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Simplicial quantum gravity in more than two dimensions

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The dynamical triangulation (DT) model of two-dimensional quantum gravity is generalized to higher dimensions. In three dimensions two updates are described which appear to allow for ergodic sampling of states, and an updating procedure which obeys detailed balance is given. The tensor-model generalization of the matrix-model formulation of higher-dimensional gravity is discussed and the relations between DT-, continuum-, and tensor-model couplings are given.

The past three years have witnessed significant progress toward the solution of two-dimensional quantum gravity.[1–3] From a string point of view, further work should involve a more complete nonperturbative understanding of present results as well as extensions to $c>1$ embedding dimensions and string field theory. From a purely quantum-gravity point of view, a generalization of the results to three and four dimensions is in order. We will begin that generalization in this paper.

First we discuss a dynamical triangulation (DT) model of simplicial gravity in more than two dimensions. Then we will discuss a tensor-model generalization of the matrix models to higher dimensions. The perturbative expansion of the tensor model generates all the DT lattice configurations (and some others as well).

DT model. Given a set $G=\{1,2,\ldots,n\}$, called the vertices of the lattice, we define a d-simplex to be a set of $d+1$ distinct elements of G. For $p<d$ we define a p-simplex to be a subset of a d-simplex with $p+1$ elements. We now define our compact, connected, simplicial d-dimensional manifold without boundary to be a set of d-simplices such that the following conditions are satisfied:

(i) Each face [$(d-1)$-simplex] is contained in exactly two d-simplices.

(ii) The d-simplices of the manifold cannot be divided into two subsets which have no faces in common.

(iii) Given any p-simplex with $p<d$, the d-simplices that contain it cannot be divided into two subsets which have no faces in common.

The last two criteria prevent disjoint or partially disjoint simplicial manifolds. An example of a portion of a two-dimensional lattice which violates (iii) is shown in Fig. 1.

What form does the action take on a DT simplicial manifold? Consider the d-dimensional continuum action

$$S_{\text{continuum}}=\int d^dx\sqrt{g}\left[\lambda-\frac{k}{2}R+\frac{c}{4}R^2\right], \tag{1}$$

where we have included just the simplest higher-derivative term. Taking the links to all have length a, the cosmological-constant term is just $\lambda V_d N_d$, where V_n is the volume of an n-simplex,

$$V_n=\frac{(n+1)^{1/2}}{n!2^{n/2}}a^n, \tag{2}$$

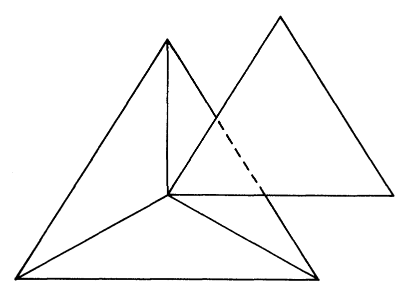

FIG. 1. A portion of a 2D lattice which violates criteria (iii) for a closed, connected, simplicial DT manifold.

and N_n is defined to be the total number of n-simplices in the lattice. The Einstein term can be written as $-kV_{d-2}\sum_{\text{hinges}\,h}\delta_h$, where V_{d-2} is the volume of a $(d-2)$-dimensional hinge (a link in 3-dimensions, triangle in 4-dimensions, etc.) and δ_h is the deficit angle associated with the hinge.[4,5] $\delta_h = 2\pi - q_h\theta_d$ where q_h is the number of d-simplices meeting at h and θ_d is the dihedral angle:

$$\theta_d = \arccos(1/d)\,. \tag{3}$$

Finally the curvature-squared term transcribes to[5] $cV_{d-2}^2\sum_{\text{hinges}\,h}\delta_h^2/V_h$, where the d-dimensional volume associated with each hinge, V_h, may be taken as $[2q_h/d(d+1)]V_d$. Putting everything together and using the identities

