Validity of the minisuperspace approximation: An example from interacting quantum field theory

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We examine the question of the validity of the minisuperspace approximation using the example of an interacting ($\lambda\Phi^4$) scalar field in a closed Robertson-Walker universe, where the scale factor and the homogeneous mode of the scalar field model the minisuperspace degrees of freedom. We explicitly compute the back reaction of the inhomogeneous modes on the minisuperspace sector using a coarse-grained effective action and show that the minisuperspace approximation is valid only when this back reaction is small.

I. INTRODUCTION

In the past few decades there has been considerable effort in the direction of building an as yet elusive quantum theory of gravity. A well-known approach is via the canonical framework in which one considers the three-geometry as a canonical coordinate and its time rate of change, the extrinsic curvature, as the conjugate momentum. On quantization, the three-geometry and its conjugate momentum are promoted to the status of operators obeying the usual canonical commutation relations and one implements the classical Hamiltonian and momentum constraints as operator equations acting on the "wave function of the Universe" $\Psi(h_{ij},\phi)$—a functional on superspace. Thus the problem of quantum cosmology manifests itself through the solution of the above equations with appropriate boundary conditions.

The quantum version of the Hamiltonian constraint, the Wheeler-DeWitt equation obeyed by Ψ, is an infinite-dimensional partial-differential equation on superspace. Apart from various difficult conceptual problems that arise regarding the interpretation of Ψ and the problem of identifying time among the various dynamical variables, solving the infinite-dimensional Wheeler-DeWitt equation in general poses a formidable technical problem. To make the problem more tractable one turns to minisuperspace quantization [1]. This refers to the technique of restricting the quantum theory of gravity to spacetimes possessing a given symmetry. In the most common examples, the assumed symmetry is spatial homogeneity, which has the advantage of reducing the infinite-dimensional superspace to a finite-dimensional minisuperspace, and hence giving a finite-dimensional Wheeler-DeWitt equation. Minisuperspace quantization was first introduced by DeWitt [2] as a method of quantizing the Friedmann universe, and was then extensively explored by Misner [1] and others [3] in the context of homogeneous anisotropic cosmologies. It was also applied to a few inhomogeneous examples [4].

With the recent revival of interest in quantum cosmology associated with new [5] proposals for boundary conditions for the wave function of the Universe, minisuperspace quantization has also received renewed attention [6]. Though the boundary conditions are formally stated in a general context, they have been implemented primarily in the context of minisuperspace models since these offer a situation where analytical calculations can be performed. However, we must assess what price we have to pay for this tremendous simplification. Since in the process of this transition we are truncating an infinite number of modes, we are ignoring nonlinear interactions of the modes among themselves and with the minisuperspace degrees of freedom. It is also well known that this truncation violates the uncertainty principle, since it implies setting the amplitudes and momenta of the inhomogeneous modes *simultaneously* to zero. Moreover, by making the transition from infinite to finite degrees of freedom we are unjustifiably circumventing the issue of divergences that are inevitably tied to a system with infinite degrees of freedom.

It would therefore be interesting to understand what influence the truncated degrees of freedom have on the minisuperspace degrees of freedom and find out under what conditions it is reasonable to consider an autonomous evolution of the minisuperspace wave function ignoring the truncated degrees of freedom. People working in the field of quantum gravity and quantum cosmology have been aware of this problem for a long time [1]. However, the first attempt to actually assess the validity of the minisuperspace approximation was made by Kuchař and Ryan [7,8]. In Ref. [7] they studied this question in the context of examples taken from ordinary quantum mechanics and field theory, and in [8] they considered the quantum cosmological example of a Taub universe embedded in a mixmaster universe. We will try to address the same question, i.e., under what conditions is it justified to neglect the influence of the truncated degrees of freedom, from a slightly different point of view.

The model we will consider is that of a self-interacting ($\lambda\Phi^4$) scalar field coupled to a closed Robertson-Walker background spacetime. The scalar field can be expanded in terms of harmonics $Q^n_{lm}(\mathbf{x})$ on S^3:

$$\Phi(\mathbf{x},t)=\sum_{nlm} f_{nlm}(t)Q^n_{lm}(\mathbf{x}) \tag{1.1}$$

where $n=1,2,3,\ldots$, $l=0,1,\ldots,n-1$, $m=-l,l+1,\ldots,l-1,l$. The coefficients f_{nlm} are time-dependent functions for homogeneous spaces. "Superspace" in this case consists of the scale factor a of the background geometry along with the infinite number of modes f_{nlm} of the scalar field. Reduction to minisuperspace in this case is realized by truncating all the $n>1$ modes of the scalar field, retaining only the lowest mode that is compatible with homogeneity. Notice that for the gravitational part, we have considered a minisuperspace truncation from the very outset; i.e., we have considered a homogeneous three-geometry.

At this point we would like to stress that the scalar field here should not be thought of as providing a matter source for the Robertson-Walker background metric, since in that case varying the action with respect to the scale factor will not give the full set of Einstein equations. In particular, the $G_{ij}=\kappa T_{ij}$ equations that constrain the energy-momentum tensor via $T_{ij}=0$ for $i\neq j$ will be missing. Instead, we would like to think of the scalar field as mimicking the inhomogeneous gravitational degrees of freedom in superspace. We are motivated to do this because linearized gravitational perturbations (gravitons) in the synchronous gauge can be shown to be equivalent to a pair of minimally coupled scalar fields [9]. Thus our "full superspace" degrees of freedom can be thought of as satisfying the Einstein equations in the sense of linearized perturbation theory on a RW background, with the inhomogeneous modes of the scalar field acting as single polarization gravitons. We deal with the scalar field rather than the gravitational degrees of freedom due to the simplicity it affords and because it is easier to include interaction. The extension to the case of gravitons should not be too difficult once we have understood this case because of their formal similarity with scalar field mentioned above. An interacting scalar field was chosen in order to be able to see how nonlinearity and mode coupling affect the back reaction of the higher modes on the lower modes.

The paper is organized as follows. In Sec. II we derive an effective Wheeler-DeWitt equation for the wave function of the minisuperspace sector, and then by assuming a WKB form for the wave function we obtain an effective Hamilton-Jacobi equation for minisuperspace from it. The minisuperspace dynamics is seen to be modified by a term arising from the back reaction of the higher modes on the minisuperspace sector given by the vacuum expectation value of the Hamiltonian for the higher modes. Section III is devoted to the explicit calculation of the back-reaction term using a coarse-grained in-in effective action. The minisuperspace approximation is seen to be justified only when this back reaction is negligible compared to the minisuperspace potential. Finally, in Sec. IV we summarize the conclusions and discuss further implications.

