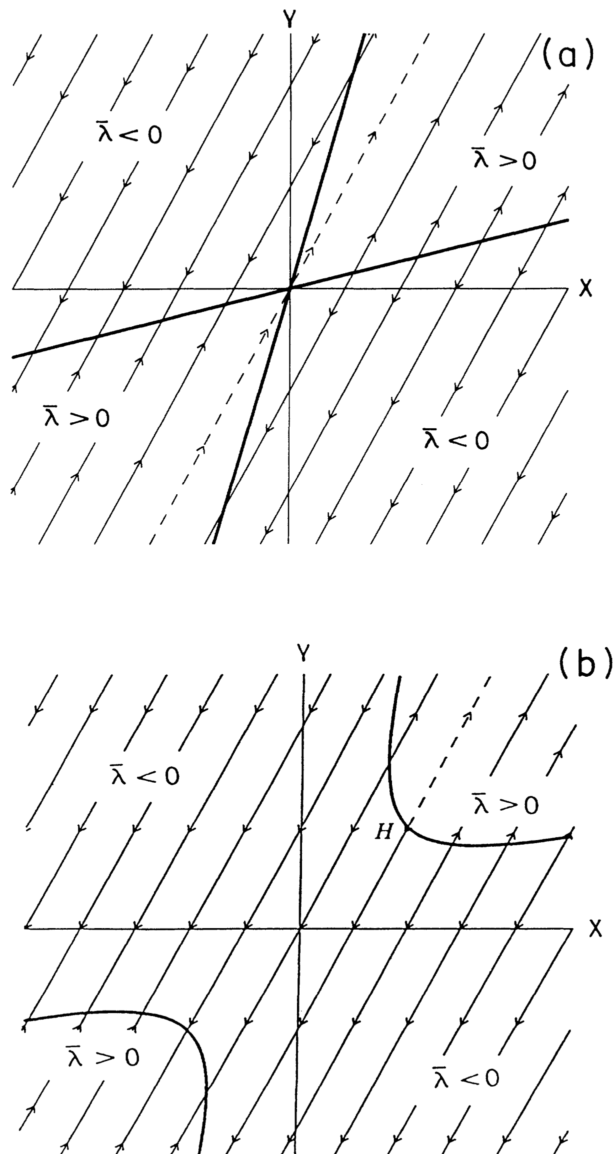


FIG. 1. The projection of the $\Lambda=0$, $\tilde{\lambda}=0$ phase space onto the X,Y plane: (a) $c_1=0$; (b) $c_1 \neq 0$. The broken line corresponds to flat space in (a) and the Schwarzschild solution in (b). The bold lines denote sections of the $\Lambda=0$, $\tilde{\lambda}=0$, $\bar{\lambda}=0$ cone.

jectories with a regular horizon in the full phase space.

To determine the global properties of the solutions it only remains to investigate the critical points at infinity.

A. Critical points at infinity in the $\Lambda=0$ subspace

The critical points at infinity which lie in the $\Lambda=0$ subspace [with $W=0$ and V determined by (2.9)], were of course obtained in I. There we found it useful to define spherical polar coordinates

$$X=\rho_1\sin\theta_1\cos\phi_1\ ,\quad Y=\rho_1\sin\theta_1\sin\phi_1\ ,\quad Z=\rho_1\cos\theta_1\ , \tag{3.6}$$

and to then bring the surface at infinity to a finite distance from the origin by the transformation

$$\rho_1=\bar{\rho}_1(1-\bar{\rho}_1)^{-1},\quad 0\le\bar{\rho}_1\le 1\ . \tag{3.7}$$

If we define a coordinate τ by $d\tau=\rho_1 d\xi=\bar{\rho}_1(1-\bar{\rho}_1)^{-1}d\xi$, then on the sphere at infinity, i.e., at $\bar{\rho}_1=1$, $d\bar{\rho}_1/d\tau=0$ identically while [24]

$$\frac{d\theta_1}{d\tau}=\cos\theta_1\left\{(n-1)\epsilon_1\cos^2\theta_1\left[\frac{n}{m}\cos\phi_1-\sin\phi_1\right]-\sin^2\theta_1\left[\sin\phi_1+\bar{P}_1\left[\frac{m-1}{m}\cos\phi_1+\sin\phi_1\right]\right]\right\}, \tag{3.8a}$$

$$\frac{d\phi_1}{d\tau}=\frac{1}{\sin\theta_1}\left[-(n-1)\epsilon_1\cos^2\theta_1\left[\cos\phi_1+\frac{n}{m}\sin\phi_1\right]+\sin^2\theta_1\bar{P}_1\left[\frac{m-1}{m}\sin\phi_1-\cos\phi_1\right]\right], \tag{3.8b}$$

where

$$\bar{P}_1=\frac{1}{m+n-1}[m(n-1)\cos^2\phi_1-2mn\cos\phi_1\sin\phi_1+n(m-1)\sin^2\phi_1]\ . \tag{3.8c}$$

Four sets of critical points are found.

(i) Four critical points, which we will denote L_1 to L_4, are located at

$$\theta_1=\frac{\pi}{2}\ ,\quad \phi_1=\arctan\left[\frac{mn\pm[mn(m+n-1)]^{1/2}}{n(m-1)}\right] \tag{3.9a}$$

or

$$X=\pm\infty\ ,\quad Y=\left[\frac{mn\pm[mn(m+n-1)]^{1/2}}{n(m-1)}\right]X,\quad Z=0\ . \tag{3.9b}$$

These are just the end points of the one-parameter family of critical points corresponding to the $\bar{\lambda}=0$ and $\tilde{\lambda}=0$.

