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ARTICLES

Constraints on the scalar-field potential in inflationary models

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In this paper, we quantify the degree of fine tuning required for successful inflationary scenarios. We define a "fine-tuning" parameter λ to be the ratio of the change in the potential ΔV to the change in the scalar field $(\Delta\psi)^4$; i.e., λ measures the required degree of flatness in the potential. For a quartic polynomial potential, the quartic coupling constant λ_q is bounded by $|\lambda_q| \lesssim 36\lambda$. For a general class of inflationary models involving a slowly rolling field, we find that the potential must be very flat, with a fine-tuning parameter $\lambda \lesssim 10^{-6}$–10^{-8}. The recently proposed "extended" inflationary scenario is even more tightly constrained, with $\lambda \lesssim 10^{-15}$.

I. INTRODUCTION

The inflationary universe model[1] was proposed to solve several cosmological problems, including the horizon problem, the flatness problem, and the monopole problem. During the inflationary epoch, the energy density of the Universe is dominated by a (nearly constant) false-vacuum energy term $\rho \simeq \rho_{\rm vac}$, and the scale factor of the Universe expands exponentially:

$$H^2 = 8\pi G\rho/3 \Longrightarrow R(t) = R(t_i)e^{\chi(t-t_i)}, \tag{1.1}$$

where $H = \dot{R}/R$ is the Hubble parameter, R is the scale factor of the Universe, $R(t_i)$ is the scale factor at the beginning of inflation, and χ ($\approx H$ during the inflationary epoch) is given by

$$\chi = \sqrt{8\pi G\rho_{\rm vac}/3}\ . \tag{1.2}$$

During this period of exponential expansion, a small causally connected region inflates to become large enough to encompass the entire observed Universe, and presumably much more. The tremendous expansion can explain the observed homogeneity and isotropy of the Universe, it can dilute the overdensity of magnetic monopoles predicted by many particle theories, and it predicts a geometrically flat (i.e., $k=0$) universe. A successful resolution to these cosmological problems requires that the scale factor must increase by at least a factor of 10^{26}, i.e., at least 60 e-foldings must occur. The period of exponential expansion must be followed by a period of thermalization, in which the vacuum energy density is converted to radiation.

In the original inflationary model,[1] the Universe supercools to a temperature $T \ll T_c$ during a first-order phase transition with critical temperature T_c. The nucleation rate for bubbles of true vacuum must be slow enough that the Universe remains in the metastable false vacuum long enough for the required ~ 60 e-foldings of the scale factor. Unfortunately, the old inflationary scenario has been shown to fail[2] because the interiors of expanding spherical bubbles of true vacuum cannot thermalize properly and produce a homogeneous radiation-dominated universe after the inflationary epoch.

Soon after the problems of the original inflationary scenario were discovered, they were overcome by the development of the new inflationary universe.[3,4] In this model it is assumed that the diagram of the effective potential (or free energy) of the inflation field ψ has a very flat plateau, and the field evolves by "slowly rolling" off the plateau. In this case, the phase transition can be second order or only weakly first order. Assuming that the scalar field in some sufficiently large region of space (approximately horizon size) settles into a state that is not too inhomogeneous and for which the average value of ψ is near the peak of the plateau, then inflation will begin. The metric in this region will (locally) approach the de Sitter form, and then the evolution of ψ is determined by the equation of motion

$$\ddot{\psi} + 3H\dot{\psi} + \Gamma\dot{\psi} + \frac{dV}{d\psi} = 0\ . \tag{1.3}$$

Here we have neglected the spatial gradient term because it is suppressed by e^{-2Ht} during the inflationary epoch. In this "slowly rolling" regime of growth, the energy density of the Universe is dominated by the vacuum contribution ($\rho \simeq \rho_{\rm vac} \gg \rho_{\rm rad}$) and the Universe expands exponentially; the $\Gamma\dot{\psi}$ term is usually unimportant during this slowly rolling phase of evolution. It is generally also true that the $\ddot{\psi}$ term is negligible, at least in the early stages of the slowly rolling regime. This means that the motion is overdamped, and is controlled entirely by the force term $(dV/d\psi)$ and the viscous damping term $(3H\dot{\psi})$. The neglect of $\ddot{\psi}$ reduces the second-order equation (1.3) to a first-order equation, resulting in solutions $\psi(t)$ for which the only undetermined parameter corresponds to a translation of the time variable. The field approaches the minimum of the potential (i.e., the true vacuum) and then oscillates about it, while the $\Gamma\dot{\psi}$ term gives rise to particle and entropy production. In this manner, a "graceful exit" to inflation is achieved.

Since the development of new inflation, a number of other models have been proposed, including the chaotic inflationary model of Linde,[5] the extended inflationary model of La and Steinhardt,[6] the hyperextended model of Steinhardt and Accetta,[7] and the double-field inflationary model of Adams and Freese.[8] Like new inflation, all of these models utilize a slowly rolling field.

The extended inflationary model[6] revives some of the aspects of the "old" inflationary models in that the inflation takes place at a supercooled first-order phase transition. The essential difference is that gravity is described not by general relativity, but by Brans-Dicke[9] theory, in which the scalar curvature $\mathcal{R}$ of gravity is coupled to a scalar field. This leads to a power-law behavior for the expansion of the scale factor, circumventing the problems of old inflation. However, studies[10,11] of bubble nucleation, collisions, and percolation place significant restrictions on the allowed range of parameters for the model. Hyperextended inflation is a generalization that utilizes more complicated couplings of the rolling field to gravity.[7] In both models, the field which couples (nonminimally) to gravity evolves through a slowly rolling equation of motion.

All known versions of inflation with slowly rolling fields tend to overproduce density fluctuations, unless the potential for the slowly rolling field is chosen very carefully. In particular, these models predict[12] density fluctuations with amplitudes given by

$$\left.\frac{\delta\rho}{\rho}\right|_{\rm hor} \simeq 0.1\frac{H^2}{\dot{\psi}}, \tag{1.4}$$

where $(\delta\rho/\rho)|_{\rm hor}$ is the amplitude of a density perturbation when its wavelength crosses back inside the horizon (more precisely, the Hubble distance) after inflation, and the right-hand side is evaluated at the time when the fluctuation crossed outside the Hubble distance during inflation. While this formula was derived originally in the context of new inflation, it applies to any model in which the end of the inflationary epoch is in some way triggered by a slowly rolling field ψ. The quantum fluctuations in the motion of the field ψ cause the hypersurface of the phase transition to be nonuniform, resulting eventually in density perturbations with magnitude given by Eq. (1.4). In the case of extended or hyperextended inflation, or double-field inflation, the perturbations of Eq. (1.4) would be superimposed with perturbations caused by the collisions of bubbles, so the expression in Eq. (1.4) would be a lower limit on the full spectrum of perturbations.

The allowable amplitude of these density perturbations is highly constrained by measurements of the isotropy of the microwave background, which indicate that

$$\left.\frac{\delta\rho}{\rho}\right|_{\rm hor} \le \delta \approx 5\times 10^{-5} \tag{1.5}$$

on scales of cosmological interest.[13] For all models of inflation with which we are familiar, the coupled constraints that the Universe must inflate sufficiently and that the density perturbations must be sufficiently small require the potential $V(\psi)$ to be very flat (see, e.g., the paper by Steinhardt and Turner[14]).