$$N_{n/m}N_m = \frac{(n+1)!}{(m+1)!(n-m)!}N_n \tag{4}$$

and

$$N_{d-1} = \frac{d+1}{2}N_d\,, \tag{5}$$

where $N_{n/m}$ is defined to be the average number of n-simplices meeting at an m-simplex,[6] we obtain the simple result

$$S_{\text{lattice}} = g_1N_d + g_2N_{d-2} + g_3\sum_h\frac{1}{q_h}\,. \tag{6}$$

With only cosmological-constant and Einstein terms in (1), $g_3=0$ and the lattice action is just a linear combination of the total number of d-simplices and the total number of hinges. Of course the simplicity of the lattice action (6) follows from the geometrical equivalence of each simplex in the DT lattice.

The exact correspondence between continuum and lattice couplings may be of some use in analytic approximations:

$$\begin{aligned} g_1 &= V_d(\lambda + kr_d + cr_d^2)\,,\\ g_2 &= -2\pi V_{d-2}(k+2cr_d)\,,\\ g_3 &= 4\pi^2cV_{d-2}r_d/\theta_d\,, \end{aligned} \tag{7}$$

where

$$r_d \equiv d^2(d-1)\sqrt{d^2-1}\,\theta_d/a^2\,. \tag{8}$$

In order to carry out numerical simulations of DT quantum gravity we require in addition to the action a satisfactory updating procedure. This updating procedure must satisfy ergodicity and sample configurations with a probability proportional to e^{-S}.

The fundamental two-dimensional (2D) updates are shown in Fig. 2, along with our generalizations for 3D. They are labeled by the numbers of d-simplices before and after the updates. Using the definition of a d-simplex, a 23 update in 3D can be defined as

$$\begin{matrix} \{i,j,k,l\} & & \{i,j,l,m\} \\ & \rightarrow & \{i,k,l,m\} \\ \{i,j,k,m\} & & \{j,k,l,m\} \end{matrix}$$

with the restriction that $\{l,m\}$ is not a 1-simplex before the update. A 32 update is then defined by reversing the arrow above and instead requiring that $\{i,j,k\}$ is not a 2-simplex. These updates do not change the topology nor the orientability of the initial lattice configuration. In 2D the updates have been rigorously proven to satisfy ergodicity for a fixed topology;[7] in 3D we have strong numerical evidence (see below) but no proof of ergodicity. The dual version of Fig. 2 is shown in Fig. 3, where faces of the DT lattice are represented by links in the dual lattice.

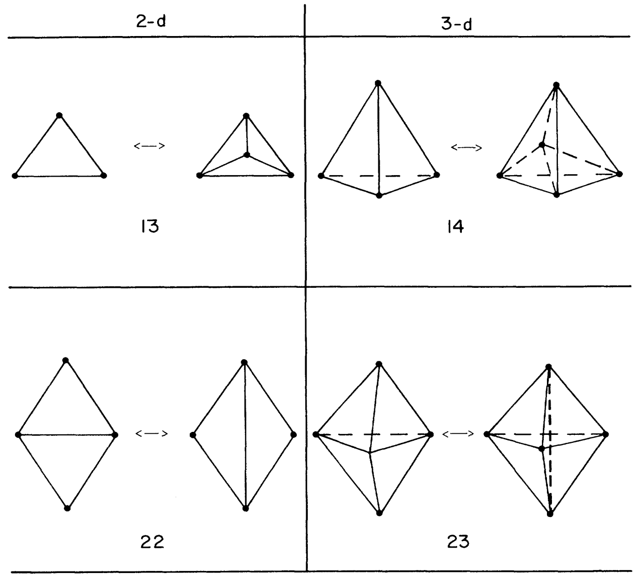

FIG. 2. Basic DT updates for two and three dimensions. They are labeled by the initial and final number of d-simplices assuming the update goes from left to right. The 23 and 32 updates are defined only if the new links and triangles did not previously exist in the DT lattice.