(ii) Two critical points, which we will denote M_1 and M_2, are located at

$$\theta_1=\frac{\pi}{2},\quad \phi_1=\arctan\left[\frac{m}{m-1}\right]\ , \tag{3.10a}$$

or

$$X=\pm\infty,\quad Y=\left[\frac{m}{m-1}\right]X,\quad Z=0\ . \tag{3.10b}$$

These of course correspond to the end points of the trajectories which lie inside the cone in the $\tilde{\lambda}=0$ subspace (cf. Fig. 1). They lie in the portion of the phase space with $\bar{\lambda}>0$.

(iii) If $\tilde{\lambda}>0$ then there are four critical points, which we will denote N_1 to N_4, which are located at

$$\theta_1=\arctan[\pm(2n^2-2n+1)^{1/2}]\ ,\quad \phi_1=\arctan\left[\frac{n-1}{n}\right] \tag{3.11a}$$

or

$$X=\pm\infty,\quad Y=\left[\frac{n-1}{n}\right]X,\quad Z=\pm\frac{X}{n}\ . \tag{3.11b}$$

These points lie in the $\bar{\lambda}=0$ subspace.

(iv) If $\bar{\lambda}>0$ then there are four critical points, which we will denote P_1 to P_4, which are located at

$$\theta_1=\arctan\{\pm[2(n-1)(m+n-1)]^{1/2}\}\ ,\quad \phi_1=\frac{\pi}{4},\frac{5\pi}{4}\ , \tag{3.12a}$$

or

$$X=\pm\infty,\quad Y=X\ ,\quad Z=\frac{\pm X}{[(n-1)(m+n-1)]^{1/2}}\ . \tag{3.12b}$$

These points lie in the portion of the phase space with $\tilde{\lambda}>0$.

The pattern of trajectories on the $\Lambda=0$ sphere at infinity are sketched in Fig. 2. Although these trajectories are unphysical it is helpful to sketch them since by continuity arguments they will determine the behavior of the physical integral curves which lie within the sphere at infinity but near its surface. We will delay a discussion of the nature of the critical points until we have listed all the other ones.

B. Critical points at infinity in the $\bar{\lambda}=0$ subspace

The critical points at infinity which lie in the $\bar{\lambda}=0$ subspace [with $Z=0$ and Y determined by (2.10)], can be studied in a similar fashion to those of the $\Lambda=0$ subspace. In particular, if we now define spherical polar coordinates ρ_2, θ_2, ϕ_2 by

$$V=\rho_2\sin\theta_2\cos\phi_2 ,$$
$$W=\rho_2\cos\theta_2 , \tag{3.13}$$
$$X=\rho_2\sin\theta_2\sin\phi_2 ,$$

and similarly make a transformation

$$\rho_2=\bar{\rho}_2(1-\bar{\rho}_2)^{-1}, \quad 0\leq\bar{\rho}_2\leq 1 , \tag{3.14}$$

we find that, at $\bar{\rho}_2=1$,

$$\frac{d\theta_2}{d\tau}=-\cos\theta_2\left\{\epsilon_2\cos^2\theta_2\left[\frac{\cos\phi_2}{m+n}+\frac{\sin\phi_2}{m}\right]+\sin^2\theta_2\left[\cos\phi_2+\bar{P}_2\left[\cos\phi_2+\frac{m-1}{m}\sin\phi_2\right]\right]\right\} , \tag{3.15a}$$

$$\frac{d\phi_2}{d\tau}=\frac{1}{\sin\theta_2}\left[\epsilon_2\cos^2\theta_2\left[\frac{\sin\phi_2}{m+n}-\frac{\cos\phi_2}{m}\right]+\sin^2\theta_2\bar{P}_2\left[\sin\phi_2-\frac{m-1}{m}\cos\phi_2\right]\right] , \tag{3.15b}$$