In this paper we attempt a systematic, quantitative study of the fine tuning necessary for any model of inflation that utilizes a slowly rolling field. Specifically, we will derive some general bounds on a "fine-tuning parameter" λ that we define by

$$\lambda \equiv \frac{\Delta V}{(\Delta\psi)^4}\ , \tag{1.6}$$

where ΔV is the decrease in the potential $V(\psi)$ during the inflationary epoch (or some specified part of it), and $\Delta\psi$ is the change in the value of the field ψ over the same period. The parameter λ is thus the ratio of the height of the potential to its width (for that part of the potential involved in the specified time period), so it measures the degree of flatness of the potential. We show in Appendix A that if the potential is a quartic polynomial, then a bound on λ (for any time period) can be used to place a bound on the usual quartic coupling constant λ_q; specifically, specifically,

$$|\lambda_q| \le 36\lambda\ , \tag{1.7}$$

where the quartic term in the Lagrangian is written as $\frac{1}{4}\lambda_q\psi^4$.

While we refer to λ as a "fine-tuning" parameter, we do not mean to suggest that a very small value of λ implies that any particular model is unacceptable. The reader will of course draw his/her own conclusions, but in our opinion it is reasonable to hope that the required small value of λ might some day be explained by a deeper understanding of the underlying particle physics.

The bulk of the paper will concern inflation involving one or more scalar fields that are not coupled to gravity, and which satisfy two conditions. First, we will assume that the evolution during the relevant time period satisfies the density perturbation constraint of Eqs. (1.4) and (1.5), leading to

$$H^2/\dot{\psi} \le 10\delta\ . \tag{1.8}$$

Second, we will assume that during the early stages of inflation, the $\ddot{\psi}$ term of Eq. (1.3) is negligible (along with the $\Gamma\dot{\psi}$ term), so that the evolution of ψ is overdamped. This leads to the equation of motion

$$3H\frac{d\psi}{dt}=-\frac{dV}{d\psi} . \tag{1.9}$$

The consistency of neglecting the $\ddot{\psi}$ term then leads to a constraint on the potential,

$$\left|\frac{d}{dt}\left[\frac{1}{3H}\frac{dV}{d\psi}\right]\right|\leq\left|\frac{dV}{d\psi}\right| , \tag{1.10}$$

which we will call the overdamping constraint. [Equation (1.10) is sometimes called the "slowly rolling" condition, but we prefer to avoid this phrase because it suggests a constraint on $\dot{\psi}$ rather than $\ddot{\psi}$. While it would be reasonable to assume that the left-hand side of Eq. (1.10) is much less ($\ll$) than the right-hand side, we will rely only on the weaker ($\leq$) constraint as shown. Note also that Eq. (1.10) is a *necessary* but not a *sufficient* condition for the negligibility of $\ddot{\psi}$—even if Eq. (1.10) were written with a $\ll$ symbol, it would not by itself imply that the $\ddot{\psi}$ term could be neglected in Eq. (1.3).] Using these two constraints, we will establish general bounds on the fine-tuning parameter λ, defined for the time period during which the constraints are valid. In addition, we will derive bounds on λ for the case of extended inflation.

This paper is organized as follows. In Sec. II we formulate the problem for what we call standard inflationary models: models involving any number of scalar fields that are not coupled to gravity, and that obey the density perturbation and overdamping constraints. This includes the standard versions of both new inflation and chaotic inflation. In this section we define notation and transform the problem into a mathematically convenient form. In Sec. III we complete the derivation by finding bounds on the fine-tuning parameter λ. We consider first the general case of scalar fields satisfying the density perturbation and overdamping constraints. We then consider the special case of a constant Hubble parameter H, such as in the double field model, for which a stronger bound can be derived. In Sec. IV we find upper limits on λ for the case of extended inflation. We then conclude (in Sec. V) with a discussion of our results and possible directions for future work.

II. FORMULATION OF THE PROBLEM FOR STANDARD INFLATIONARY MODELS

In this paper, we will derive a set of upper bounds on the degree of fine tuning required for a fairly general class of inflationary scenarios which utilize a slowly rolling field ψ. The degree of fine tuning will be measured by the parameter λ, defined by Eq. (1.6). In this section we will consider an inflationary scenario involving an arbitrary number of scalar fields that are not coupled to gravity, and which satisfy the density perturbation constraint of Eq. (1.8) and the overdamping constraint of Eq. (1.10). We will assume that both constraints hold for a period of N *e*-foldings.

In general, we want the density perturbation constraint to apply for physical size scales (at the present epoch) in the range 3000 Mpc (the horizon size) down to about 1 Mpc (the size scale corresponding to a galactic mass). This range spans a factor of 3000 in physical size and corresponding to $N=\ln(3000)\approx 8$ *e*-foldings of the inflationary epoch.[15] Thus, the physical requirements of inflation imply that $N\approx 8$.

The relevant time variable for an inflationary epoch is the number n of *e*-foldings since the beginning of the epoch. We will therefore adopt a new time variable x defined by

$$dx\equiv\frac{dn}{N}=\frac{Hdt}{N} . \tag{2.1}$$

The variable x thus ranges from 0 to 1 during the relevant time period. We will also introduce the notation

$$F(x)\equiv-\frac{dV}{d\psi} , \tag{2.2}$$

where the letter F is chosen to suggest a force.

In the newly defined notation, the overdamping constraint is written as

$$\left|H\frac{d}{dx}\left[\frac{F}{H}\right]\right|\leq 3NF , \tag{2.3}$$

and the density perturbation constraint is

$$3H^3/F\leq 10\delta . \tag{2.4}$$

Furthermore, the quantities ΔV and $\Delta\psi$ can be written as

$$\Delta V=\frac{N}{3}\int_0^1(F^2/H^2)dx , \tag{2.5a}$$

$$\Delta\psi=\frac{N}{3}\int_0^1(F/H^2)dx . \tag{2.5b}$$

We have chosen our sign convention so that ΔV is a positive quantity and so that $x=0$ at the beginning of the constrained time period.

Since it is convenient to deal with dimensionless quantities, we rescale the functions $F(x)$ and $H(x)$. To do this, we let $\bar{x}$ denote the value of x in the range [0,1] such that the quantity H^3/F, which appears in the density perturbation constraint, is maximized. (If the maximum is not unique, we can choose $\bar{x}$ to be any of the maxima.) We then introduce the dimensionless functions

$$f(x)\equiv F(x)/\bar{F} , \tag{2.6a}$$

$$h(x)\equiv H(x)/\bar{H} , \tag{2.6b}$$

where the overbar refers to the value of the function at $x=\bar{x}$. Using the definition (1.6) of λ with Eqs. (2.5) and (2.6), one has

$$\lambda=\frac{3}{N^3}\left[\frac{3\bar{H}^3}{\bar{F}}\right]^2 J[f,h] , \tag{2.7}$$

where

$$J[f,h]\equiv\frac{\int_0^1(f^2/h^2)dx}{\left[\int_0^1(f/h^2)dx\right]^4} . \tag{2.8}$$

But the density perturbation constraint implies that $3\bar{H}^3/\bar{F}\leq 10\delta$, and so

$$\lambda \le \frac{300\delta^2}{N^3} J[f,h] \ . \tag{2.9}$$

Notice that λ is proportional to δ^2 and inversely proportional to N^3 (although the limits on the functional J also depend on N—see Sec. III).

