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Sum rules for Higgs bosons

J. F. Gunion

*Institute for Theoretical Physics, University of California, Santa Barbara, California 93106 and Department of Physics, University of California, Davis, California 95616**

H. E. Haber

*Institute for Theoretical Physics, University of California, Santa Barbara, California 93106 and Santa Cruz Institute for Particle Physics, University of California, Santa Cruz, California 95064**

J. Wudka

Department of Physics, University of California, Davis, California 95616

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We exhibit the sum rules for Higgs-boson couplings required by unitarity. We give explicit results and applications for an $SU(2)\times U(1)$ electroweak gauge theory with arbitrary Higgs multiplets. As an example, we examine the constraints on nonminimal Higgs sectors should a single neutral Higgs boson with the standard-model coupling strength to the ZZ channel be observed in Z decays.

I. INTRODUCTION

The construction of consistent quantum field theories containing massive vector bosons presented a major challenge to theorists in the 1960s. In general, such models are nonrenormalizable. Moreover, the tree-level amplitudes for scattering processes exhibit bad high-energy behavior; i.e., they increase with the center-of-mass energy, in violation of unitarity. It was later shown that spontaneously broken gauge theories [with the possible addition of massive U(1) vector bosons] are the only renormalizable and consistent quantum field theories involving massive-vector-boson states. Soon after, it was established that such theories are also the unique class of theories involving vector bosons in which the tree-level amplitudes for all scattering processes do not grow with the center-of-mass energy.[1,2] Such a result is possible because relations among parameters of the theory imposed by the gauge invariance are precisely what are needed to cancel out bad high-energy behavior among tree-level Feynman diagrams. That is, the required cancellation of bad high-energy behavior leads to a series of sum-rule relations among various coupling constants of the theory. Some of these sum rules were examined more explicitly in Ref. 3. Nevertheless, the actual physical content of these sum rules has remained somewhat obscure and is difficult to extract from the existing literature. Since experimental accessibility of Higgs bosons is now significant, it seems important to give in more explicit fashion the various sum rules, and demonstrate their utility in a few selected scenarios.

In the literature, the unitarity constraints have been utilized in two different ways. First, one must demand that all tree-level amplitudes exhibit good high-energy behavior; i.e., they must approach a constant value (or else vanish) at infinite energy. Second, unitarity imposes an upper bound on the value of this constant. This bound can be translated into a bound on (a combination of) Higgs-boson masses[4] or Higgs-boson self-couplings.[5,3] However, in this paper we are only interested in the implications of the former requirement, namely the sum rules for Higgs-boson couplings that arise as a result of the cancellation of high-energy growth of the tree-level amplitudes.

In Sec. II we present the general unitarity constraints which follow from the $2\to 2$ tree-level scattering processes involving scalar and vector bosons in an arbitrary spontaneously broken gauge theory. In Sec. III we write down the additional unitarity constraints which arise when one adds fermions to the theory. We then turn to practical applications of these unitarity sum rules in the case of the standard $SU(2)\times U(1)$ electroweak theory with conventional fermion multiplets but with arbitrary Higgs multiplets. In Sec. IV we explicitly write out the Higgs-boson-coupling sum rules which follow from unitarity for tree-level amplitudes involving three or four vector bosons. (Sum rules resulting from the scattering of two vector bosons into scalar bosons are relegated to the Appendix.) In Sec. V the corresponding sum rules involving Higgs-fermion couplings are examined. Specific applications of these sum rules that may be relevant for experiments in the near future are discussed in Sec. VI. In particular, we examine the constraints on nonminimal Higgs sectors should a single neutral Higgs boson (ϕ^0) with the standard-model coupling strength to the ZZ channel be observed in $Z\to f\bar{f}+\phi^0$ or $e^+e^-\to Z+\phi^0$. Our conclusions are given in Sec. VII.