II. EFFECTIVE WHEELER-DeWITT EQUATION FOR MINISUPERSPACE

Let us start with the classical action of the gravitational and matter fields. Our system is a self-interacting scalar field coupled to a closed Robertson-Walker background. The metric is given by

$$ds^2=l_P^2(dt^2-a^2d\Omega^2)$$
$$=l_P^2a^2(d\eta^2-d\Omega^2)\ , \tag{2.1}$$

where $d\Omega^2$ represents the line element on S^3, $l_P^2=2/3\pi m_P^2$ is included in the metric for simplification of computations, and $\eta=\int dt/a$ is the conformal time. The gravitational and matter actions are given by

$$S_g=\tfrac{1}{2}\int d\eta\, a^2\left[1-\frac{a'^2}{a^2}\right] \tag{2.2}$$

$$S_m=-\tfrac{1}{2}\int\sqrt{-g}\;d^4x\left[\Phi\Box\Phi+\tfrac{1}{2}m^2\Phi^2+\frac{\lambda}{4!}\Phi^4+\tfrac{1}{2}\xi R\,\Phi^2\right]\ , \tag{2.3}$$

where $\Box$ is the Laplace-Beltrami operator on the metric $g_{\mu\nu}$, m is the mass of the scalar field, $\xi=0$ and $\frac{1}{6}$ correspond to the minimal and conformal coupling respectively. Expanding Φ in scalar spherical harmonics on S^3,

$$\Phi=\frac{\Phi_0(t)}{(2\pi^2)^{1/2}}+\sum_{nlm}f_{nlm}Q^n_{lm}(x)\ , \tag{2.4}$$

where $Q^n_{lm}(x)$ are scalar spherical harmonics on S^3, $n=2,3,\ldots,\infty$, $l=0,1,\ldots,n-l$, $m=-l,-l+1,\ldots,l-1,l,\ldots$ Using the following transformations and redefinitions, $\Phi_0=\chi_0/(a\sigma)$, $f_n=\tilde{f}_n/(a\sigma)$, $m^2\to m^2/\sigma^2$, $\lambda\to\lambda/2\pi^2$, the matter action for conformal coupling, $\xi=\frac{1}{6}$, can be written as

$$S_m=\int d\eta\left[\frac{1}{2}(\chi_0'^2-m^2a^2\chi_0^2)-\frac{\lambda}{4!}\chi_0^4-\frac{1}{2}\sum_k\tilde{f}_k\left[\frac{d^2}{d\eta^2}+k^2\right]\tilde{f}_k-\frac{1}{2}\sum_k m^2a^2\tilde{f}_k^2\right.$$
$$\left.-\frac{\lambda}{4!}\left[6\sum_k\chi_0^2\tilde{f}_k^2+4\sum_{klm}\alpha_{klm}\tilde{f}_k\tilde{f}_l\tilde{f}_m+\sum_{klmn}\beta_{klmn}\tilde{f}_k\tilde{f}_l\tilde{f}_m\tilde{f}_n\right]\right]\ , \tag{2.5}$$

where

$$\alpha_{klm}=\int d^3x\sqrt{\det\Omega}\,Q_l(x)Q_m(x)Q_k(x)\ ,$$
$$\beta_{klmn}=\int d^3x\sqrt{\det\Omega}\,Q_l(x)Q_m(x)Q_k(x)Q_n(x)\ , \tag{2.6}$$

$\det\Omega$ is the determinant of the metric on S^3 and we have taken the single subscript on the Q's to collectively denote the quantum numbers (nlm). From now on we will drop the tildes on the f's to simplify notation. From

(2.2) and (2.5), the canonical momenta are given by

$$\pi_a = \frac{\partial L}{\partial a'} = -a' , \tag{2.7}$$

$$\pi_{\chi_0} = \frac{\partial L}{\partial \chi_0'} = \chi_0' , \tag{2.8}$$

$$\pi_{f_n} = \frac{\partial L}{\partial f_n'} = f_n' , \tag{2.9}$$

where L is the Lagrangian defined by $\int d\eta\, L = S_g + S_m$ and the prime denotes differentiation with respect to η.

Using Eqs. (2.7), (2.8), and (2.9) the Hamiltonian can be written as

$$H = -\frac{1}{2}a'^2 + \frac{1}{2}\chi_0'^2 + \frac{1}{2}\sum_n f_n'^2 + V_0(a,\chi_0) + V(a,\chi_0,f_n) , \tag{2.10}$$

where

$$V_0(a,\chi_0) = -\frac{1}{2}a^2 + \frac{1}{2}m^2 a^2 \chi_0^2 + \frac{\lambda}{4!}\chi_0^4 \tag{2.11}$$

and

$$\begin{aligned} V(a,\chi_0,f_n) = {} & \frac{6\lambda}{4!}\sum_k \chi_0^2 f_k^2 + \frac{\lambda}{4!}\sum_{klm}\alpha_{klm} f_k f_l f_m \chi_0 \\ & + \frac{1}{2}\sum_k (k^2 + m^2 a^2) f_k^2 \\ & + \frac{\lambda}{4!}\sum_{klmn}\beta_{klmn} f_k f_l f_m f_n . \end{aligned} \tag{2.12}$$

The Wheeler-DeWitt equation for this Hamiltonian is given by

$$H\Psi = 0 \tag{2.13}$$

where the momenta are replaced by the operators in the usual way. This leads to

$$\left[\frac{1}{2}\frac{\partial^2}{\partial a^2} - \frac{1}{2}\frac{\partial^2}{\partial \chi_0^2} - \frac{1}{2}\sum_n \frac{\partial^2}{\partial f_n^2} + V_0 + V\right]\Psi(a,\chi_0,f_n) = 0 . \tag{2.14}$$

There is, of course, the usual factor-ordering problem in making the transition to quantum mechanics. We will choose a factor ordering such that the kinetic term appears as the Laplace-Beltrami operator on superspace. We have chosen the special case of conformal coupling to simplify the structure of the Wheeler-DeWitt equation, but it is not difficult to extend this to arbitrary ξ [10]. We will make some approximations on the above equation. The first one consists of dropping the α_{klm} and β_{klmn} terms, assuming them to be small and retaining terms only up to quadratic order in the f_n's. Previously most authors [11,12] have considered models with free scalar fields on a Robertson-Walker background. In those cases there is no interaction between the lowest mode of the scalar field and the higher modes. In our case this interaction is present even with the above approximation. Retaining only the quadratic terms in the f_n's allows us to write the wave function as a direct product of wave functions for individual f_n's:

$$\Psi(a,\chi_0,f_n) = \Psi_0(a,\chi_0)\prod_n \Psi_n(a,\chi_0,f_n) , \tag{2.15}$$

with

$$\begin{aligned} V(a,\chi_0,f_n) &= \frac{6\lambda}{4!}\sum_n \chi_0^2 f_n^2 + \frac{1}{2}\sum_n (m^2 a^2 + n^2) f_n^2 \\ &= \sum_n V_n(\chi_0,f_n,a) . \end{aligned} \tag{2.16}$$