In order to derive an upper bound on the fine-tuning parameter, we must derive an upper bound on the functional $J[f,h]$, subject to the constraints

$$h^3(x)/f(x) \le 1 \quad \forall x \in [0,1] \tag{2.10a}$$

and

$$\left| \frac{1}{f}\frac{df}{dx} - \frac{1}{h}\frac{dh}{dx} \right| \le 3N \quad \forall x \in [0,1] \ . \tag{2.10b}$$

The first of these inequalities is related to the density perturbation constraint, but more precisely it follows from the rescaling of the functions relative to the maximum value of H^3/F. The second inequality is the overdamping constraint. We also know that there exists a point $\bar{x} \in [0,1]$ such that

$$f(\bar{x}) = h(\bar{x}) = 1 \ . \tag{2.10c}$$

Notice that for the fiducial case of a constant Hubble parameter ($h=1$) and a linear potential (so that $f=1=$const), we have $J=1$. For this case, the fine-tuning parameter $\lambda \le 300\delta^2 N^{-3}$; i.e., the parameter depends on the square of the density perturbation limit δ and the inverse cube of the number N of e-foldings. In order to obtain numerical values for our limits, we will consider a representative case in which $\delta \approx 5\times 10^{-5}$ and $N \approx 8$; the resulting fiducial value of λ is $\sim 10^{-9}$.

III. CONSTRAINTS ON STANDARD INFLATIONARY MODELS

Using the formulation of the problem given above, we will now find upper limits to the fine-tuning parameter. We will first consider the general problem, in which both f and h are arbitrary functions to be chosen independently. We will then consider the special case of a constant Hubble parameter ($h=1$), for which a somewhat stronger bound can be derived.

A. The general problem

The fine-tuning parameter λ is bounded by Eq. (2.9), so we must derive an upper bound for the functional $J[f,h]$ of Eq. (2.8), subject to the constraints of Eqs. (2.10).

It is convenient to define

$$p(x) \equiv \left[\frac{f(x)}{h(x)} \right]^{1/2} , \tag{3.1}$$

leading to the equivalent problem of finding an upper bound on

$$J[p,h] = \frac{\int_0^1 p^4 dx}{\left[\int_0^1 (p^2/h) dx \right]^4} , \tag{3.2}$$

subject to the density perturbation constraint

$$p(x) \ge h(x) \tag{3.3a}$$

and the overdamping constraint

$$\left| \frac{1}{p}\frac{dp}{dx} \right| \le \tfrac{3}{2} N \ . \tag{3.3b}$$

In addition, we know that there exists a point $\bar{x} \in [0,1]$ such that

$$p(\bar{x}) = h(\bar{x}) = 1 \ . \tag{3.3c}$$

In order to maintain some physical intuition, one should remember that $p \propto \sqrt{d\psi/dt}$.

One can see immediately that the functional $J[p,h]$ is maximized by choosing $h(x)=p(x)$ to saturate the density perturbation constraint (3.3a):

$$J[p,h] \le K[p] \equiv \frac{\int_0^1 p^4 dx}{\left[\int_0^1 p \, dx \right]^4} \ . \tag{3.4}$$

The maximization of $K[p]$ is reasonably intuitive, since one can think of $p(x)$ as a probability distribution. In this language one is trying to maximize the fourth moment of the distribution while keeping the normalization fixed. It is well known that this is accomplished by choosing the distribution to be as nonuniform as possible, and in this case the limit on nonuniformity is imposed by the constraint (3.3b). Thus, one expects the optimum to be a function which is as sharply peaked as the overdamping constraint allows. We will show that this expectation is correct.

Formally, one can attempt to maximize $K[p]$ by calculating its functional derivative, initially ignoring the constraint (3.3b). One finds

$$\frac{\delta K}{\delta p(x)} = \frac{4}{\langle p \rangle^5} [\langle p \rangle p^3(x) - \langle p^4 \rangle] , \tag{3.5}$$

where we have used the notation

$$\langle \cdots \rangle \equiv \int_0^1 \cdots dx \ . \tag{3.6}$$

Thus, the functional K can be increased by increasing $p(x)$ for any x such that $p(x) > p_m$ and/or lowering $p(x)$ for any x such that $p(x) < p_m$, where

$$p_m = \left[\frac{\langle p^4 \rangle}{\langle p \rangle} \right]^{1/3} . \tag{3.7}$$

Thus, a stationary point of $K[p]$ can be found by setting $p(x)=$const, from which it follows that $p(x)=p_m$. This stationary point is consistent with the constraint (3.3b), but it is easily shown that it is a minimum and not a maximum. To see this, we can calculate the second-order variation of K:

$$K[p(x)=p_m+\delta p(x)] = 1 + 6\frac{\langle \delta p^2 \rangle - \langle \delta p \rangle^2}{p_m^2} + O(\delta p^3) \ . \tag{3.8}$$

The second-order variation is therefore positive unless the variance of $\delta p(x)$ vanishes; this happens only if $\delta p(x)=$const, in which case the functional K is unchanged to all orders.

Since the vanishing of $\delta K/\delta p(x)$ leads to a minimum but not a maximum of $K[g]$, we next look at the boundaries of the allowed region of function space. We consider therefore a trial function that saturates the constraint (3.3b):

$$p(x)=p_*(x)\equiv p_0 e^{-3Nx/2} , \tag{3.9}$$

where p_0 is an arbitrary constant. (Note that $p=p_0e^{+3Nx/2}$ would also saturate the constraint, and that it would give the same value for the functional $K[p]$. Thus there is no need to consider both the growing and decaying exponentials.) To see if $p_*(x)$ might maximize $K[p]$, we consider a small variation

$$p(x)=p_*(x)+\delta p(x) . \tag{3.10}$$

We are interested only in variations that satisfy the constraint (3.3b), which implies that

$$\frac{d}{dx}[e^{3Nx/2}\delta p(x)]\geq 0 . \tag{3.11}$$

The variation of $K[p]$ can then be calculated by using Eq. (3.5):

$$K[p_*+\delta p]=K[p_*]+\int_0^1\frac{\delta K}{\delta p(x)}[p_*]\delta p(x)dx , \tag{3.12}$$

where

$$\frac{\delta K}{\delta p(x)}[p_*]=\frac{81N^4}{16p_0(1-e^{-3N/2})^5}\times[4(1-e^{-3N/2})e^{-9Nx/2}-(1-e^{-6N})] . \tag{3.13}$$

To see the role of the constraint (3.11), we can integrate by parts to get

$$K[p_*+\delta p]=K[p_*]-\int_0^1 dx\, W(x)\frac{d}{dx}[e^{3Nx/2}\delta p(x)] , \tag{3.14}$$

where

$$\begin{aligned} W(x)&=\int_0^x\frac{\delta K}{\delta p(x')}e^{-3Nx'/2}dx' \\ &=Z[(1-e^{-3N/2})(1-e^{-6Nx}) \\ &\quad -(1-e^{-6N})(1-e^{-3Nx/2})] , \end{aligned} \tag{3.15}$$

and

$$Z=\frac{27N^3}{8p_0(1-e^{-3N/2})^5} . \tag{3.16}$$

Note that $W(0)=W(1)=0$, so there are no surface terms in Eq. (3.14). It is shown in Appendix B that $W(x)\geq 0$, so one can see from Eqs. (3.11) and (3.14) that $K[p]$ is decreased by any allowed variation, and hence $p_*(x)$ is at least a local maximum.