II. UNITARITY CONSTRAINTS ON PROCESSES INVOLVING SCALAR AND VECTOR BOSONS

The unitarity constraints on which we shall focus in this section derive from the tree-level scattering amplitudes for $AA\to AA$, $AA\to A\phi$, and $AA\to\phi\phi$, where

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A is any vector boson, and ϕ is any Higgs boson. We will display the sum rules for these three cases in terms of the Feynman rules for various trilinear and quartic interactions. Our conventions are as follows. We indicate vector bosons by indices a,b,c,d,e and Higgs bosons by indices i,j,k; the indices label boson states of definite electric charge Q. By convention, we shall use $\bar{a}$ and $\bar{i}$ to label the antiparticle states of opposite charge to a and i, respectively. Vector bosons are defined so as to have normal Hermiticity: e.g., $W^+=(W^-)^*$ and $Z=Z^*$. The Hermiticity properties of a given Higgs boson are specified by a phase η defined by $(\phi_i^Q)^*=\eta_i\phi_{\bar{i}}^{-Q}$, where Q is the charge of the Higgs boson. Associated with gauge boson a is Lorentz index α, and so forth. The Feynman rules (with all momenta assumed to be incoming) are

$$\begin{aligned}
&A_aA_bA_c:\ ig_{abc}[(p_a-p_b)_\gamma g_{\alpha\beta}+(p_b-p_c)_\alpha g_{\beta\gamma}\\
&\qquad\qquad +(p_c-p_a)_\beta g_{\gamma\alpha}]\ ,\\
&A_aA_b\phi_i:\ ig_{abi}g_{\alpha\beta}\ ,\\
&A_a\phi_i\phi_j:\ ig_{aij}(p_i-p_j)_\alpha\ ,\\
&A_aA_b\phi_i\phi_j:\ ig_{abij}g_{\alpha\beta}\ ,
\end{aligned}\tag{2.1}$$

where g_{abc} is real and antisymmetric in its indices, g_{abi} is symmetric under interchange of a and b, g_{aij} is antisymmetric under interchange of i and j, and g_{abij} is symmetric under interchange of a and b as well as under interchange of i and j. In addition, the various couplings defined in Eq. (2.1) are nonzero only if electric charge is conserved at the vertex. Finally, Hermiticity implies

$$g_{abi}=\eta_i g^*_{\bar{a}\bar{b}\bar{i}}\ ,\tag{2.2}$$

$$g_{aij}=\eta_i\eta_j g^*_{\bar{a}\bar{j}\bar{i}}=-\eta_i\eta_j g^*_{\bar{a}\bar{i}\bar{j}}\ .\tag{2.3}$$

If CP is conserved in the Higgs sector, then the Higgs phases can be chosen so that g_{aij}, g_{abi}, and g_{abij} are all real. In this convention,[6] for any neutral Higgs state i, $\eta_i=\pm1$ is simply the CP quantum number of ϕ_i^0. Consequently, using Eq. (2.2) and the reality of g_{abi}, we see that $g_{a\bar{a}i}=0$ if $\eta_i=-1$; i.e., there is no tree-level coupling of CP-odd neutral Higgs bosons to vector-boson pairs.[7,8] Similarly, Eq. (2.3) implies that a $Z\phi_i^0\phi_j^0$ coupling can be nonzero only if ϕ_i^0 and ϕ_j^0 have opposite CP quantum numbers. Finally, we note that for charged Higgs bosons, the sign of η is model dependent and depends in part upon the representation of the Higgs bosons involved.

Using Eqs. (2.1) we compute the tree-level scattering amplitudes $AA\to AA$, $AA\to A\phi$, and $AA\to\phi\phi$. By demanding the cancellation of bad high-energy behavior, as described in Sec. I, we obtain three master sum rules:

$A_aA_b\to A_cA_d$:

$$\sum_e{}'\left[g_{abe}g_{cd\bar{e}}\left[M_e^2+\frac{(M_a^2-M_b^2)(M_c^2-M_d^2)}{M_e^2}\right]-g_{ade}g_{cb\bar{e}}\left[M_e^2+\frac{(M_a^2-M_d^2)(M_c^2-M_b^2)}{M_e^2}\right]\right]$$
$$-\sum_e g_{ace}g_{bd\bar{e}}(M_a^2+M_b^2+M_c^2+M_d^2-2M_e^2)=\sum_k\eta_k(g_{abk}g_{cd\bar{k}}-g_{adk}g_{bc\bar{k}})\ ;\tag{2.4}$$