The idea is to obtain from (2.14) an "effective" Wheeler-DeWitt equation for the minisuperspace wave function of the form

$$(H_0 + \Delta H)\Psi_0(a,\chi_0) = 0 , \tag{2.17}$$

where H_0 represents the minisuperspace Hamiltonian given by

$$H_0 = \frac{1}{2}\frac{\partial^2}{\partial a^2} - \frac{1}{2}\frac{\partial^2}{\partial \chi_0^2} + V_0 \tag{2.18}$$

and ΔH represents the influence of the higher modes. If the second term in (2.17) were absent, one would have an autonomous evolution of the minisuperspace wave function unaffected by the presence of the other modes. Examination of this term will therefore enable us to comment on the validity of the minisuperspace description.

The Laplacian on minisuperspace is given by

$$\nabla^2 = -\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial \chi_0^2} . \tag{2.19}$$

The Wheeler-DeWitt equation (2.14) can therefore be written as

$$\left[-\frac{1}{2}\nabla^2 - \frac{1}{2}\sum_n \frac{\partial^2}{\partial f_n^2} + V_0(a,\chi_0) + \sum_n V_n(a,\chi_0,f_n)\right]\Psi = 0 . \tag{2.20}$$

The first approximation many authors have made is to assume that Ψ_n varies slowly with the minisuperspace variables. Following their derivation ([13]–[16]) and using (2.23) in (2.20) one can arrive at

$$[-\tfrac{1}{2}\nabla^2 + V_0(a,\chi_0)]\Psi_0 = -\sum_n \langle H_n \rangle \Psi_0 \tag{2.21}$$

and

$$\left[\frac{\nabla\Psi_0}{\Psi_0}\right]\cdot\nabla\Psi_n = (H_n - \langle H_n \rangle)\Psi_n , \tag{2.22}$$

where H_n corresponds to the Hamiltonian for the nth mode and the expectation value $\langle H_n \rangle = \int df_n \Psi_n^* H_n \Psi_n$. The second approximation consists of specializing the ansatz (2.15) to the case where Ψ_0 has the WKB form, i.e.,

$$\Psi = e^{iS(a,\chi_0)}\prod_n \Psi_n(a,\chi_0,f_n) , \tag{2.23}$$

which can be used in regions of superspace where the wave function oscillates rapidly. Using (2.23) in (2.22) and (2.21) we then get

$$\frac{1}{2}(\nabla S)^2+V_0=-\sum_n \langle H_n\rangle \tag{2.24}$$

and

$$i(\nabla S)\cdot\nabla\Psi_n=(H_n-\langle H_n\rangle)\Psi_n \ . \tag{2.25}$$

Defining "time" along the classical WKB trajectories via $\nabla S\cdot\nabla=d/d\eta$, equation (2.22) can be written as

$$i\frac{d\Psi_n}{d\eta}=(H_n-\langle H_n\rangle)\Psi_n \ . \tag{2.26}$$

This represents a Schrödinger equation for the wave functions Ψ_n modified by a back-reaction term. The problem of determining the appropriate vacuum state for the field is equivalent to finding a solution to (2.26) with appropriate boundary conditions. We will concentrate mainly on Eq. (2.24), which can be interpreted as the Hamilton-Jacobi equation for the minisuperspace variable with a back-reaction term. Identifying $\partial S/\partial a=\pi_a$ and $\partial S/\partial\chi_0=\pi_{\chi_0}$, Eq. (2.24) reads

$$\tfrac{1}{2}\pi_a^2-\tfrac{1}{2}\pi_{\chi_0}^2+V_0(a,\chi_0)=-\sum_n\langle H_n\rangle \ . \tag{2.27}$$

Substituting for the canonical momenta we get

$$\tfrac{1}{2}a'^2-\tfrac{1}{2}\chi_0'^2+V_0(a,\chi_0)=-\sum_n\langle H_n\rangle \ . \tag{2.28}$$

Let us now compare Eq. (2.28) with (2.17). Without the back-reaction term Eq. (2.28) is equivalent to the classical Hamiltonian constraint equation for a conformally coupled homogeneous scalar field χ_0 in a closed Friedmann universe with scale factor a. The back-reaction term $\sum_n\langle H_n\rangle=\int d^3x\sqrt{h_{ij}}\langle T_{00}\rangle$ can be identified with that appearing in the time-time component of the usual semiclassical Einstein equation. The origin of this term is in the quantum fluctuations of the higher modes in superspace. In this restricted framework, by choosing the WKB form of the minisuperspace wave function we are examining the influence of the higher modes on the minisuperspace sector in the limit that the minisuperspace modes behave classically. This equation has also been considered in Refs. [13]–[19], but the focus there was not the question of justification of the minisuperspace approximation. The back-reaction term has been calculated explicitly to our knowledge only in Ref. [14] for the case of a free spinor field in a de Sitter background for a de Sitter-invariant vacuum state implied by the Hartle-Hawking boundary conditions.

At this point we would like to make some comments on the derivation of the above semiclassical Einstein equation from the Wheeler-DeWitt equation. We used the WKB approximation to approach this limit, merely for simplicity of description. However, it has also been pointed out [16,18] that this approach, adopted by several authors [12]–[17], does not lead *uniquely* to the standard back-reaction term in Eq. (2.28). This term appears to be among many choices allowed by the derivation. This is related to the ambiguity in dividing the total phase of the wave function into the part associated with the minisuperspace wave function and the part associated with the higher modes [16,18].

It was then suggested [16,19,30] that the appropriate way to derive the back reaction in a unique way starting from the Wheeler-DeWitt equation was to construct the Wigner distribution function for the WKB wave function and notice that it is peaked around the trajectories in phase space given by the standard semiclassical Einstein equations. More recently these derivations were criticized in [20,21], and it was realized that the issue of recovering the classical behavior in certain degrees of freedom is far more subtle and requires additional considerations of destruction of interference via "decoherence." In a more complete and consistent approach one can recover the semiclassical Einstein equations under more re strictive conditions from the peak of the WKB Wigner distribution function constructed from a reduced density matrix incorporating decoherence [21]. Since this more correct approach is based on a reduced density matrix for the minisuperspace variables, in the strictest sense it is probably inappropriate to discuss this issue using pure-state wave functions for the minisuperspace sector. For our purpose here we will assume that the Eq. (2.28) is valid and can be justified on a more rigorous basis separately.