It is a little harder to show that $p_*(x)$ provides a global maximum for $K[p]$, so we relegate this derivation to Appendix C. Making use of the result, however, we have

$$J[p,h]\leq K[p_*]=\frac{27N^3}{32}\left[\frac{1-e^{-6N}}{(1-e^{-3N/2})^4}\right] . \tag{3.17}$$

Since we are interested in reasonably large values of N, the factor in large parentheses can be approximated by 1. For the case of $N=8$, for example, this results in an error of relative magnitude 2.5×10^{-5}. With this level of accuracy one could combine Eq. (3.17) with Eq. (2.9) to obtain

$$\lambda\leq\frac{2025}{8}\delta^2\approx 6.3\times10^{-7} , \tag{3.18}$$

where we choose $\delta=5\times10^{-5}$ to obtain the numerical value.

Since we have gone through a complicated series of transformations in order to simplify the problem, it is not obvious what form the physical functions should have in order to maximize λ. It is therefore instructive to transform the optimal solution back to the original notation. After some algebra one finds that the optimal potential is simply

$$V(\psi)=\tfrac{1}{4}\lambda_q(\psi-\psi_0)^4+V_0 , \tag{3.19}$$

where

$$\lambda_q=\frac{2025}{2}\delta^2 , \tag{3.20}$$

and ψ_0 and V_0 are arbitrary constants that may be chosen to be zero. The optimal behavior for the Hubble parameter is

$$H(t)=\frac{\tilde{H}}{1+\frac{3}{2}\tilde{H}t} , \tag{3.21}$$

where

$$\tilde{H}=(\tfrac{2}{9}\lambda_q)^{1/2}M , \tag{3.22}$$

and M is an arbitrary constant with the units of mass. The beginning and ending times of the inflationary period are given by $t_B=0$ and

$$t_E=\frac{2}{3\tilde{H}}(e^{3N/2}-1) . \tag{3.23}$$

During this time the scalar field ψ evolves according to

$$\psi(t)=-\frac{M}{1+\frac{3}{2}\tilde{H}t}+\psi_0 . \tag{3.24}$$

Notice that the potential in Eq. (3.19) is typical of a chaotic inflationary model, but chaotic inflation predicts a time dependence of the Hubble parameter that is different from Eq. (3.21).

B. Constraint with constant Hubble parameter

We now consider the restricted problem in which the Hubble parameter is constant during the inflationary epoch, i.e.,

$$h(x)=1 \quad \forall x\in[0,1] . \tag{3.25}$$

This form of the problem is applicable, for example, to the model of "double field inflation" which has recently been proposed.[8] In this case we will not solve the optimization problem completely, but instead we will derive a bound on λ.

For this case the functional J can be written

$$J[f]=\frac{\int_0^1 f^2 dx}{\left[\int_0^1 f\,dx\right]^4} . \tag{3.26}$$

Since f is a bounded function, it must attain a maximum value f_* over the interval [0,1]. Since f is also positive definite, we can bound the numerator in Eq. (3.26) by

$$\int_0^1 f^2 dx \le f_* \int_0^1 f\,dx , \tag{3.27}$$

which implies an upper bound on the functional J of the form

$$J[f]\le \frac{f_*}{\left[\int_0^1 f\,dx\right]^3} . \tag{3.28}$$

We can now find an upper limit to the functional $J[f]$ by placing a lower limit on the functional

$$L[f]\equiv \int_0^1 f\,dx . \tag{3.29}$$

We first use the constraint [Eq. (2.10a)] from density perturbations that $f \ge 1$, so

$$L[f]\ge 1 . \tag{3.30}$$

Now let x_* denote the value of x where $f(x)$ attains its maximum value f_*. Since the function $f(x)$ cannot decrease faster than allowed by the overdamping constraint (2.10b), the integral in Eq. (3.29) must obey the bound

$$\int_0^1 f\,dx \ge f_* \left[\int_0^{x_*} dx\, e^{-3N(x_*-x)} + \int_{x_*}^1 dx\, e^{-3N(x-x_*)}\right] = \frac{f_*}{3N}\left[2-e^{-3Nx_*}-e^{-3N(1-x_*)}\right] . \tag{3.31}$$

Using elementary calculus to show that the function above attains its minimum value when $x_*=0$ or $x_*=1$, one has

$$L[f]\ge \frac{f_*}{3N}(1-e^{-3N})\approx \frac{f_*}{3N} . \tag{3.32}$$

In using the approximate equality, we are being slightly imprecise. In most cases of interest, however, $N\gtrsim 8$, so the neglected term is $\lesssim 10^{-10}$.

We now have two lower limits on the functional $L[f]$ [see Eqs. (3.30) and (3.32)]. Both of these limits must apply for any value of f_*. These lower limits on $L[f]$ imply the following upper limits on $J[f]$:

$$J[f]\le f_* \quad \text{and} \quad J[f]\le \frac{27N^3}{f_*^2} . \tag{3.33}$$

A bound independent of f_* can be obtained by multiplying the square of the first inequality by the second, and then taking the cube root:

$$J[f]\le 3N . \tag{3.34}$$

We now have the desired upper limit on the fine-tuning parameter:

$$\lambda \le \frac{900\delta^2}{N^2}\approx 3.5\times 10^{-8} , \tag{3.35}$$

where the approximate equality corresponds to the choice of parameters $N=8$ and $\delta=5\times 10^{-5}$. Note that this bound is about a factor of 20 more stringent than the bound obtained in the general case.

IV. EXTENDED INFLATION

For the case of extended inflation,[6,11] the action of the theory has the form

$$A=A(\psi,\phi)=\int d^4x\sqrt{-g}\left[\frac{1}{8b}\psi^2\mathcal{R}-\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - V(\psi)+\mathcal{L}(\phi)\right] , \tag{4.1}$$

where $1/8b$ is the nonminimal coupling coefficient,[16] $\mathcal{R}$ is the scalar curvature, ψ is the rolling field that couples to gravity, and ϕ is the matter field(s) that drives the inflation. For internal consistency we are using a notation different from the original authors—our ψ corresponds to the ϕ of the original papers, and our ϕ corresponds to the original σ. In addition, we have chosen to use a standard normalization for both the ψ terms in the Lagrangian and for $\mathcal{L}(\phi)$, so our ψ is equal to $\phi/\sqrt{16\pi}$ in the original notation. (Ref. 17, on the other hand, uses the same normalization that we do.) Since the value of the fine-tuning parameter λ depends on the normalization of ψ, it is important to us that the normalization is standard.

Notice that the bounds for the fine-tuning parameter λ derived in the previous sections do not apply in this case, since the coupling of ψ to gravity gives it an equation of motion more complicated than Eqs. (1.3) or (1.9). We do not have a general argument that covers cases of this type, but we can still place a bound on λ for the extended model by means of an explicit, model-dependent calculation.

For extended inflation, the time dependence of the field ψ and the scale factor R can be solved directly.[6] Let

$$\bar{H}\equiv\left[\frac{8\pi}{3}\right]^{1/2}\frac{M_F^2}{m_P} , \tag{4.2}$$

where m_P is the effective Planck mass at the beginning of the inflationary epoch (taken as $t\equiv 0$), and M_F^4 is the energy density of the σ-field false vacuum. Then the relevant solution can be written

$$\psi(t)=\left[\frac{b}{2\pi}\right]^{1/2} m_P\left[1+\frac{\bar{H}t}{\alpha}\right] , \tag{4.3a}$$

$$R(t)=(1+\bar{H}t/\alpha)^{b+1/2} , \tag{4.3b}$$

where $\alpha\equiv\sqrt{(3+2b)(5+6b)/12}$.