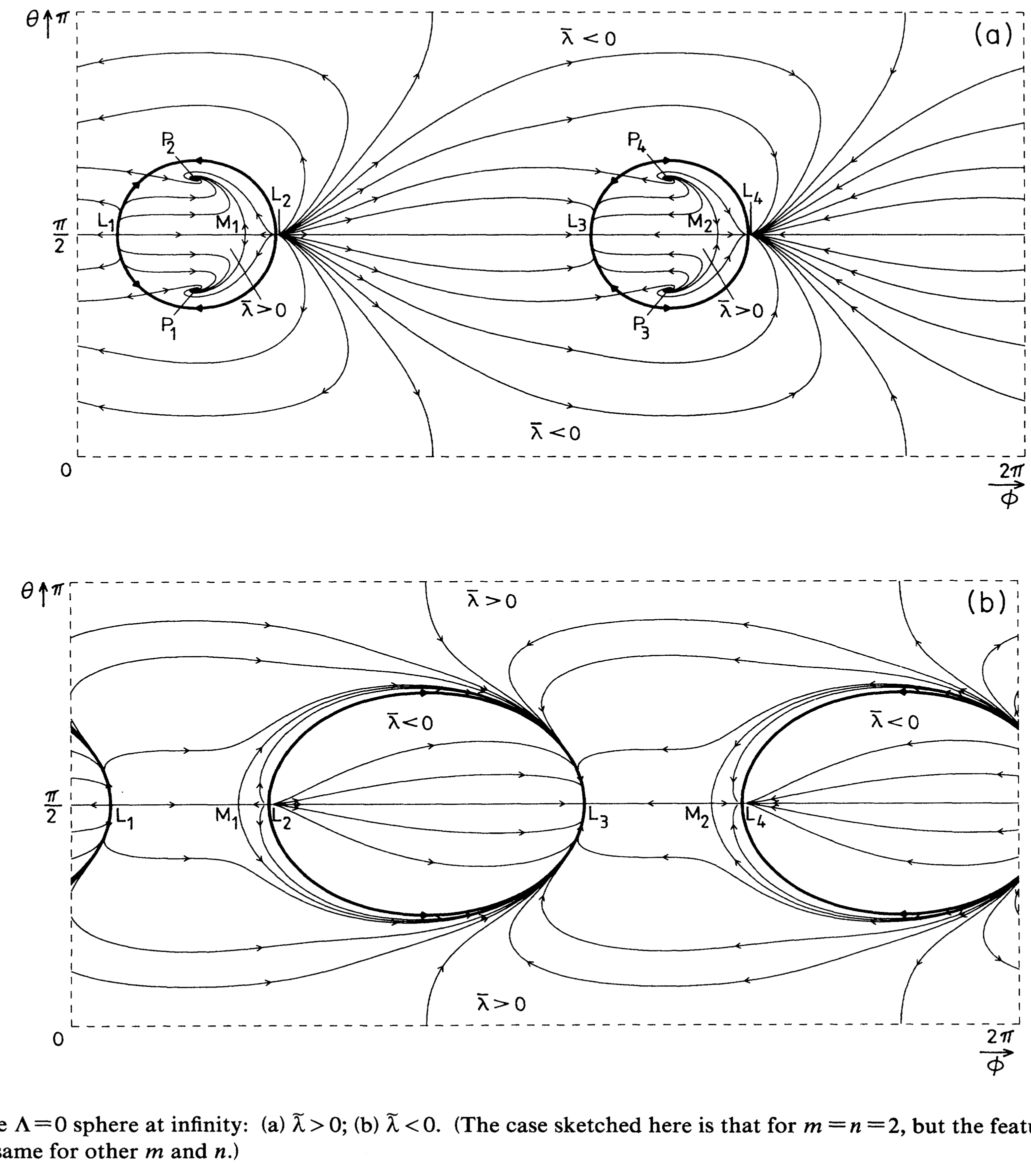


FIG. 2. The $\Lambda=0$ sphere at infinity: (a) $\bar{\lambda}>0$; (b) $\bar{\lambda}<0$. (The case sketched here is that for $m=n=2$, but the features of the phase space are the same for other m and n.)

where $d\tau=\rho_2 d\xi=\bar{\rho}_2(1-\bar{\rho}_2)^{-1}d\xi$ and

$$\bar{P}_2=\frac{1}{n+1}[(m-1)(m+n)\cos^2\phi_2 -2m(m+n)\cos\phi_2\sin\phi_2 +m(m+n+1)\sin^2\phi_2] . \quad (3.15c)$$

There are once again four distinct sets of critical points.

(i) Four critical points, which we will denote L_5 to L_8, are located at

$$\theta_2=\frac{\pi}{2} , \quad \phi_2=\arctan\left[\frac{m(m+n)\pm[m(m+n)(n+1)]^{1/2}}{m(m+n+1)}\right] \quad (3.16a)$$

or

$$X=\pm\infty , \quad V=\left[\frac{m(m+n+1)}{m(m+n)\pm[m(m+n)(n+1)]^{1/2}}\right]X , \quad W=0 . \quad (3.16b)$$

Similarly to the points L_{1-4} these points are the end points of the one-parameter family of critical points corresponding to $\Lambda=0$ and $\tilde{\lambda}=0$. In the full five-dimensional phase space the points L_{1-4} and L_{5-8} will be members of a one-parameter set of critical points which correspond to the intersection of the $\Lambda=0$, $\tilde{\lambda}=0$, $\bar{\lambda}=0$ surface with the sphere at infinity.

(ii) We of course reobtain the critical points M_1 and M_2 which lie in the $\tilde{\lambda}=0$, $\Lambda=0$ subspace. In terms of θ_2 and ϕ_2 they are located at

$$\theta_2=\frac{\pi}{2}, \quad \phi_2=\arctan\left[\frac{m-1}{m}\right] , \quad (3.17a)$$

or

$$X=\pm\infty, \quad V=\left[\frac{m}{m-1}\right]X, \quad W=0 . \quad (3.17b)$$

(iii) If $\Lambda<0$ then there are four critical points, which we will denote Q_1 to Q_4, which are located at

$$\theta_2=\arctan\left[\pm\left[\frac{2(n+1)}{m+n}\right]^{1/2}\right], \quad \phi_2=\frac{\pi}{4},\frac{5\pi}{4} , \quad (3.18a)$$

or

$$X=\pm\infty, \quad V=X, \quad W=\pm X\left[\frac{m+n}{n+1}\right]^{1/2} . \quad (3.18b)$$

These points lie in the portion of the phase space with $\bar{\lambda}<0$.

(iv) If $\Lambda<0$ then there are four critical points, which we will denote R_1 to R_4, which are located at

$$\theta_2=\arctan\left[\pm\left[\frac{m+n}{m+n+1}+\frac{m+n+1}{m+n}\right]^{1/2}\right] , \quad \phi_2=\arctan\left[\frac{m+n}{m+n+1}\right] , \quad (3.19a)$$

or

$$X=\pm\infty , \quad V=\left[\frac{m+n+1}{m+n}\right]X , \quad W=\pm X\left[\frac{m+n+1}{m+n}\right]^{1/2} . \quad (3.19b)$$

These points lie in the $\bar{\lambda}=0$ subspace.

In Fig. 3 we sketch the pattern of trajectories on the $\tilde{\lambda}=0$ sphere at infinity.