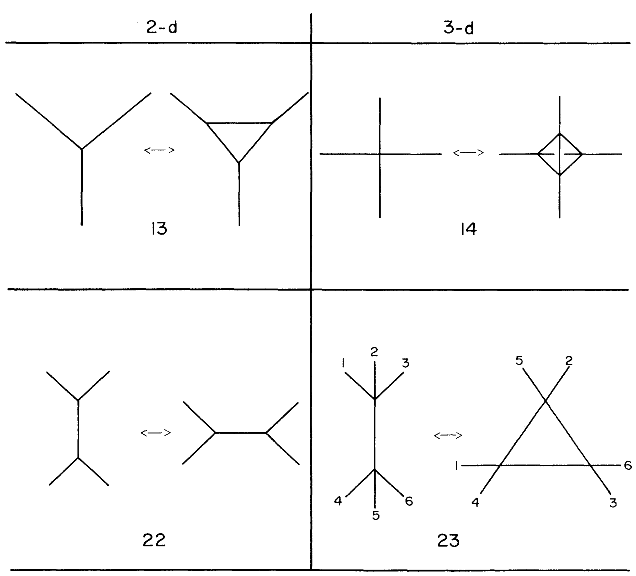

FIG. 3. Dual versions of the basic updates in two and three dimensions. Links (faces of the DT lattice) have been labeled in the 23 update.

These updates generalize to higher dimensions. For example in 4D a 15 update can be obtained by adding a site in the interior of a 4-simplex and connecting that site to all the others. A 4D 24 update involves replacing two 4-simplices attached at a face by four 4-simplices which share a common link joining two sites which were not originally a 1-simplex. These two 4D updates have at least a chance of being ergodic because in 4D there are three constraints between the five N_n for fixed topology,[6] leaving only two independent N_n. Thus ergodicity requires at least two types of updates, the same as in 3D.

What is our evidence for ergodicity in 3D?[8] We generated all 47 of the configurations with the Betti numbers of a sphere (1,0,0,1) which satisfied (i), (ii), and (iii) and had a number of sites less than or equal to 8. These configurations, which contained as many as 20 tetrahedrons, were all reachable by the 14 and 23 updates (and their inverses) described above starting from any single such configuration. In fact, all configurations with fixed $N_0 \leq 8$ were able to be reached from any one configuration with that value of N_0 and just the 23 updates. In addition we generated 55 configurations with 100 sites and toroidal topology using the random lattice algorithm[6] and checked that they could all be connected to a single configuration (and therefore to each other) by the 14 and 23 updates. Of course a rigorous proof of ergodicity would be much more satisfying than the numerical evidence described here.

The next important question is that of detailed balance. Though at first glance this may appear problematic, detailed balance can be satisfied in the following way. Keep a list of all the clusters of five sites which form two or three tetrahedrons as shown in the 23 update of Fig. 2, and for which the 23 or 32 update is allowed. Call these clusters A or B depending on whether there are two or three tetrahedrons. Update by choosing a cluster at random from the list and replacing it by an A with probability

$$P_A = \frac{e^{-S_A}}{e^{-S_A} + e^{-S_B}} \qquad (9)$$

and by a B with probability $1 - P_A$. Updating clusters from the list in this way is equivalent to (but exceedingly more efficient than) repeatedly choosing 5 sites at random from the lattice and updating only if the 5 sites form an A or B cluster. Since $e^{-S_A} P(A \to B) = e^{-S_B} P(B \to A)$ and other configurations in an ensemble are left alone, this updating scheme obeys detailed balance; i.e., an ensemble of configurations distributed as e^{-S} will remain distributed as e^{-S} after updating. It is computationally important that keeping the list of clusters current involves only local operations after each update.

The 14 and 41 updates can be similarly carried out. Define clusters as consisting of nonelementary tetrahedrons made up of four elementary tetrahedrons (≡state B) or elementary tetrahedrons not embedded in a nonelementary tetrahedron (≡state A). This subdivides the lattice manifold into disjoint volumes, each of which is in either state A or state B. Choose at random from the list of clusters and update as in (9) above.