Before going on, we write down several relations for future reference. From Eqs. (4.3) it follows that R is related to ψ by

$$R=\left[\left[\frac{2\pi}{b}\right]^{1/2}\frac{\psi}{m_P}\right]^{b+1/2} . \tag{4.4}$$

Since ψ determines the temporary value of the Planck mass, we can write

$$\psi(0)=\sqrt{b/2\pi}\,m_P , \tag{4.5a}$$

$$\psi(t_e)=\sqrt{b/2\pi}\,M_E , \tag{4.5b}$$

where t_e denotes the time at the end of the inflationary epoch, and M_E is the value of the Planck mass at that time. M_E is usually assumed to be near the present value of the Planck mass M_P, but in this calculation we will allow for the possibility that it might be significantly different. (Such a difference would occur if the field ψ continued to evolve after the end of inflation, settling into the minimum of its potential at a later time.) Combining Eqs. (4.4) and (4.5), one has

$$\frac{R(t_e)}{R_B}=\left[\frac{M_E}{m_P}\right]^{b+1/2} , \tag{4.6}$$

where $R_B\equiv 1$ denotes the value of the scale factor at the beginning of inflation ($t=0$). It will also be useful to have an expression for the Hubble parameter:

$$H\equiv\frac{\dot{R}}{R}=\frac{\kappa\overline{H}}{1+(\overline{H}t/\alpha)}=\frac{\kappa\overline{H}}{R^{1/(b+1/2)}} , \tag{4.7}$$

where

$$\kappa\equiv\frac{b+1/2}{\alpha} . \tag{4.8}$$

We now want to invoke the condition of sufficient inflation in order to determine the required number of e-foldings during the inflationary epoch. Following Ref. 11, the condition of sufficient inflation can be written

$$\frac{1}{H_0R_0}\le\frac{1}{H_BR_B} , \tag{4.9}$$

where $H_B=\kappa\overline{H}$ is the value of the Hubble parameter at the beginning of inflation, and H_0 and R_0 are the values of the Hubble parameter and scale factor, respectively, at the present epoch. If we assume that the Universe has evolved adiabatically since the end of inflation, then

$$\frac{R(t_e)}{R_0}\approx\frac{T_0}{M_F} , \tag{4.10}$$

and the condition of sufficient inflation becomes

$$\frac{R(t_e)}{R_B}\ge\frac{H_B}{H_0}\frac{T_0}{M_F} . \tag{4.11}$$

We then write the Hubble parameters as

$$H_B^2=\frac{8\pi}{3}\frac{\kappa^2M_F^4}{m_P^2}\quad\text{and}\quad H_0^2=\frac{8\pi}{3}\frac{\beta^2T_0^4}{M_P^2} , \tag{4.12}$$

where we have written the present energy density of the Universe as $\beta^2T_0^4$. Parametrizing the present Hubble parameter and radiation temperature by

$$H_0=h_0\times 100\ \text{km sec}^{-1}\text{Mpc}^{-1}=2.13h_0\times 10^{-42}\ \text{GeV} , \tag{4.13a}$$

$$T_0=\tau_0\times 2.74\ \text{K}=2.36\tau_0\times 10^{-13}\ \text{GeV} , \tag{4.13b}$$

we will take $h_0=\frac{1}{2}$ and $\tau_0=1$ to obtain

$$\beta\approx 162h_0/\tau_0^2\approx 80 . \tag{4.14}$$

We thus obtain the condition of sufficient inflation:

$$\frac{R(t_e)}{R_B}\ge\frac{\kappa M_F}{\beta T_0}\frac{M_P}{m_P} . \tag{4.15}$$

(This result agrees with Ref. 11, except that the authors of Ref. 11 approximated β and κ by unity.)

Combining Eqs. (4.6) and (4.15), one has

$$e^N\equiv\frac{R(t_e)}{R_B}\ge\left[\frac{\kappa M_F}{\beta T_0}\frac{M_P}{M_E}\right]^{(b+1/2)/(b-1/2)} . \tag{4.16}$$

Using Eq. (4.4) to relate the ratio of scale factors to the ratio of scalar fields, the condition for sufficient inflation can be rewritten as

$$\frac{\psi(t_e)}{\psi(0)}\ge\left[\frac{\kappa M_F}{\beta T_0}\frac{M_P}{M_E}\right]^{1/(b-1/2)} . \tag{4.17}$$

Combining this bound with Eq. (4.5a), one has

$$\Delta\psi\ge\left[\frac{b}{2\pi}\right]^{1/2}m_P\left[\left[\frac{\kappa M_F}{\beta T_0}\frac{M_P}{M_E}\right]^{1/(b-1/2)}-1\right] . \tag{4.18}$$

For later use we note that Eq. (4.17) can also be combined with Eq. (4.5b) to obtain

$$\Delta\psi\ge\left[\frac{b}{2\pi}\right]^{1/2}M_E\left[1-\left[\frac{\beta T_0}{\kappa M_F}\frac{M_E}{M_P}\right]^{1/(b-1/2)}\right] . \tag{4.19}$$

We must now place a bound on the change in the potential. Since ΔV is the change in the potential of the ψ field during inflation, and since the energy density is dominated by the false vacuum energy density M_F^4 of the second field ϕ, we have

$$\Delta V(\psi)\le M_F^4 . \tag{4.20}$$

We can thus express a limit on the fine-tuning parameter through

$$\lambda\le\frac{4\pi^2}{b^2}\frac{M_F^4}{m_P^4}\left[\left[\frac{\kappa M_F}{\beta T_0}\frac{M_P}{M_E}\right]^{1/(b-1/2)}-1\right]^{-4} . \tag{4.21}$$

In order to obtain a bound on λ, we must limit the range of the energy scale M_F. We could invoke the obvious limit that $M_F\le m_P$, i.e., the energy scale of the inflation field cannot exceed the Planck mass.[18] However, the vacuum energy density M_F^4 must also exceed the kinetic energy of the de Sitter space fluctuations in both ψ and ϕ. Using this constraint,[11] the bound is found to take the slightly stronger form

$$M_F^4 \leq \left[\frac{3}{8\pi}\right]^2 m_P^4 \ . \tag{4.22}$$

Combining Eqs. (4.21) and (4.22), we obtain the limit on the fine-tuning parameter

$$\lambda \leq \left[\frac{3}{4b}\right]^2 \left[\left[\frac{\kappa M_F}{\beta T_0}\frac{M_P}{M_E}\right]^{1/(b-1/2)} - 1\right]^{-4} . \tag{4.23}$$

The weakest bound will occur for the smallest allowed value of the inflation scale M_F. Assuming that baryogenesis can take place at the weak scale but not below it, we take $M_F \geq 100$ GeV. As discussed in Ref. 11, the parameter b is constrained to lie in the range $1.5 \leq b \leq 25$ in order to achieve a successful extended inflationary scenario. For the upper end of this range, $b=25$, we obtain

$$\lambda \leq 3.2\times 10^{-5} \ , \tag{4.24a}$$

where we have taken $M_E = M_P$ for the numerical example. For the lower end of the range $b=1.5$ we obtain

$$\lambda \leq 9.7\times 10^{-52} \ . \tag{4.24b}$$

If, on the other hand, one assumes that inflation takes place at the grand-unified-theory (GUT) scale of $M_F = 10^{14}$ GeV, then these bounds become stronger:

$$\begin{aligned} &\lambda \leq 1.3\times 10^{-7} \quad (\text{for } b=25), \\ &\lambda \leq 9.7\times 10^{-100} \quad (\text{for } b=1.5) \ . \end{aligned} \tag{4.25}$$

To limit ΔV we have so far used only the mild constraint that the de Sitter space quantum fluctuations in ψ and ϕ cannot contribute more to the energy density than M_F^4. A significantly stronger bound can be obtained by applying the density perturbation constraint of Eq. (1.8): $H^2/\dot{\psi} \leq 10\delta$.