C. Other critical points at infinity

In the full five-dimensional phase space we can of course define five-dimensional spherical polar coordinates similarly to (3.7) and (3.13), and proceed in a similar fashion. However, the expressions one obtains are very cumbersome, and since it is impossible to sketch the four-sphere at infinity the spherical polars method provides no advantages over the more widely used Poincaré-sphere mapping technique. We will therefore use the latter approach. We set

$$X=\frac{\pm1}{\varepsilon}, \quad Y=\frac{\pm y}{\varepsilon} , \quad Z=\frac{\pm z}{\varepsilon} , \quad V=\frac{\pm v}{\varepsilon}, \quad W=\frac{\pm w}{\varepsilon} , \quad (3.20)$$

and examine the limit $\varepsilon\to0$. The differential equations become

$$\pm\frac{d\varepsilon}{d\tau}=-\varepsilon\left[\frac{-\epsilon_2}{m}w^2+\frac{n}{m}(n-1)\epsilon_1 z^2-(m-1)\frac{p}{m}\right] , \quad (3.21a)$$

$$\pm\frac{dy}{d\tau}=\frac{\epsilon_2}{m}w^2 y-(n-1)\epsilon_1 z^2\left[1+\frac{n}{m}y\right] +p\left[(m-1)\frac{y}{m}-1\right] , \quad (3.21b)$$

$$\pm\frac{dz}{d\tau}=\frac{\epsilon_2}{m}w^2 z+z\left[y-\frac{n}{m}(n-1)\epsilon_1 z^2\right]+(m-1)\frac{pz}{m} , \quad (3.21c)$$

$$\pm\frac{dv}{d\tau}=\epsilon_2 w^2\left[\frac{v}{m}-\frac{1}{m+n}\right]-\frac{n}{m}(n-1)\epsilon_1 z^2 v +p\left[(m-1)\frac{v}{m}-1\right] , \quad (3.21d)$$

$$\pm\frac{dw}{d\tau}=w\left[v+\frac{\epsilon_2}{m}w^2\right]-\frac{n}{m}(n-1)\epsilon_1 z^2 w+(m-1)\frac{pw}{m} , \quad (3.21e)$$

where

$$p \equiv m(m+1)+2mny+n(n+1)y^2 -2(m+n)(m+ny)v+(m+n)(m+n-1)v^2 , \quad (3.21f)$$

$d\tau=\varepsilon^{-1}d\xi$, and the overall plus or minus sign refers to the choice of the same sign in (3.20).

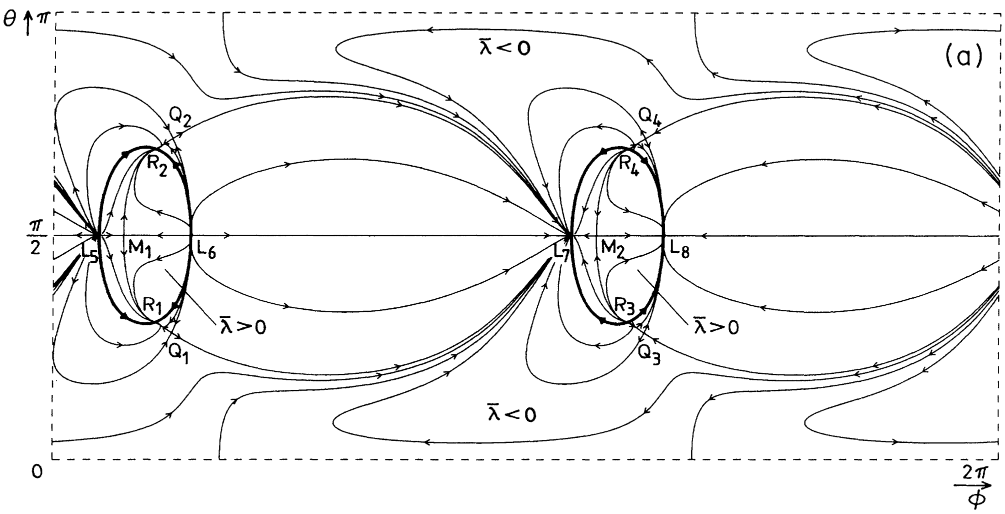
In addition to the critical points given in Secs. III A and III B we find three more sets of points.

(i) The first set is just the extension of the critical points L_{1-8} to the one-parameter family of critical points which coincide with the intersection of the $\Lambda=0$, $\tilde{\lambda}=0$, $\bar{\lambda}=0$ surface and the sphere at infinity. We shall denote the whole set by $L(y)$, where

$$\frac{mn-[mn(m+n-1)]^{1/2}}{n(m-1)} \le y \le \frac{mn+[mn(m+n-1)]^{1/2}}{n(m-1)} . \quad (3.22)$$

The points are located at

$$X=\pm\infty, \quad Y=yX, \quad Z=0 ,$$
$$V=\left[\frac{(m+n)^{1/2}(m+ny)\pm[2mny-n(m-1)y^2-m(n-1)]^{1/2}}{(m+n)^{1/2}(m+n-1)}\right]X, \quad W=0 . \quad (3.23)$$