Results of our simulations in 3D will be described elsewhere.

Tensor model. Performing efficient numerical simulations directly on the DT model is not trivial. Because of the variable connectivity of the lattice, vectorization and parallel processing are somewhat difficult, particularly in higher dimensions. In addition 2D simulations were plagued by extremely long autocorrelation times.[9] It may be that the original Regge calculus formalism is more suitable than the DT model for numerical simulations.[10,11] Regge calculus employs a fixed connectivity lattice with variable link lengths and appears to have much shorter autocorrelation times in 2D.[11] If, however, there is a tensor model correspondence in dimensions greater than two, there is the hope that analytic or semianalytic results can be obtained as was the case in 2D. In addition numerical simulations could be directly performed on the tensor model.

Consider the following tensor model:[12]

$$S_{\text{tensor}} = \frac{1}{2}\sum_{ijk} M_{ijk} M^*_{ijk} + \frac{g}{\sqrt{N}} \sum_{ijkl} M_{ijk} M_{kjl} M_{kli} M_{ilj} \,, \qquad (10)$$

where

$$M_{ijk} = M_{kij} = M_{jki} = M^*_{jik} = M^*_{ikj} = M^*_{kji}, \qquad i,j,k,l = 1,\ldots,N \,.$$

The perturbative expansion of the vacuum energy for this model involves linking up vertices associated with four indices. In the dual language this is equivalent to connecting up tetrahedrons at their faces, where each tetrahedron is associated with four sites (see Fig. 4). In this way, it is seen that every 3D DT simplicial manifold is a connected vacuum diagram of (10). Unfortunately there are other connected vacuum diagrams such as the one shown in Fig. 5. It is easy to see that there is no way to associate tetrahedrons with the vertices, and triangles with the links of Fig. 5 in order to obtain a DT simplicial manifold which satisfies rules (i), (ii), and (iii). In 2D such "illegal" diagrams were tadpoles and self-energy insertions and, therefore, could be removed from the tensor model by the addition of counterterms. Hence there was good

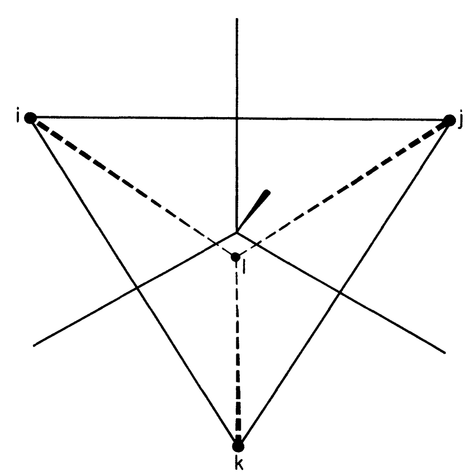

FIG. 4. Vertex of the tensor model for three dimensions, and its dual.

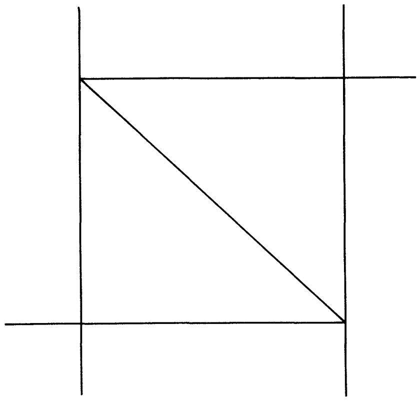

FIG. 5. An example of a portion of a Feynman diagram of the tensor model (10), which does not correspond to a legal 3D DT simplicial manifold.

reason to believe that the critical behavior of the DT and tensor models would be the same. Eventually numerical evidence arrived in support of that belief.[9] In higher dimensions, however, the situation appears slightly more obscure. One may argue that including in the path integral for quantum gravity a sum over the additional "degenerate" manifolds generated by (10) will likely not change the universality class, if indeed there is any universal behavior at all. But detailed comparisons between the DT and tensor models in higher dimensions must be carried out to test this hypothesis.