For the sake of definiteness, we will evaluate the density perturbation constraint for perturbations on the scale of the present Hubble distance. This requires us to evaluate $H^2/\dot{\psi}$ at the time t_* when these perturbations crossed the Hubble distance during inflation, so t_* is determined by

$$\frac{1}{H(t_*)R(t_*)} = \frac{1}{H_0 R_0} \ . \tag{4.26}$$

To evaluate the right-hand side of this equation, use the adiabaticity condition (4.10) to eliminate R_0, and then use Eq. (4.6) to eliminate $R(t_e)$. Use Eq. (4.12) to eliminate H_0, and Eq. (4.7) to eliminate $R(t_*)$. The resulting equation can be solved for $H(t_*)$:

$$H(t_*) = \kappa\left[\frac{8\pi}{3}\right]^{1/2}\frac{M_F^2}{M_E}\left[\frac{\kappa M_F}{\beta T_0}\frac{M_P}{M_E}\right]^{1/(b-1/2)} . \tag{4.27}$$

$\dot{\psi}$ can be obtained by differentiating Eq. (4.3a), using Eq. (4.2) for $\overline{H}$. The result is

$$\dot{\psi} = \left[\frac{4b}{3}\right]^{1/2}\frac{M_F^2}{\alpha} \ , \tag{4.28}$$

an equation that holds for any time during the inflationary epoch. With the two equations above, the density perturbation constraint can be written as

$$\left.\frac{\delta\rho}{\rho}\right|_{\text{hor}} \simeq 0.1\frac{4\pi\alpha\kappa^2}{\sqrt{3b}}\frac{M_F^2}{M_E^2}\left[\frac{\kappa M_P}{\beta T_0}\frac{M_F}{M_E}\right]^{2/(b-1/2)} \leq \delta \ , \tag{4.29}$$

which in turn can be written as a bound on M_F/M_E:

$$\frac{M_F}{M_E} \leq \left[\frac{\beta T_0}{\kappa M_P}\right]^{1/(b+1/2)} \times \left\{10\frac{\sqrt{3b}}{4\pi\alpha\kappa^2}\delta\right\}^{(b-1/2)/[2(b+1/2)]} . \tag{4.30}$$

[Equation (4.29) agrees with the results of Ref. 11, except for minor differences: the earlier authors set $\beta \approx \kappa \approx 1$ and $M_E \approx M_P$, they used Eq. (1.4) with a coefficient of order 1 instead of order 0.1, and they calculated $\dot{\psi}$ without applying a correction to compensate for the nonstandard normalization that they used for ψ.] For $b=1.5$ we find $M_F < (M_E/M_P)\times 1500$ GeV, and for $b=25$ we find $M_F < (M_E/M_P)\times 3.8\times 10^{15}$ GeV.

The bound on M_F/M_E can be turned into a bound on λ by combining Eqs. (4.20) and (4.19) to obtain

$$\lambda \leq \frac{4\pi^2}{b^2}\frac{M_F^4}{M_E^4}\left[1 - \left[\frac{\beta T_0}{\kappa M_P}\frac{M_E}{M_F}\right]^{1/(b-1/2)}\right]^{-4} . \tag{4.31}$$

Since the expression on the right-hand side is monotonically increasing with M_F over the range of interest,[19] the bound on λ can be obtained by substituting the bound (4.30) for M_F/M_E into Eq. (4.31). The resulting bound is then

$$\begin{aligned} &\lambda \leq 8.8\times 10^{-16} \quad (\text{for } b=25) \ , \\ &\lambda \leq 4.1\times 10^{-63} \quad (\text{for } b=1.5) \ . \end{aligned} \tag{4.32}$$

Note that unlike the bound (4.23), this bound is independent of M_E, the value of the Planck mass at the end of inflation.

Thus the constraint on λ for extended inflation is in fact somewhat stronger than for standard inflationary models. Intuitively, the stronger constraint can be attributed to the fact that the Hubble parameter H is decreasing with time. Since the conditions at the end of inflation are essentially fixed by present conditions, the Hubble parameter in extended inflation models must be larger than normal at early times. The $H^2/\dot{\psi}$ fluctuations are therefore more difficult to control, leading to the stronger constraint.

In the calculation above, we assumed that the density perturbations in the extended inflation model could be calculated using Eq. (1.4), which is not strictly accurate when the field ψ is coupled to gravity. A valid calculation of the density perturbations can be performed by working in the Einstein conformal frame.[17] The resulting correction factor, however, is of order unity and will not significantly affect the results of this paper.

V. DISCUSSION

In this paper, we have defined a fine-tuning parameter λ that quantifies the degree of flatness required for scalar field potentials in order to achieve successful inflation. This parameter is appropriate for any inflationary scenario which involves a "slowly rolling" scalar field. In this study, we have also isolated the functional dependence of λ on the parameters of the problem [see, e.g., Eqs. (2.9), (3.18), (3.35), (4.23), (4.30), and (4.31)]. In particular, $\lambda \sim \delta^2$ for models in which the scalar fields are not coupled to gravity, so future limits on microwave background anisotropy[20] will tighten our constraints on λ as the square of $\delta \sim \Delta T/T$. For the extended inflation model λ varies as a fractional power of δ, where the power varies from 1 to 2 over the allowed range of parameters.

We find that for a variety of cases, the fine-tuning parameter is confined to be quite small. For any inflationary scenario in which the scalar fields are not coupled to gravity and satisfy the density perturbation and overdamping constraints [Eqs. (1.8) and (1.10)], we find that $\lambda \leq 6.3 \times 10^{-7}$. If the model also has a constant value of the Hubble parameter H during inflation, then $\lambda \leq 3.5 \times 10^{-8}$. We have also analyzed specifically the standard form of extended inflation and find bounds that depend sensitively on the value of the Brans-Dicke parameter[16] b. The parameter is restricted to the range[11] $1.5 \leq b \leq 25$, and we find that $\lambda \leq 4.1 \times 10^{-63}$ for the lower end of this range, and $\lambda \leq 8.8 \times 10^{-16}$ at the upper end. Thus we find that extended inflation models require a higher degree of fine tuning than standard models, and that models with small values of b require particularly large amounts of fine tuning.

Our derivation for the case of extended inflation relies on the precise form of the proposed model, and is not nearly as general as our treatment for models in which the scalar field is not coupled to gravity. Perhaps this approach can be generalized in future work. The case of hyperextended inflation is also very interesting; but since this model allows for a wide variety of non-linear couplings of the ψ field to the scalar curvature, we have not yet investigated it.[21]

As mentioned in the Introduction, we do not mean to suggest that a very small value of λ implies that any particular model is unacceptable. It is well known that there are some very small dimensionless numbers in nature: the ratio of the weak energy scale to the Planck scale is about 10^{-17}, and Yukawa coupling constant for the electron is about 10^{-6}. Since inflation seems to be the only way to avoid the horizon, flatness, and magnetic-monopole problems, it seems appropriate to hope that particle theory will one day explain whatever fine tuning of parameters is necessary for inflation to occur. In particular, we do not feel that the strong constraint on extended inflationary models is any cause to abandon them. These models represent a genuinely novel approach for the ending of inflation, and the consequences should be thoroughly pursued.