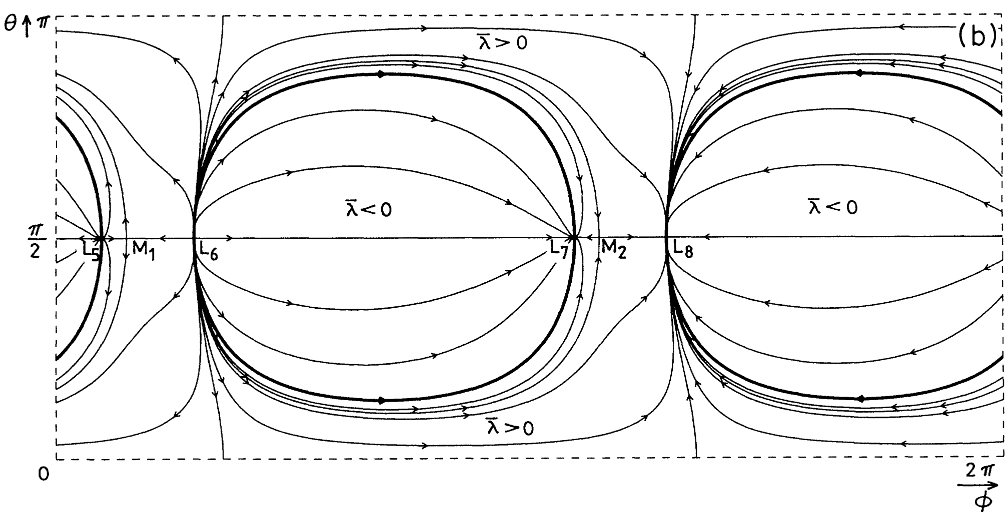


FIG. 3. The $\tilde{\lambda}=0$ sphere at infinity: (a) $\Lambda<0$; (b) $\Lambda>0$. (The case sketched here is that for $m=n=2$, but the features of the phase space are the same for other m and n.)

(ii) If $\Lambda<0$ and $\tilde{\lambda}<0$ then there are eight critical points, which we will denote S_1 to S_8, located at

$$
\begin{aligned}
&X=\pm\infty, \quad Y=X ,\\
&Z=\frac{\pm X}{(n-1)^{1/2}} ,\\
&V=X, \quad W=\pm X(m+n)^{1/2} .
\end{aligned}
\tag{3.24}
$$

These are points for which $P=0$ (but $\bar{\lambda}\neq 0$). They occur in the portion of the phase space for which $\bar{\lambda}<0$.

(iii) If $\Lambda<0$ and $\tilde{\lambda}<0$ then there are eight critical points, which we will denote T_1 to T_8, located at

$$
\begin{aligned}
&X=\pm\infty, \quad Y=\left[\frac{m+1}{m}\right]X ,\\
&Z=\frac{\pm X}{m}\left[\frac{m+1}{n-1}\right]^{1/2}, \quad V=\left[\frac{m+1}{m}\right]X ,\\
&W=\frac{\pm X}{m}[(m+1)(m+n)]^{1/2} .
\end{aligned}
\tag{3.25}
$$

These points lie in the $\bar{\lambda}=0$ subspace.

To complete the analysis we must also check the possibility of the existence of critical points at infinity with $X=0$. This may be done, for example, by setting

$$
\begin{aligned}
&W=\frac{\pm 1}{\varepsilon}, \quad X=\frac{\pm x}{\varepsilon} ,\\
&Y=\frac{\pm y}{\varepsilon} ,\\
&Z=\frac{\pm z}{\varepsilon}, \quad V=\frac{\pm v}{\varepsilon} ,
\end{aligned}
\tag{3.26}
$$

and investigating the resulting differential equations as $\varepsilon\to 0$, similarly to above. (Since the $W=0$ case corresponds to the spherical polars analysis of Sec. III A, we know that there are no critical points at infinity with both $W=0$ and $X=0$.) The analysis reveals that there are no new critical points in addition to those already listed.

D. Asymptotic form of solutions

We find that as trajectories approach the various critical points at infinity $\xi\to\xi_0=\text{const}$, except for the points $L(y)$ when $\xi\to-\infty$ for $X\to+\infty$, and $\xi\to+\infty$ for $X\to-\infty$. The points $L(y)$, which lie on the edge of the $\Lambda=0$, $\tilde{\lambda}=0$, $\bar{\lambda}=0$ surface, of course correspond to $r\to 0$ (the possibility $r\to\text{const}$ being excluded at infinity). These points have exactly the same properties as the corresponding points at a finite distance from the origin: the points with $X>0$ ($X<0$) repel (attract) a three-dimensional set of trajectories. However, the trajectories in question here are the entirely unphysical ones which are confined to the surface at infinity.

In the remaining cases we find that the limit $\xi\to\xi_0=\text{const}$ corresponds to $r\to\infty$ for all points except S_{1-8}. For these points $r\to\text{const}$, $\kappa\sigma\sim\text{const}$, and although the metric functions diverge the curvature invariants are finite, indicating the presence of an horizon.

In Table I we display the asymptotic form of the metric functions (2.15), and of the scalar field, for integral curves which approach each of the remaining six sets of points [25]. In order to use these properties to classify the various solutions we must first of all determine the nature of the various critical points. It is straightforward to evaluate the eigenvalue spectrum for small perturbations at infinity if the coordinates (3.20) are used. In Table II we display the eigenvalues for the points with $X>0$. For the corresponding points with $X<0$ the sign of the eigenvalues is simply reversed. Thus the dimension of the set $\mathcal{A}$ of trajectories which are attracted, as given in the third column, becomes the dimension of the set of trajectories which are repelled for the corresponding points with $X<0$.

With regard to the points $M_{1,2}$, N_{1-4}, and P_{1-4} the picture of the phase space that emerges is unchanged from that of I. The dimension of $\mathcal{A}$ is greater by one than the dimension of the corresponding set of trajectories in I. However, this is merely due to the effect of the "redundant" coordinate V which we were able to integrate out in I. The important point to note is that none of the trajectories with nonzero Λ are attracted to any of the points $M_{1,2}$, N_{1-4}, and P_{1-4}. In particular, the points $M_{1,2}$, which have realistic asymptotics, are not reached.