One can easily generalize (10) to d dimensions. The number of indices of the tensors becomes d (the number of sites in a face of a d-simplex) and the number of factors in the interaction term becomes $d+1$ (the number of faces in a d-simplex). Each perturbative diagram comes with a factor of $g/\sqrt{N}$ for each d-simplex and a factor of N for each site (from traces over the indices) resulting in a weight for each diagram of

$$w=\left(\frac{g}{\sqrt{N}}\right)^{N_d} N^{N_0} . \tag{11}$$

Let us now restrict our attention to diagrams which form simplicial manifolds in the sense of (i), (ii), and (iii). As stated above, it is hoped that this restriction will not change the continuum behavior of the theory. For these diagrams there are relations between the N_n which allow us to eliminate some of them.[6] The result is that the weight of each simplicial manifold generated by the tensor model is

$$\begin{aligned} w&=g^{N_2}N^{\chi}, \quad d=2\,, \\ w&=g^{N_3}N^{\chi+N_1-3N_3/2}, \quad d=3\,, \\ w&=g^{N_4}N^{\chi+N_2/2-3N_4/2}, \quad d=4\,, \end{aligned} \tag{12}$$

where

$$\chi=\sum_{n=0}^{d}(-1)^n N_n \tag{13}$$

is the topologically invariant Euler number *which is 0 in odd dimensions.* The action corresponding to these weights is given by $w=e^{-S}$. Thus we arrive at the surprising result that in 3D the action for the simplicial manifolds generated by the tensor model is precisely the sum of an Einstein term, a cosmological-constant term and nothing else. In the notation of (6),

$$\begin{aligned} g_1&=\tfrac{1}{2}\ln N-\ln g, \quad g_2=-\ln N, \quad g_3=0 \quad \text{for } d=2\,, \\ g_1&=\tfrac{3}{2}\ln N-\ln g, \quad g_2=-\ln N, \quad g_3=0 \quad \text{for } d=3\,. \end{aligned} \tag{14}$$

The 4D model, on the other hand, generates an Einstein term, a cosmological-constant term, and a topological term, i.e.,

$$g_1=\tfrac{3}{2}\ln N-\ln g, \quad g_2=-\tfrac{1}{2}\ln N \quad \text{for } d=4\,, \tag{15}$$

and instead of an R^2 term in the action we obtain a term $g_\chi \chi$ where $g_\chi=-\ln N$ and in four dimensions χ can be written as

$$\chi=\frac{1}{32\pi^2}\int d^4x\sqrt{g}\,(R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2) \quad \text{for } d=4\,. \tag{16}$$

In terms of the Einstein coupling and cosmological constant in the continuum action (1),

$$\begin{aligned} k&=\frac{\ln N}{2\pi}, \quad \lambda=\frac{1}{V_2}(-\ln g) \quad \text{for } d=2\,, \\ k&=\frac{\ln N}{2\pi a}, \quad \lambda=\frac{1}{V_3}(0.3245\ln N-\ln g) \quad \text{for } d=3\,, \\ k&=\frac{\ln N}{\pi\sqrt{3}a^2}, \quad \lambda=\frac{1}{V_4}(0.4511\ln N-\ln g) \quad \text{for } d=4\,. \end{aligned} \tag{17}$$

It remains to be seen whether the continuum limits of the $d=3$ and 4 theories can be sensibly defined as in two dimensions.[3]

Conclusion. We have generalized the dynamical triangulation model to arbitrary dimension and proposed an updating scheme for $d=3$. We discussed the related tensor models and calculated the action which corresponds to them in $d=3$ and 4. It appears the foundation is now in place for analytic and numerical studies of higher-dimensional quantum gravity in the DT- and tensor-model formalisms.

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[12]We claim no credit for Eq. (10), which was presented to us by H. Ooguri. Versions have also been considered by J. Ambjorn, M. Douglas, N. Sasakura, and perhaps others.