As defined in this paper, λ is a ratio of energy densities [i.e., $\Delta V/(\Delta\psi)^4$]. However, if we consider the energy *scale* as the relevant physical quantity, we should constrain the ratio of energy scales; this ratio is $\lambda^{1/4}$ and is constrained to be less than 10^{-3} or maybe 10^{-2}. Moreover, since the ratio of the GUT scale to the Planck scale is $\sim 10^{-3}$ and these two energy scales appear in the problem, the required "flatness" of inflationary potentials could be a generic outcome of physics at these energies.

Although we do not want to discard inflationary models because they require very small values of λ, we point out that some types of fine tuning might be considered preferable to others. That is, some types of fine tuning might seem more likely to be susceptible to an explanation from fundamental particle physics. For example, a model in which λ is related to fine tunings that already occur in particle physics models, such as the ratio of the GUT scale to the Planck scale, might be considered preferable to a model that requires a genuinely new fine tuning. In the model of Ref. 22, these mass scales may arise naturally in inflation with pseudo-Nambu-Goldstone bosons. Alternatively, one might consider a model which works for any λ in the range $0 \leq \lambda \leq 10^{-8}$ to be preferable to a model which requires $\lambda \approx 10^{-8}$. Hyperextended inflation has this appealing property, since inflation can apparently occur with no potential at all for the ψ field. In this case one might look for a fundamental symmetry to force the potential to vanish.

Much of the analysis in this paper has relied on the simultaneous use of the density perturbation and overdamping constraints. The two constraints, however, are not at all on an equal footing. The density perturbation constraint is clearly necessary for consistency with observation, but the overdamping constraint is really a matter only of computational simplicity. Thus, it might be interesting to investigate models in which the overdamping constraint is relaxed. This generalization would lead to complications, however, since the relevant solutions to the differential equations would then depend on an additional parameter. It can be said at least that our present derivation makes only minimal use of the overdamping constraint. If the overdamping constraint holds for just a single e-folding during the same period that the density perturbation constraint holds, then Eq. (3.17) leads to a significant bound. Note also that Eq. (1.7) means that a bound on λ for any segment of the potential, no matter how small, is enough to place a bound on the quartic coefficient in the potential.

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APPENDIX A: BOUND ON THE QUARTIC COUPLING CONSTANT

In the text we showed how to bound the fine-tuning parameter λ defined by

$$\lambda \equiv \frac{\Delta V}{(\Delta\psi)^4} , \tag{A1}$$

where ΔV and $\Delta\psi$ are the change in the potential and in the scalar field value, respectively, during the slow-rolling portion of the inflationary period. In this appendix we show that if $V(\psi)$ can be described by a quartic polynomial

$$V(\psi) = \text{const} + \alpha\psi + \beta\psi^2 + \gamma\psi^3 + \epsilon\psi^4 , \tag{A2}$$

then the quartic coupling ϵ is bounded by

$$|\epsilon| < 9\lambda . \tag{A3}$$

One normally writes the quartic term as $\frac{1}{4}\lambda_q\psi^4$, so $|\lambda_q| \le 36\lambda$. The inequality relies only on the assumption that $V(\psi)$ is monotonically decreasing over the interval, a necessary feature of the slowly rolling solution [see Eq. (1.9)].

Without loss of generality we can assume that the ψ interval extends from 0 to $\Delta\psi$, since the origin of ψ can always be shifted. While such a shift will change the values of α, β, and γ, the value of ϵ will be unaffected. Then

$$\lambda = \frac{V(0) - V(\Delta\psi)}{(\Delta\psi)^4} = -\frac{\alpha + \beta\Delta\psi + \gamma(\Delta\psi)^2 + \epsilon(\Delta\psi)^3}{(\Delta\psi)^3} . \tag{A4}$$

Redefining the parameters by

$$\tilde{\alpha} \equiv \frac{\alpha}{\lambda(\Delta\psi)^3}, \quad \tilde{\beta} \equiv \frac{\beta}{\lambda(\Delta\psi)^2} , \quad \tilde{\gamma} \equiv \frac{\gamma}{\lambda\Delta\psi}, \quad \tilde{\epsilon} \equiv \frac{\epsilon}{\lambda} , \tag{A5}$$

Eq. (A4) becomes

$$\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} + \tilde{\epsilon} = -1 . \tag{A6}$$

In this notation the monotonicity of the potential can be written

$$\frac{1}{\lambda(\Delta\psi)^3}\frac{dV}{d\psi} = \tilde{\alpha} + 2\tilde{\beta}x + 3\tilde{\gamma}x^2 + 4\tilde{\epsilon}x^3 \le 0 , \tag{A7}$$

where $x \equiv \psi/\Delta\psi$ can lie anywhere in the range

$$0 \le x \le 1 . \tag{A8}$$

Using Eq. (A6) to eliminate $\tilde{\gamma}$ from Eq. (A7), one has

$$\tilde{\alpha}(1-3x^2) + \tilde{\beta}x(2-3x) + \tilde{\epsilon}x^2(4x-3) \le 3x^2 . \tag{A9}$$

The consequences of this relation can be extracted by considering special values of x. Choosing, for example, to look at $x = 0$, $\frac{1}{3}$, $\frac{2}{3}$, and 1, one obtains

$$\tilde{\alpha} \le 0 , \tag{A10a}$$

$$18\tilde{\alpha} + 9\tilde{\beta} - 5\tilde{\epsilon} \le 9 , \tag{A10b}$$

$$-9\tilde{\alpha} - 4\tilde{\epsilon} \le 36 , \tag{A10c}$$

$$-2\tilde{\alpha} - \tilde{\beta} + \tilde{\epsilon} \le 3 . \tag{A10d}$$

By combining (A10a) and (A10c), one has immediately that $\tilde{\epsilon} \ge -9$. Similarly, (A10b) and (A10d) can be combined to give $\tilde{\epsilon} \le 9$. It then follows that $|\tilde{\epsilon}| \le 9$, and the result (A3) follows by undoing the parameter definition (A5).

The quartic coupling is the most interesting because it is dimensionless, but bounds can similarly be obtained for the other parameters in the potential. By straightforward manipulations involving Eqs. (A6) and (A10), one can show

$$-8 \le \tilde{\alpha} < 0 , \tag{A11a}$$

$$-12 \le \tilde{\beta} \le 22 , \tag{A11b}$$

$$-30 \le \tilde{\gamma} \le 30 . \tag{A11c}$$

It is often useful to know how strong a bound is, in the sense of knowing how closely the bound could be saturated. For that purpose we have carried out a numerical search, and discovered that the bound on $|\tilde{\epsilon}|$ can in fact be saturated from both sides. The value $\tilde{\epsilon} = -9$ can be obtained with $\tilde{\alpha} = 0$ and $\tilde{\beta} = -8$, and the value $\tilde{\epsilon} = 9$ can be obtained with $\tilde{\alpha} = -4$ and $\tilde{\beta} = 14$. Although the extrema were discovered numerically, it is nonetheless straightforward to check analytically that, with these assignments, Eq. (A9) holds for all $x \in [0,1]$.

Unlike the bound (A3), the bounds (A11) cannot be saturated. Numerical studies show that $\tilde{\alpha}$ must lie in the range of about -6 to 0, $\tilde{\beta}$ must be in the range of about -9 to 16, and $\tilde{\gamma}$ must lie in the range of about -20 to $16\frac{1}{2}$.