With regard to the trajectories for nonzero Λ, the first point to note is that if $\Lambda>0$ then there are no solutions

TABLE I. Asymptotic form of solutions for trajectories approaching critical points at infinity which correspond to the limit $r\to\infty$.

	Values of $\Lambda,\tilde{\lambda},\bar{\lambda}$	e^{2u}	e^{2v}	$e^{2\kappa\sigma}$
$M_{1,2}$	$\Lambda=0,\ \tilde{\lambda}=0,\ \bar{\lambda}>0$	Const	Const	Const
N_{1-4}	$\Lambda=0,\ \tilde{\lambda}>0,\ \bar{\lambda}=0$	r^2	$r^{2m/n}$	$r^{m\sqrt{m+n}/\sqrt{n}}$
P_{1-4}	$\Lambda=0,\ \tilde{\lambda}>0,\ \bar{\lambda}>0$	$r^{2n/(m+n)}$	Const	$r^{m\sqrt{n}/\sqrt{m+n}}$
Q_{1-4}	$\Lambda<0,\ \tilde{\lambda}=0,\ \bar{\lambda}<0$	$r^{2(m+n)/n}$	Const	$r^{m\sqrt{m+n}/\sqrt{n}}$
R_{1-4}	$\Lambda<0,\ \tilde{\lambda}=0,\ \bar{\lambda}=0$	r^2	$r^{-2m/(m+n)}$	$r^{m\sqrt{n}/\sqrt{m+n}}$
T_{1-8}	$\Lambda<0,\ \tilde{\lambda}<0,\ \bar{\lambda}=0$	r^2	r^{-2}	Const

for which an asymptotic region exists. The trajectories simply leave the $\Lambda=0$, $\tilde{\lambda}=0$, $\bar{\lambda}=0$ surface in the $X>0$ region of the phase space at a point corresponding to a singularity (with $r=0$), or exceptionally to a horizon, and return to a similar point on the $\Lambda=0$, $\tilde{\lambda}=0$, $\bar{\lambda}=0$ surface in the $X>0$ portion of the phase space. This is identical to the case of the $\tilde{\lambda}<0$ and $\bar{\lambda}<0$ trajectories with $\Lambda=0$, as was observed in I.

If $\Lambda<0$ then the points R_{1-4} are reached by all trajectories with both $\bar{\lambda}\geq 0$ and $\tilde{\lambda}\geq 0$, and by some trajectories in each case that at least one of the constants $\bar{\lambda}$ or $\tilde{\lambda}$ is negative. In particular, all possibly interesting Kaluza-Klein black-hole trajectories, with $\bar{\lambda}>0$ and $\tilde{\lambda}\geq 0$, end at these points. Most of these trajectories begin at a naked singularity, but a four-dimensional subset begins on a regular horizon, as discussed above. We see from Table I, however, that none of the solutions is asymptotically flat and that the scalar field diverges as $e^{2\kappa\sigma}\sim r^{m\sqrt{n}/\sqrt{m+n}}$. Thus no new regular black-hole solutions are obtained.

The trajectories which reach the points Q_{1-4} and T_{1-8} do not provide any new solutions of immediate physical interest since the spacetimes have $\bar{\lambda}<0$ in the first case, and $\tilde{\lambda}<0$ in the second case. It would appear that the separatrices of trajectories which reach Q_{1-4} divide $\bar{\lambda}<0$ solutions which reach the points R_{1-4} from other $\bar{\lambda}<0$ trajectories which have no asymptotic region, similarly to the $\Lambda>0$ case. By symmetry the trajectories which reach T_{1-8} similarly divide the $\tilde{\lambda}<0$ solutions into two such classes also. Given the large dimensionality of the phase space, a complete characterization of all solutions with $\bar{\lambda}<0$ or $\tilde{\lambda}<0$ is not immediately obvious though, particularly since the nature of the separatrices of trajectories which end on the points S_{1-8} seems obscure. Since none of these solutions is physical, however, they will not concern us here.

IV. CONCLUSION

As expected from the no-hair theorems a model containing just Einstein gravity plus a cosmological constant in higher dimensions leads to no nontrivial solutions in four dimensions which represent an asymptotically flat black hole with a regular horizon, if metrics of the form (1.1) are considered. If $\Lambda<0$ solutions with a regular horizon do exist, but they are not asymptotically flat and the scalar field corresponding to the radius of the extra dimensions diverges at spatial infinity (cf. points R_{1-4} in Table I). Most solutions also have a naked singularity, however. The solutions of Kim and Cho [14] are

TABLE II. Nature of critical points at infinity. In the second column the eigenvalues for small perturbations which are degenerate have the degeneracy listed in parentheses. In the third column the dimension of $\mathcal{A}$, $d_{\mathcal{A}}$, is listed. The values of y and v listed are defined by (3.22) and $V=vX$ in (3.23).