With these cautionary notes we see that Eq. (2.28) can be put in one-to-one correspondence with Eq. (2.17) with Ψ_0 replaced by the WKB wave function and ΔH identified with $\sum_n\langle H_n\rangle$. We would now like to calculate this back-reaction term explicitly in order to examine the validity of the minisuperspace approximation.

III. BACK REACTION OF HIGHER MODES ON THE MINISUPERSPACE SECTOR

In this section we will explicitly calculate the back reaction of the higher modes on the minisuperspace sector as given in Eq. (2.28). We will do this using an effective action for the minisuperspace variables where the higher modes are integrated out. Since in choosing which fields are to be integrated out in the path integral we use a high-low "momentum" splitting rather than the background field versus fluctuation field splitting used in the usual effective action, we refer to this as a coarse-grained effective action. We notice that the right-hand side of Eq. (2.28), which we wish to evaluate, involves an expectation value of the Hamiltonian. In particular, we will be interested in obtaining the expectation value in a vacuum state of the field. The specific vacuum state is determined by the boundary conditions on the Schrödinger equation. It has been shown [22] that in order to generate vacuum expectation values from the effective action (rather than in-out matrix elements) one should use closed-time-path (CTP) in-in version of the effective action rather than the better known in-out version. We will therefore be using an in-in coarse grained effective action. The reader is referred to Ref. [22] for details of the CTP formalism and to [23] for an introduction to the coarse-grained effective action.

The coarse-grained effective action is defined as

$$\exp[iS_{\text{eff}}(a^+,\chi_0^+,f_k^+)]=\int \mathcal{D}f_k^+\mathcal{D}f_k^-\exp\{i[S(a^+,\chi_0^+,f_k^+)-S^*(a^-,\chi_0^-,f_k^-)]\}$$
$$=\int \mathcal{D}f_k^+\mathcal{D}f_k^-\exp\{i[S_0(a^+,f_k^+)+S_{\text{int}}(a^+,\chi_0^+,f_k^+)-S_0^*(a^-,f_k^-)-S_{\text{int}}^*(a^-,\chi_0^-,f_k^-)]\}\ , \tag{3.1}$$

where

$$S_0=-\frac{1}{2}\sum_k \int d\eta\, f_k\left[\frac{d^2}{d\eta^2}+k^2\right]f_k \tag{3.2}$$

and

$$S_{\text{int}}=-\frac{1}{2}\int d\eta\left[m^2a^2f_k^2-\frac{\lambda}{4!}\left[6\sum_k \chi_0^2 f_k^2+4\sum_{klm}\alpha_{klm}f_kf_lf_m+\sum_{klmn}\beta_{klmn}f_kf_lf_mf_n\right]\right]. \tag{3.3}$$

S^* indicates that in this functional integral, m^2 carries an $i\epsilon$ term. a^+,χ_0^+,f_k^+ are the fields in the positive time branch running from $\eta=-\infty$ to $+\infty$ and a^-,χ^-,f_k^- are the corresponding fields in the negative time branch running from $\eta=+\infty$ to $-\infty$. The path integral is over field configurations that coincide at $t=\infty$. $\mathcal{D}f_k$ symbolizes the functional integration measure over the amplitudes of the higher modes of the scalar field. We will eventually be interested in obtaining effective equations of motion from this effective action which can be put in correspondence with the Einstein-Hamiltonian-Jacobi equation derived in the previous section in the limit where the scale factor and the zero mode of the scalar field becomes classical. The effective equations of motion will be given by

$$\left.\frac{\delta\tilde{S}_{\text{eff}}}{\delta a^+}\right|_{\substack{a^+=a^-=a\\ \chi_0^+=\chi_0^-=\chi_0}}=0 \tag{3.4}$$

and

$$\left.\frac{\delta\tilde{S}_{\text{eff}}}{\delta \chi_0^+}\right|_{\substack{a^+=a^-=a\\ \chi_0^+=\chi_0^-=\chi_0}}=0\ , \tag{3.5}$$

where $\tilde{S}_{\text{eff}}=S_{\text{eff}}+S_{g+}+S_{\chi_0^+}$ and S_{g+} and $S_{\chi_0^+}$ are the classical actions for the gravitational and lowest-mode matter fields, respectively. Since we are interested in deriving these equations of motion, we need to compute only those terms in the effective action that involve the positive fields. Terms containing only negative fields will not contribute to the equations of motion.

Thus the effective action essentially involves functionally integrating over the $k>1$ modes. As we briefly mentioned before, this is different from the conventional background-field effective action. In that case the field is split into a part representing a classical background and another part representing the quantum fluctuations about this background. The effective action is then calculated by functionally integrating over the fluctuation field which contains the full spectrum of momentum modes. Since the background field is taken to be classical and fixed, the full generating functional does not contain an integration over the background field. In our case, however, the full generating functional contains a functional integral over the lowest mode of the field, but since in obtaining Eq. (3.4) and (3.5), we are considering extremal configurations of the field, this corresponds to the limit in which χ_0 is regarded as a classical field. This is equivalent to evaluating the functional integral in the saddle-point approximation, which is the appropriate approximation to consider if we want to compare results with Eq. (2.25). The action has been split into the S_0 and S_{int} in this particular manner because we will regard S_0 as the free action and S_{int} a small perturbation on it. We will calculate S_{eff} perturbatively regarding m^2 and λ as small parameters. We have assumed conformal coupling here for simplicity, but one can also regard $(\xi-\frac{1}{6})$ as a perturbation parameter and consider small deviations from conformal coupling. From the definition of the effective action, we have

$$iS_{\text{eff}}=\ln\langle \exp[iS_{\text{int}}(a^+,\chi_0^+,f_k^+)-iS_{\text{int}}^*(a^-,\chi_0^-,f_k^-)]\rangle\ , \tag{3.6}$$

where the vacuum state considered in the expectation value is the conformal vacuum defined when the mass and coupling constant are switched off in the infinite past. The vacuum here refers only to the higher modes. Equation (3.6) can be expanded in a perturbation series in the following manner:

$$S_{\text{eff}}\simeq\langle S_{\text{int}}(a^+,\chi_0^+,f_k^+)\rangle-\langle S_{\text{int}}(a^-,\chi_0^-,f_k^-)\rangle+\frac{i}{2}\langle S_{\text{int}}(a^+,\chi_0^+,f_k^+)^2\rangle+\frac{i}{2}\langle S_{\text{int}}(a^-,\chi_0^-,f_k^-)^2\rangle$$
$$-i\langle S_{\text{int}}(a^+,\chi_0^+,f_k^+)S_{\text{int}}(a^-,\chi_0^-,f_k^-)\rangle+i\langle S_{\text{int}}(a^+,\chi_0^+,f_k^+)\rangle\langle S_{\text{int}}(a^-,\chi_0^-,f_k^-)\rangle$$
$$-\frac{i}{2}\langle S_{\text{int}}(a^+,\chi_0^+,f_k^+)\rangle^2-\frac{i}{2}\langle S_{\text{int}}(a^-,\chi_0^-,f_k^-)\rangle^2+\dots\ . \tag{3.7}$$

where the * superscripts are omitted for compactness. The second, fourth, and eighth term need not be calculated for the reasons given before. Our approximation scheme will be to truncate the series at one-loop order and retain terms up to second order in coupling constants. The higher mode coupling terms involving α_{klm} and β_{klmn} will not appear to one-loop order. This is consistent with the linearized approximation introduced in the Hamiltonian in the previous section. We should mention here that though we are restricting ourselves to a one-loop truncation for simplicity, the effective-action formalism allows an extension to higher-loop calculations in a systematic approximation scheme. The α_{lmn} and β_{klmn} terms will contribute to these orders the effect of nonlinear mode couplings, which was the main motivation for using this model. It is, however, not very clear how these higher-loop approximations translate to the Hamiltonian scheme described in Sec. II.

Since we have used conformally related fields in a conformally static spacetime, the free propagators can be read off from the quadratic part of the action. In the CTP effective action, in addition to the Feynman propagator, the Dyson and Wightman propagators appear. We have

$$\begin{aligned}\langle f_k^+(\eta)f_k^+(\eta')\rangle &= i\frac{\delta_{kk'}}{2\pi}\int d\omega \frac{e^{i\omega(\eta-\eta')}}{\omega^2-k^2+i\epsilon}\,,\\ \langle f_k^-(\eta)f_k^-(\eta')\rangle &= i\frac{\delta_{kk'}}{2\pi}\int d\omega \frac{e^{i\omega(\eta-\eta')}}{\omega^2-k^2-i\epsilon}\,,\\ \langle f_k^+(\eta)f_k^-(\eta')\rangle &= i\frac{\delta_{kk'}}{2\pi}\int d\omega\, e^{i\omega(\eta-\eta')}\\ &\qquad\times[2\pi i\delta(\omega^2-k^2)\theta(\omega)]\,.\end{aligned} \tag{3.8}$$

Using these propagators we can evaluate the terms in the effective action given in Eq. (3.7) to quadratic order. By Wick's theorem only the connected Feynman diagrams will contribute. We will consider the contributions to Eq. (3.7) term by term using Eq. (3.8). The first term gives

$$\langle S_{\rm int}(a^+,\chi_0^+,f_k^+)\rangle = \frac{-i}{2}\int d\eta\{\tilde m^{+2}(\eta)+\tfrac{1}{2}\lambda\chi_0^{+2}\}\times\sum_{k=2}^{\infty}\int\frac{d\omega}{2\pi}\frac{1}{\omega^2-k^2+i\epsilon}\,, \tag{3.9}$$

where the lowest mode is omitted in the sum and $\tilde m^{\pm 2}=m^2a^{\pm 2}$. Note that the superscripts $\pm$ belong to a,χ_0,f_k,m,M, and should not be taken as their exponents. Evaluating the sum using ζ-function regularization gives

$$\langle S_{\rm int}(a^+,\chi_0^+,f_k^+)\rangle = \frac{13}{48}\int d\eta[\tilde m^{+2}(\eta)+\tfrac{1}{2}\lambda\chi_0^{+2}(\eta)] \tag{3.10}$$

The second term in (3.10) is the analog of the topological mass terms found by Ford and Toms [24]. This term is characteristic of the nontrivial topology and vanishes in the limit of flat spatial sections. We next consider the contribution from

$$\begin{aligned}&\frac{i}{2}\{\langle S_{\rm int}(a^+,\chi_0^+,f_k^+)^2\rangle-[\langle S_{\rm int}(a^+,\chi_0^+,f_k^+)\rangle]^2\}\\ &\quad=\left[\frac{\lambda^2}{16}\int d\eta_1 d\eta_2\chi_0^{+2}(\eta_1)\chi_0^{-2}(\eta_2)+\frac{\lambda}{4}\int d\eta_1 d\eta_2\tilde m^{+2}(\eta_1)\chi_0^{+2}(\eta_2)+\frac{1}{4}\int d\eta_1 d\eta_2\tilde m^{+2}(\eta_1)\tilde m^{+2}(\eta_2)\right]\\ &\qquad\times\left\{\int\frac{d\omega''}{2\pi}e^{i\omega''(\eta_1-\eta_2)}\left[\frac{-i}{2}\sum_k\int\frac{d\omega}{2\pi}\left[\frac{1}{(\omega''-\omega)^2-k^2+i\epsilon}\right]\left[\frac{1}{\omega^2-k^2+i\epsilon}\right]\right]\right\}\end{aligned} \tag{3.11}$$

and the contribution from

$$\begin{aligned}&-i\langle S_{\rm int}(a^+,\chi_0^+,f_k^+)S_{\rm int}(a^-,\chi_0^-,f_k^-)\rangle+i\langle S_{\rm int}(a^+,\chi_0^+,f_k^+)\rangle\langle S_{\rm int}(a^-,\chi_0^-,f_k^-)\rangle\\ &\quad=2\left[\frac{3\lambda^2}{96}\int d\eta_1 d\eta_2\chi_0^{+2}(\eta_1)\chi_0^{-2}(\eta_2)+\frac{\lambda}{16}\int d\eta_1 d\eta_2[\tilde m^{+2}(\eta_1)\chi_0^{-2}(\eta_2)+\tilde m^{-2}(\eta_1)\chi_0^{+2}]\right.\\ &\qquad\left.+\frac{1}{8}\int(d\eta_1 d\eta_2\tilde m^{+2}(\eta_1)\tilde m^{-2}(\eta_2)\right]\\ &\qquad\times\int\frac{d\omega''}{2\pi}e^{i\omega(\eta_1-\eta_2)}\theta(\omega'')\left[i\sum_k\int\frac{d\omega}{2\pi}[2\pi i\delta(\omega^2-k^2)\theta(\omega')][2\pi i\delta((\omega-\omega')^2-k^2)]\right]\,.\end{aligned} \tag{3.12}$$