APPENDIX B: POSITIVITY OF $W(x)$

In this appendix we show that $W(x) \ge 0$, where $W(x)$ is defined by Eq. (3.15). If we let

$$A \equiv e^{3N/2} \tag{B1}$$

and

$$B \equiv e^{3N(1-x)/2} , \tag{B2}$$

then $W(x)$ can be rewritten as

$$W(x) = ZA^{-5}[(A-1)(A^4-B^4) - (A^4-1)A - B)] \tag{B3}$$

$$= ZA^{-5}(A-1)(B-1)(A-B)(A^2+B^2 + AB + A + B + 1) . \tag{B4}$$

In the form above, $W(x)$ is manifestly positive semidefinite.

APPENDIX C: GLOBAL MAXIMIZATION OF $K[p]$

In Sec. III we showed that the function $p_*(x)$ defined by Eq. (3.9) is a local maximum of the functional $K[p]$

defined in Eq. (3.2), subject to the constraint of Eq. (3.3b). In this appendix we show that it is in fact the global maximum—it is unique up to the possibility of multiplying by an arbitrary constant or performing a reflection $x \to 1-x$. To show this we will consider an arbitrary trial function $p_1(x)$ that is not proportional to $p_*(x)$ or $p_*(1-x)$, and we will show that it can be modified in a way that will increase $K[p]$ while still maintaining the constraint.

The first step in the modification procedure is to calculate the quantity p_m of Eq. (3.7):

$$p_m^3 = \frac{\int_0^1 p_1^4(x)dx}{\int_0^1 p_1(x)dx} . \tag{C1}$$

It was shown in Sec. III that $K[p]$ is increased by increasing $p(x)$ for any x such that $p(x) > p_m$ and/or lowering $p(x)$ for any x such that $p(x) < p_m$.

In order to show that a variation increasing $K[p]$ can be carried out in a manner consistent with the constraint, it is useful to first perform surgery on the function. If $p_1(1) > p_1(0)$, then define a new function $p_1'(x) \equiv p_1(1-x)$, so $p_1'(0) \ge p_1'(1)$. We will drop the primes, but will assume without loss of generality that

$$p_1(0) \ge p_1(1) . \tag{C2}$$

Now define a new function $p_2(x)$, as shown schematically in Fig. 1. Figure 1(a) shows the function $p_1(x)$, divided into segments at the intersections of the curve with the line $p = p_m$. Thus, for each segment one has either $p_1(x) \ge p_m$ or $p_1(x) \le p_m$. The new function $p_2(x)$ is defined by reordering the segments of $p_1(x)$, so that now all the segments with $p(x) \ge p_m$ precede the segments for which $p(x) \le p_m$. Segments for which $p(x) = p_m$ are allowed and can be placed anywhere, but we recall that we have already shown in Sec. III that the constant function is not the maximum of $K[p]$. Note that the value of p_m given by Eq. (C1) is unchanged by the transformation from $p_1(x)$ to $p_2(x)$. Let x_m denote the value of x which is the border between the segments with $p_2(x) \ge p_m$ and the segments with $p_2(x) \le p_m$. The bottom line is that we have constructed a function $p_2(x)$ that satisfies the constraint, and for which

$$K[p_2] = K[p_1] \tag{C3}$$

and

$$\begin{aligned} p_2(x) &\ge p_m \quad \text{for } x < x_m \\ p_2(x) &\le p_m \quad \text{for } x > x_m . \end{aligned} \tag{C4}$$

The final step in the argument is to notice that since the curve $p_2(x)$ is by hypothesis not proportional to $p_*(x)$, there must be places in which the overdamping constraint holds as an inequality. Suppose, for example, that such a region occurs for $x < x_m$. One can then increase the function in this region, thereby increasing the value of $K[p]$. The increase is carried out so that $p(x)$ is held fixed at the right end of the region, but is increased everywhere else in the region by an amount that varies continuously. To maintain continuity one multiplies the

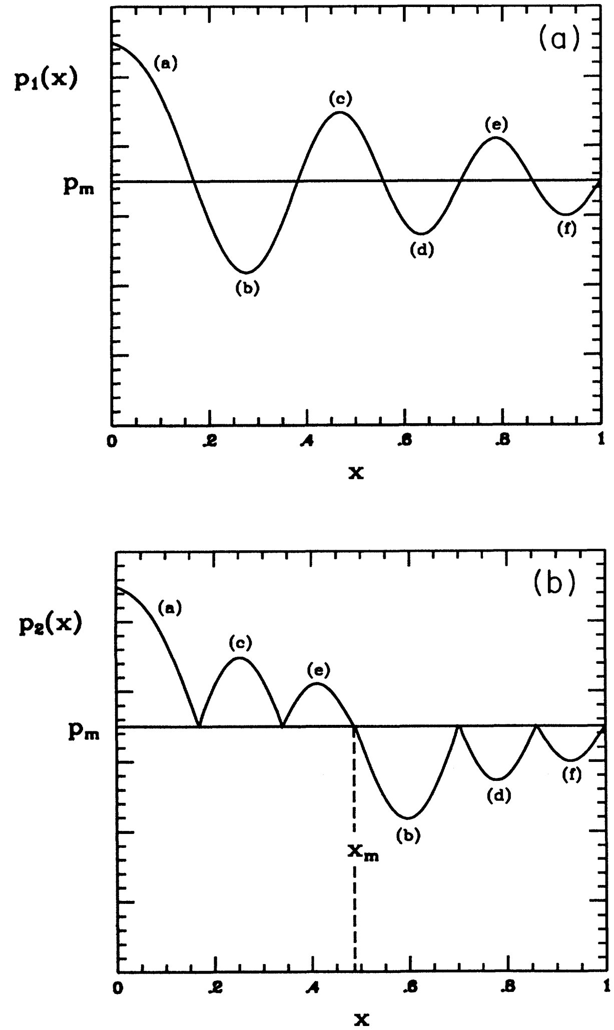

FIG. 1. A transformation of the function $p_1(x)$. The original function $p_1(x)$ is shown in (a), and the transformed function $p_2(x)$ is shown in (b). The transformation, which is explained in the text, is used to show that a global maximum to the constrained variation problem has been found.

function to the left of the region by a constant amount, increasing the value of $K[p]$ even more. Note that multiplication by a constant does not affect the constraint (3.3b). If a region for which the constraint holds as an inequality occurs to the right of x_m, one can modify the function by a similar transformation in which the function is decreased.

Thus, if $p_1(x)$ is not proportional to $p_*(x)$ or its reflection, it is always possible to construct the function $p_2(x)$ with the same value of K, and then to increase K by the transformations described in the previous paragraph, maintaining the constraint condition in all cases. It follows that p_* (or a function proportional to it or its reflection) is the global maximum of the constrained variational problem.

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[19]Differentiation shows that the right-hand side of Eq. (4.31) is monotonically increasing in M_F provided that $(M_F M_P/\beta T_0 M_{\mathrm{E}}) > \kappa^{-1}[(b+\frac{1}{2})/(b-\frac{1}{2})]^{b-1/2}$. The right-hand side of this inequality is found numerically to have a maximum value (in the range $b>\frac{3}{2}$) of 2.762, so the monotonicity condition holds whenever $M_F > 2.762\beta(M_E/M_P)T_0 \approx 10^{-10}(M_E/M_P)$ GeV.

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