	Eigenvalues (with degeneracies)	$d_{\mathcal{A}}$	Nature of the set $\mathcal{A}$ of trajectories attracted
$L(y)$	$0,(2);2;y;v$	0	
$M_{1,2}$	$-1,(3);\frac{1}{m-1},(2)$	3	All $\Lambda=0$, $\tilde{\lambda}=0$, $\bar{\lambda}>0$ trajectories
N_{1-4}	$\frac{-(n-1)}{n},(3);\frac{1}{n};\frac{2}{n}$	3	All $\Lambda=0$, $\tilde{\lambda}>0$, $\bar{\lambda}=0$ trajectories
P_{1-4}	$\frac{-1}{2}\left[1\pm\left[\frac{m+n-9}{m+n-1}\right]^{1/2}\right]$; $-1,(2);\frac{1}{m+n-1}$	4	All $\Lambda=0$, $\tilde{\lambda}>0$, $\bar{\lambda}>0$ trajectories
Q_{1-4}	$-1,(2);\frac{-1}{n+1}$; $\frac{1}{2}\left[-1\pm\left[\frac{n+9}{n+1}\right]^{1/2}\right]$	4	Separatrix of trajectories in portion of phase space with $\Lambda<0$, $\bar{\lambda}<0\}$
R_{1-4}	$\frac{-(m+n+1)}{m+n},(3);\frac{-2}{m+n};\frac{-1}{m+n}$	5	All $\Lambda<0$, $\bar{\lambda}\geq 0$ trajectories; 5-d subset of $\Lambda<0$, $\bar{\lambda}<0$ trajectories
S_{1-8}	$-2,(2);-1;1,(2)$	3	Separatrix of trajectories in portion of phase space with $\Lambda<0$, $\bar{\lambda}<0$
T_{1-8}	$\frac{-(m+1)}{m},(2);\frac{-2}{m}$; $\frac{1}{2m}[-(m+1)\pm\sqrt{(m+1)(m+9)}]$	4	Separatrix of trajectories in portion of phase space with $\Lambda<0$, $\tilde{\lambda}<0$

discounted because they necessarily lead to a choice of $\bar{\lambda} \leq 0$ (cf. the Appendix).

This result may seem superficially discouraging for higher-dimensional theories. After all one would expect uncharged black holes to exist in the Universe as a result of stellar collapse, and without extra charges there is no obvious way of including a scalar field corresponding to the compactified dimensions, while at the same time circumventing the no-hair theorems. However, one could optimistically take the view that the problem is just an artifact of the extremely simple nature of the model we have considered here. The inclusion of dilaton fields, or of additional powers of the Riemann curvature in the higher-dimensional action, or other even more sophisticated schemes, may lead to more interesting results.

APPENDIX

We will present here the corrected version of the integration of the field equations Kim and Cho [14] attempted. We will use their coordinates, which are similar to (2.15) for the physical part of the higher-dimensional metric:

$$ds_D^2 = \exp\left[\frac{-4n\kappa\sigma}{m\sqrt{n(m+n)}}\right](-B\,dt^2 + A\,dr^2 + r^2\bar{g}_{ij}d\bar{x}^i d\bar{x}^j) + \exp\left[\frac{4\kappa\sigma}{\sqrt{n(m+n)}}\right]\tilde{g}_{ab}d\tilde{y}^a d\tilde{y}^b\,, \quad (\text{A1})$$

where $A = A(r)$, $B = B(r)$, and $\sigma = \sigma(r)$, but we retain our definitions of σ and Λ, which differ from those of Kim and Cho by a factor of 2. Rather than restricting $\bar{g}_{ij}$ we shall continue to take it to be an arbitrary m-dimensional Einstein space metric [cf. (1.1b)]. The internal space is Ricci flat ($\tilde{\lambda} = 0$).

The Einstein equations with a Λ term in D dimensions for the metric (A1) are

$$\frac{-\ddot{B}}{2A} + \frac{\dot{B}}{4A}\left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right] - \frac{m\dot{B}}{2rA} = \frac{2\Lambda B}{m}\exp\left[\frac{-4\alpha\kappa\sigma}{m}\right], \quad (\text{A2a})$$

$$\frac{\ddot{B}}{2B} - \frac{\dot{B}}{4B}\left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right] - \frac{m\dot{A}}{2rA} = \frac{-2\Lambda A}{m}\exp\left[\frac{-4\alpha\kappa\sigma}{m}\right] - \frac{4\kappa^2\dot{\sigma}^2}{m}, \quad (\text{A2b})$$

$$\frac{(m-1)}{r^2}(1 - \bar{\lambda}A) + \frac{1}{2r}\left[\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right] = \frac{-2\Lambda A}{m}\exp\left[\frac{-4\alpha\kappa\sigma}{m}\right], \quad (\text{A2c})$$

$$\kappa\ddot{\sigma} + \frac{1}{2}\kappa\dot{\sigma}\left[\frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{2m}{r}\right] = -\alpha\Lambda A\exp\left[\frac{-4\alpha\kappa\sigma}{m}\right], \quad (\text{A2d})$$

where an overdot is equivalent to d/dr, and $\alpha = \sqrt{n}/\sqrt{m+n}$.

Following Kim and Cho [14] we adopt the ansatz

$$r = \beta e^{2\gamma\kappa\sigma}\,, \quad (\text{A3})$$

where β and γ are constants. By adding (A2b) and A/B times (A2a) one then obtains an equation which may be integrated to give

$$AB = r^{2\delta}\,, \quad (\text{A4})$$

where $\delta = 1/(m\gamma)^2$. (We have used the freedom to rescale t to absorb an unphysical constant.) If we now substitute (A4) into (A2c) we obtain

$$\frac{d}{dr}(Br^{m-1-\delta}) = r^{\delta+m-2}\left[(m-1)\bar{\lambda} - \frac{2\Lambda r^2}{m}\exp\left[\frac{-4\alpha\kappa\sigma}{m}\right]\right]. \quad (\text{A5})$$