These are the only terms that will contribute to the part of the effective action that is required to derive the equations of motion. The details for evaluating the integrals using appropriate regularization schemes is given in the Appendix. The integrals are evaluated by rotating ω to Euclidean space and rotating back to physical space at the end. On carrying out the integrations, Eq. (3.11) can be written as

$$\begin{aligned}&\frac{i}{2}[\langle S_{\rm int}(a^+,\chi_0^+,f_k^+)^2\rangle-(\langle S_{\rm int}(a^+,\chi_0^+,f_k^+)\rangle)^2]\\ &\quad=\frac{1}{n-4}\left[\int d\eta\frac{1}{32}\tilde m^{+4}(\eta)+\frac{1}{128}\chi_0^{+4}(\eta)+\frac{\lambda}{32}\tilde m^{+2}(\eta)\chi_0^{+2}(\eta)\right]+\frac{1}{16}\int d\eta M^{+4}\ln\mu a\\ &\qquad+\frac{1}{32}\int d\eta_1 d\eta_2 M^{+2}(\eta_1)K(\eta_1-\eta_2)M^{+2}(\eta_2)\,,\end{aligned} \tag{3.13}$$

where $M^{\pm 2} = \tilde{m}^{\pm 2} + \frac{1}{2}\lambda\chi_0^{\pm 2}$, and the first three terms represent the divergent contribution as $n \to 4$ which is to be absorbed in the renormalization of the classical action. The counterterms required for this turn out to agree with those calculated by previous authors [25]. μ is a quantity of mass dimension representing the scale at which quantities are renormalized. $K(\eta_1 - \eta_2)$ represents the finite nonlocal contribution and is given by

$$K(\eta_1-\eta_2) = \int \frac{d\omega}{2\pi} e^{i\omega(\eta_1-\eta_2)} \left[\frac{-2}{4-\omega^2} - \ln\left[\frac{4-\omega^2}{4}\right] + \mathcal{J}_4 \right] \tag{3.14}$$

$$\mathcal{J}_4 = i\int_0^1 dx \int_0^\infty dt \left[\frac{(1+it)}{[(1+it)^2 - \omega^2 x(1-x)]^{3/2}} - \text{c.c.} \right] \times \frac{1}{e^{2\pi t}-1} \tag{3.15}$$

The kernel reduces to the flat-space counterpart calculated previously [26] in the limit $a \to \infty$. Using the Cutkowsky rules as in Ref. [22] we find that the ω integral in Eq. (3.12) including the i factor is simply twice the imaginary part of the ω' integral in (3.11). Using this fact and defining from (3.14),

$$K(\omega) = \left[\frac{-2}{4-\omega^2} - \ln\left[\frac{4-\omega^2}{4}\right] + \mathcal{J}_4(\omega) \right] . \tag{3.16}$$

The contribution from Eq. (3.12) can be written as

$$\begin{aligned} &-i\langle S_{\rm int}(a^+,\chi_0^+,f_k^+) S_{\rm int}(a^-,\chi_0^-,f_k^-)\rangle \\ &\quad + \langle S_{\rm int}(a^+,\chi_0^+,f_k^+)\rangle\langle S_{\rm int}(a^-,\chi_0^-,f_k^-)\rangle \\ &\qquad = \frac{1}{32}\int d\eta_1 d\eta_2 M^{+2}(\eta_1)\bar{K}(\eta_1-\eta_2)M^{-2}(\eta_2) , \end{aligned} \tag{3.17}$$

where

$$\bar{K}(\eta_1-\eta_2) = 2\int \frac{d\omega}{2\pi} e^{i\omega(\eta_1-\eta_2)} \mathrm{Im}K(\omega)\theta(\omega) \tag{3.18}$$

and

$$\mathrm{Im}K(\omega) = \pi\theta(\omega^2-4) . \tag{3.19}$$

The contribution from (3.12) does not modify the divergent terms but contributes only to the finite nonlocal part. So the full renormalized effective action (omitting terms that involve negative fields only) can be written as

$$\begin{aligned} \tilde{S}_{\rm eff} = S_{g+} &+ \frac{1}{2}\int d\eta(\chi_0'^2 - \tilde{m}^{+2}\chi_0^{+2}) - \frac{\lambda}{4!}\int d\eta\, \chi_0^{+4} \\ &+ \frac{13\lambda}{48}\int d\eta\, M^{+2} + \frac{1}{16}\int d\eta\, M^{+4} \\ &+ \frac{1}{32}\int d\eta_1 d\eta_2 M^{+2}(\eta_1) K(\eta_1-\eta_2) M^{+2}(\eta_2) \\ &+ \frac{1}{32}\int d\eta_1 d\eta_2 M^{+2}(\eta_1)\bar{K}(\eta_1-\eta_2)M^{-2}(\eta_2) , \end{aligned} \tag{3.20}$$

where the coupling constants have their renormalized values. S_{g+} represents the classical gravitational part of the action. We have not included the contribution of the trace anomaly terms since they do not appear to this order of approximation. We are now in a position to derive the effective equations of motion from this effective action. Following Eqs. (3.4) and (3.5) these are given by

$$\begin{aligned} a'' + a - m^2 a\chi_0^2 + \tfrac{1}{4}m^2 a(m^2 a + \tfrac{1}{2}\lambda\chi_0^2)\ln\mu a + \frac{1}{16a}M^4 \\ + \frac{m^2}{8}\int d\eta_1 a(\eta)\mathcal{K}(\eta-\eta_1)M^2(\eta_1) = 0 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \chi_0'' + m^2 a^2\chi_0 + \frac{\lambda}{6}\chi_0^3 - \frac{13\lambda}{48}\chi_0 - \frac{\lambda}{8}M^2\chi_0\ln\mu a \\ - \frac{\lambda}{16}\int d\eta_1 \chi_0(\eta)\mathcal{K}(\eta-\eta_1)M^2(\eta_1) = 0 , \end{aligned} \tag{3.22}$$

where $M^2 = m^2 a^2 + \frac{1}{2}\lambda\chi_0^2$, and

$$\mathcal{K}(\eta-\eta_1) = K(\eta-\eta_1) + \bar{K}(\eta-\eta_1) . \tag{3.23}$$