This equation can be integrated, taking account of (A3), and we find

$$B = r^{2\delta}\left[\frac{(m-1)\bar{\lambda}}{\delta+m-1} - \frac{M}{r^{\delta+m-1}} - \frac{2\Lambda r^2}{m[\delta+m+1-2\alpha/(m\gamma)]}\exp\left[\frac{-4\alpha\kappa\sigma}{m}\right]\right], \quad (\text{A6})$$

where M is an arbitrary constant, with a similar expression for A on account of (A4). To check how the various constants are restricted by the constraint equation we must now substitute our solutions for A, B, and σ back into (A2b) or (A2d). After a little algebra (A2b) becomes

$$\delta(m-1)\bar{\lambda}r^{2\delta-1} - 2\left[\delta - \frac{\alpha}{2\gamma}\right]\frac{\Lambda}{m}\beta^{2\alpha/(m\gamma)}r^{2[\delta-\alpha/(m\gamma)]} = 0\,, \quad (\text{A7})$$

while (A2d) similarly becomes

$$\frac{1}{2\gamma}(m-1)\bar{\lambda}r^{2\delta-1} + \left[\alpha - \frac{1}{m\gamma}\right]\frac{\Lambda}{m}\beta^{2\alpha/(m\gamma)}r^{2[\delta-\alpha/(m\gamma)]} = 0\,, \quad (\text{A8})$$

If $m > 1$ there are two possibilities for satisfying each of these equations.

1. $\bar{\lambda}=0$, $\alpha=\dfrac{1}{m\gamma}$

Thus

$$\exp\left[\frac{2\sqrt{m+n}\,\kappa\sigma}{m\sqrt{n}}\right]=\frac{r}{\beta}\,, \tag{A9a}$$

and the $(m+2)$-dimensional metric is

$$\hat{g}_{\alpha\beta}d\hat{x}^{\alpha}d\hat{x}^{\beta}=-r^2\Delta\,dt^2+\frac{dr^2}{\Delta r^{2m/(m+n)}}+r^2\bar{g}_{ij}d\bar{x}^{i}d\bar{x}^{j}\,, \tag{A9b}$$

where

$$\Delta=\frac{-2(m+n)\Lambda}{m^2(m+n+1)}-\frac{M}{r^{m(m+n+1)/n}}\,. \tag{A9c}$$

We therefore simply recover the solution (2.18), up to some trivial rescalings. Alternatively, if instead of (2.17) and (2.19) we make the choices

$$A_1=\frac{mA_0^{m+1}A_2^{n}}{m+n},\quad \left[\frac{A_2}{A_0}\right]^{n/(m+n)}=\frac{m+n}{m}\,, \tag{A10}$$

when $c_0=c_2=(m+n)c_3=\frac{1}{2}|C|$ and

$$A_1=\frac{(m+n)\hat{\lambda}}{mA_0^{m+1}A_2^{n}},\quad \left[\frac{A_2}{A_0}\right]^{n/(m+n)}=\frac{m+n}{m}\,, \tag{A11}$$

when $c_0=c_2=(m+n)c_3=-\frac{1}{2}|C|$, then instead of (2.18a) and (2.18b) we have (A9b) and (A9c) with $M=C/(m+n-1)$.

2. $\alpha=m\gamma$, $\beta^2=(m-1)(m+n)\bar{\lambda}/(2\Lambda)$

Thus

$$\exp\left[\frac{2\sqrt{n}\,\kappa\sigma}{m\sqrt{m+n}}\right]=\left[(m-1)(m+n)\frac{\bar{\lambda}}{2\Lambda}\right]^{1/2}r \tag{A12a}$$

and the $(m+2)$-dimensional metric is

$$\hat{g}_{\alpha\beta}d\hat{x}^{\alpha}d\hat{x}^{\beta}=-r^{2(m+n)/n}\Delta\,dt^2+\frac{dr^2}{\Delta}+r^2\bar{g}_{ij}d\bar{x}^{i}d\bar{x}^{j}\,, \tag{A12b}$$

where

$$\Delta=r^{2(m+n)/n}\left[-\frac{n^2(m-1)\bar{\lambda}}{m^2(n+1)}-\frac{M}{r^{m(n+1)/n}}\right]\,. \tag{A12c}$$

We require that $\bar{\lambda}<0$ (and hence $\Lambda<0$) in order that the metric have the correct signature. Consequently this solution is just a special case of the solutions which approach the critical points Q_{1-4} at infinity [cf. (3.18)]. A solution also exists for $\bar{\lambda}>0$, $\Lambda>0$, and $0<r<\{-m^2(n+1)M/[n^2(m-1)\bar{\lambda}]\}^{n/[m(n+1)]}$, if M is taken to be negative, but this solution of course has no asymptotic region.

Kim and Cho required $\bar{\lambda}=1$ and $m=2$ to obtain black-hole-type solutions. However, this possibility is not admitted by the solutions (A9) and (A12) (if we also require that an asymptotic region be defined).

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[23] Strictly speaking the modulus signs in Secs. 3 and 4 of Ref. [10] should be removed and replaced by a similar understanding.

[24] We note that there is a spurious overall factor of $\sin^2\theta$ in the second of Eqs. (5.10) in Ref. [10] which should be removed, while the factor of $\tilde{\lambda}$ in Eqs. (5.10) should be replaced by ϵ.

[25] In terms of the coordinate $\tilde{r}$ [cf. (1.7)], the asymptotic behavior appropriate to the corresponding metric functions and scalar fields for the reduction to $(n+2)$ dimensions, viz., $e^{2\tilde{u}}=\bar{r}^{2m/n}e^{2\mu}$, $e^{2\tilde{v}}=\bar{r}^{2m/n}e^{2\nu}(d\bar{r}/d\tilde{r})^2$, and $e^{2\kappa\tilde{\sigma}}=\bar{r}^{\sqrt{m(m+n)}}$, is given by making the following exchanges in Table I: $r\rightarrow\tilde{r}$, $m\leftrightarrow n$, $M_{1,2}\leftrightarrow N_{1-4}$, and $Q_{1-4}\leftrightarrow T_{1-8}$.