It can be demonstrated using Eqs. (3.14), (3.18), and (3.19) that $\mathcal{K}$ is real [22]. So the equations of motion generated from this effective action are real. Note that Eqs. (3.21) and (3.22) are very similar to the equations obtained by Hu and Zhang [26] in their section B(c). The differences come from the terms that are special to the topology of the three-sphere here and the nonlocal terms that were truncated in their case. The lowest mode field χ_0 plays a role similar to the background field though the interpretations are different as discussed before. Following Refs. [26] and [27] one can obtain a first integral from (3.21) and (3.22) since the effective Lagrangian has no explicit dependence on η. This is given by

$$\begin{aligned} \frac{1}{2}a'^2 - \frac{1}{2}\chi_0'^2 + \frac{1}{2}m^2 a^2\chi_0^2 - \frac{1}{2}a^2 + \frac{\lambda}{4!}\chi_0^4 \\ - \frac{13\lambda}{96}\chi_0^2 - \frac{13}{48}m^2 a^2 - \frac{1}{16}M^2\ln\mu a \\ + \frac{1}{32}\int d\eta_1 M^2(\eta)\mathcal{K}(\eta-\eta_1)M^2(\eta_1) = E , \end{aligned} \tag{3.24}$$

where E is a constant. We assume $E=0$, which is equivalent to having no quanta of higher modes in the initial state. Equation (3.24) is then equivalent to the effective G_{00} Einstein equation or the Einstein-Hamilton-Jacobi equation plus back reaction that we derived in the previous section. Comparing Eqs. (3.24) and (2.28) we can identify the back-reaction piece as

$$\begin{aligned} \sum_n \langle H_n\rangle = -\frac{13\lambda}{96}\chi_0^2 - \frac{13\lambda}{48}m^2 a^2 - \frac{1}{16}M^2\ln\mu a \\ - \frac{1}{32}\int d\eta_1 M^2(\eta)\mathcal{K}(\eta-\eta_1)M^2(\eta_1) . \end{aligned} \tag{3.25}$$

Since $\mathcal{K}$ is real the back-reaction term given above is real. This also indicates that it represents a genuine expectation value in the "in" vacuum state rather than an in-out matrix element generated using the in-out effective action. In the above derivation we have obtained the expectation value of the Hamiltonian from the coarse-grained effective action. This quantity is the same as the one that would be obtained in (2.28) by a self-consistent solution of (2.26) and (2.28) using the boundary conditions on the wave function appropriate to that for a conformal vacuum "in" state. In the case of a massless free field with a small nonconformal coupling in a flat Robertson-Walker background, it has been explicitly demonstrated [22] that $\langle T_{\mu\nu} \rangle$ in the conformal in-vacuum state computed from the CTP effective action is indeed equivalent to that calculated using a self-consistent equation of motion approach [28] up to local terms. The effective action seems a more efficient tool for generating the nonlocal back-reaction term.

Since Eq. (3.24) is the "effective" Wheeler-DeWitt equation for the minisuperspace sector within our approximation scheme the condition for validity of this approximation can be stated as

$$\sum_n \langle H_n \rangle \ll V_0 \; . \tag{3.26}$$

where by the left-hand side we mean the regulated value given by (3.25). The third term in Eq. (3.25) involving the nonlocal kernel has been demonstrated as signifying dissipative behavior in closely related models [29]. This dissipative behavior in turn has been related to particle production by the dynamical background geometry in semiclassical gravity models which provides a systematic damping of the source [30]. In our case this can be interpreted as scalar particles in the higher modes being produced as a result of the dynamical evolution of the minisuperspace degrees of freedom. These generate back reaction that modifies the minisuperspace evolution. We can therefore think of this term as introducing dissipation in the minisuperspace sector due to interaction with the higher modes that are integrated out. One can justifiably think of autonomous minisuperspace evolution only when this dissipation is small. Since we have used the scalar field modes to simulate higher gravitational modes these considerations can also be directly extended to include gravitons. A similar idea has been discussed by Padmanabhan and Singh [31] in a linearized gravity model where they claim that in order that the minisuperspace approximation be valid, the rate of production of gravitons should be small.

IV. DISCUSSION

In this paper we have discussed the issue of the validity of the minisuperspace truncation in the model of a massive $\lambda\Phi^4$ scalar field conformally coupled to a closed RW background. We calculated the back reaction of the higher modes on the midisuperspace sector and found that the minisuperspace truncation can be justified only when condition (3.26) is satisfied, i.e, when the minisuperspace potential dominates the back-reaction term. Thus the oftentimes expedient way of treating the minisuperspace degrees of freedom as a closed system is incomplete, and in the more general cases incorrect.

There are still several questions to be resolved. We originally motivated to find the influence of the truncated modes on the *quantized* minisuperspace sector. However, in the calculation we invoked the ansatz (2.23) and carried out the analysis in the WKB limit and considered the situation where the minisuperspace variables behave classically. The WKB approximation is of limited validity and it is important to go beyond it to be able to analyze the full quantum behavior of the minisuperspace sector where there is no time defined in general. One will probably also have to go beyond the effective action technique in that case since there will be no classical background available. One possible approach is that taken by Kuchar and Ryan in [7] where they analyze a lower-dimensional minisuperspace model embedded in a higher- (but finite-) dimensional minisuperspace model in a regime where exact solutions for both models exist. But one would also like to understand the effect of discarding an infinite number of inhomogeneous modes, which is the actual situation in full quantum gravity. In this context it may be interesting to analyze some midisuperspace models such as the Gowdy model [4] which contain infinite degrees of freedom, but for which known solutions exist.

Even within the framework of our approximation scheme we had mentioned in Sec. II some limitations of the derivation of the effective Hamilton Jacobi or semiclassical Einstein equations from the Wheeler-DeWitt equation using the WKB ansatz. To obtain classical behavior in the minisuperspace degrees of freedom one also needs "decoherence," which requires working with reduced density matrices rather than pure state wave functions. It would be interesting to see if the criterion for validity of the minisuperspace approximation can be formulated in the framework of reduced density matrices from an effective master equation rather than a Wheeler-DeWitt equation. This may also clarify the role of the back reaction as a dissipative effect on the minisuperspace sector.

In the Hamiltonian derivation of the effective Wheeler-DeWitt equation we assumed that the Ψ_n's vary slowly with the minisuperspace variables. It is not clear how this approximation translates to the effective action calculation. One would also like to know how to incorporate the boundary conditions proposed in [5] in our effective action framework. Another important question is related to the regularization and renormalization of divergences. We have regulated and renormalized the divergences in a covariant manner in the effective action after the limit that the scale factor behaves classically has been taken, i.e., in the usual framework of quantum field theory in curved spacetime. However, since the radiative corrections introduce new dynamical variables in the Wheeler-DeWitt operator [32] it is not clear that there is a smooth transition between handling divergences at the level of the Wheeler-DeWitt equation and in the semiclassical limit. This question has also been raised in Ref. [14] and it has been indicated that the naive noncovariant regularization and covariant regularization results do not agree.

Finally, it would be interesting to see the connection with the work of Ryan and Kuchař for quantum-mechanical examples [6]. Their condition of vanishing transition matrix elements seems closely related to our assumption of slow variation of the higher mode wave functions with the minisuperspace variables.

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APPENDIX

The following integral appears in E. (3.11):

$$I=\int \frac{d\omega}{2\pi}\int \frac{d\omega'}{2\pi}e^{i(\omega+\omega')(\eta-\eta')}\times\sum_{\kappa}\frac{1}{\omega'^2-\kappa^2+i\epsilon}\frac{1}{\omega^2-\kappa^2+i\epsilon} \tag{A1}$$

On rotating $\eta_1,\eta_2,\omega,\omega'$ to Euclidean space, we can write

$$I=\int \frac{d\omega''}{2\pi}e^{i\omega''(\eta_1-\eta_2)}\times\sum_{\kappa}\int\frac{d\omega}{2\pi}\frac{1}{[(\omega''-\omega)^2+\kappa^2][\omega^2+\kappa^2]} \tag{A2}$$

where $\omega''=\omega+\omega'$.

Let us continue to n dimensions replacing $d\omega/2\pi$ by $d^{n-3}k/(2\pi)^{n-3}$ where finally we will take the limit $n\to 4$. Notice that we are adding flat rather than curved dimensions but this choice is left arbitrary in dimensional regularization.

Let

$$I_1=\sum_{k}\int\frac{d^{n-3}k}{[(k-k')^2+\kappa^2][k^2+\kappa^2]} . \tag{A3}$$

Using Feynman parameters we can rewrite I_1 as

$$\begin{aligned}I_1&=\sum_{\kappa}\frac{1}{(2\pi)^{n-3}}\int_0^1 dx\int\frac{d^{n-3}k}{[k^2+\kappa^2+k''^2x(1-x)]^2}\\&=\frac{\Gamma(\frac{7}{2}-n/2)}{(4\pi)^{n-3/2}}\int_0^1 dx\left[\sum_{\kappa=1}^{\infty}\frac{\kappa^2}{[\kappa^2+k''^2x(1-x)]^{7/2-n/2}}-\frac{1}{[1+k''^2x(1-x)]^{7/2-n/2}}\right] .\end{aligned} \tag{A4}$$

The second term in (A4) is just the lowest mode contribution subtracted out. Let us concentrate on the first term and evaluate the sum contained in it. In the appendix of Ref. [33] we have the sum

$$\sum_{\kappa=1}^{\infty}\frac{\kappa^2}{(\kappa^2+\sigma)^r}=Z(r,\sigma) . \tag{A5}$$

This function has poles at $r=\frac{3}{2}-n'$, where $n'=0,1,\ldots$. This can be put in one-to-one correspondence with the sum in the first term in (A4). In this case one has a pole at $n=4$, about which we have the following Laurent series expansion following Ref. [33]:

$$Z(\tfrac{7}{2}-n/2,\sigma)=\frac{2Z_{-1}}{4-n}+Z_0(0,\sigma)+O(n-4) \tag{A6}$$

where $\sigma=k''^2x(1-x)$. Z_{-1} and A_0 are given as

$$\begin{aligned}Z_{-1}(0,\sigma)&=\frac{1}{2} ,\\Z_0(0,\sigma)&=-1-\ln\frac{1+\sqrt{1+\sigma}}{2}+\frac{1}{\sqrt{1+\sigma}}\\&\quad+\frac{1}{2(1+\sigma)^{3/2}}+I_0(\sigma) ,\end{aligned} \tag{A7}$$

where

$$I_0(\sigma)=i\int_0^{\infty}dt\left[\frac{(1+it)^2}{[(1+it)^2+\sigma^2]^{3/2-n}}-\text{c.c.}\right]\frac{1}{e^{2\pi t}-1} . \tag{A8}$$

The Z_{-1} piece contains the divergence at $n=4$. After absorbing the divergence by renormalizing the coupling constants one sets $n=4$ in the finite part. As we can see from (3.11), the factor containing I is common to three terms. The divergences in the first, second, and third terms are absorbed in the renormalization of λ, m, and the cosmological constant Λ in the standard manner. Using the value of Z_{-1} from (A7) it is seen that the counterterms are exactly the same as previously obtained in Ref. [25]. The contribution from Z_{-1} is local. Let us next consider the contribution $\bar{I}$ to the first term in I from the Z_0 piece. This is given by

$$\bar{I}=\frac{1}{8}\int\frac{d\omega''}{2\pi}e^{i\omega''(\eta_1-\eta_2)}\int_0^1 dx\, Z_0(0,\omega^2x(1-x)) . \tag{A9}$$

In addition to the contribution in (A9), there is also the lowest mode contribution in I_1 to be subtracted out. Defining

$$\mathcal{J}_1=\int_0^1 dx\ln\left[\frac{1+\sqrt{1+\omega^2x(1-x)}}{2}\right] , \tag{A10}$$

$$\mathcal{J}_2=\int_0^1\frac{dx}{\sqrt{1+\omega^2x(1-x)}} , \tag{A11}$$

$$\mathcal{J}_3 = \int_0^1 \frac{dx}{2[1+\omega^2 x(1-x)^{3/2}]} , \tag{A12}$$

$$\mathcal{J}_4 = \int_0^1 dx\, I_0(\sigma) , \tag{A13}$$

and noticing that the zero-mode contribution is precisely $2\mathcal{J}_3$, we can write the full contribution of the Z_0 term to I as

$$\frac{1}{8}\int \frac{d\omega}{2\pi} e^{i\omega(\eta_1-\eta_2)}(-1-\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4)$$
$$=\frac{1}{8}\int \frac{d\omega}{2\pi} e^{i\omega(\eta_1-\eta_2)}\left[\frac{-2}{4+\omega^2}-\ln\left[\frac{4+\omega^2}{4}\right]+\mathcal{J}_4(\omega)\right] . \tag{A14}$$

In order to evaluate the integrals in (3.11) we had been working in Euclidean space, but in the final step we must rotate back to physical spacetime after performing this rotation, the nonlocal kernel $\mathcal{K}(\eta_1-\eta_2)$ in (3.13) is given by

$$K(\eta_1-\eta_2) = \int \frac{d\omega}{2\pi} e^{i\omega(\eta_1-\eta_2)}\left[\frac{-2}{4-\omega^2}-\ln\left[\frac{4-\omega^2}{4}\right]+\mathcal{J}_4\right] . \tag{A15}$$

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