$A_aA_b\to A_c\phi_i$:

$$\sum_e{}'\left[g_{abe}g_{\bar{e}ci}\left[\frac{M_a^2-M_b^2+M_e^2}{2M_e^2}\right]-g_{ace}g_{\bar{e}bi}\left[\frac{M_a^2-M_c^2+M_e^2}{2M_e^2}\right]-g_{bce}g_{\bar{e}ai}\right]=\sum_k\eta_k(g_{cik}g_{ab\bar{k}}-g_{bik}g_{ac\bar{k}})\ ;\tag{2.5}$$

$A_aA_b\to\phi_i\phi_j$:

$$\sum_k\eta_k g_{aik}g_{b\bar{k}j}-\tfrac{1}{2}g_{abij}+\tfrac{1}{4}\sum_e{}'\frac{g_{aei}g_{\bar{e}bj}}{M_e^2}-\tfrac{1}{2}\sum_e g_{abe}g_{\bar{e}ij}=0\ .\tag{2.6}$$

In Eqs. (2.4)–(2.6), the prime on a gauge-boson sum indicates that only gauge bosons with nonzero mass are to be summed over. Note that due to the conventions described above, our results differ slightly from those given in Refs. 1 and 2. We will examine specific consequences of these general sum rules in Sec. IV.

III. UNITARY CONSTRAINTS ON PROCESSES INVOLVING FERMIONS

The unitary constraints on which we shall focus in this section derive from the tree-level scattering amplitudes for $FF\to AA$ and $FF\to A\phi$, where F is any fermion. We will display the sum rules for these two cases in terms of the Feynman rules for the relevant trilinear interactions. We augment the conventions of Sec. II by labeling fermion states with indices m,n,p, and the corresponding antiparticle states with $\bar{m},\bar{n},\bar{p}$. We define the required fermionic couplings as

$$\begin{aligned}
&A_a\overline{f_m}f_n:\ i\gamma_\mu(g^L_{a\bar{m}n}P_L+g^R_{a\bar{m}n}P_R)\ ,\\
&\phi_i\overline{f_m}f_n:\ i(g^L_{i\bar{m}n}P_L+g^R_{i\bar{m}n}P_R)\ ,
\end{aligned}\tag{3.1}$$

where $P_{R,L}\equiv(1\pm\gamma_5)/2$ are the usual right- and left-handed projection operators, and all particles are taken to be entering the vertex. As before, the various couplings defined in Eq. (3.1) are nonzero only if electric charge is conserved at the vertex. Note that Hermiticity implies

$$g^{L,R}_{a\bar{m}n}=g^{L,R*}_{\bar{a}\bar{n}m}\ ,\qquad g^{L,R}_{i\bar{m}n}=\eta_i g^{R,L*}_{i\bar{n}m}\ . \tag{3.2}$$

If CP is conserved, then the phase conventions of the fields can be chosen such that $g^{L,R}_{a\bar{m}n}$ and $g^{L,R}_{i\bar{m}n}$ are real. As a result, for CP-conserving Higgs-fermion interactions, $g^{L}_{i\bar{n}n}=\eta_i g^{R}_{i\bar{n}n}$, which implies that CP-even ($\eta_i=+1$) neutral Higgs bosons have only scalar couplings to $f\bar{f}$, whereas CP-odd ($\eta_i=-1$) neutral Higgs bosons have only pseudoscalar couplings to $f\bar{f}$.

Using Eqs. (3.1), we compute the tree-level scattering amplitudes for $FF\to AA$ and $FF\to A\phi$. Individual terms in the amplitude for $FF\to AA$ can grow as fast as E^2, as the center-of-mass energy E increases. The cancellation of the terms which grow as E^2 is guaranteed by the gauge group structure of the fermion representations.[2] In our notation, these conditions simply read

$$\sum_p (g^L_{b\bar{m}p}g^L_{a\bar{p}n}-g^L_{a\bar{m}p}g^L_{b\bar{p}n})=\sum_e g_{abe}g^L_{\bar{e}\,\bar{m}n}\ , \tag{3.3}$$

and an identical equation obtained by replacing L by R in Eq. (3.3).

Next, we demand the cancellation of terms in the amplitudes for $FF\to AA$ which grow as E. We are then led to the sum rules

$\bar{f}_m f_n \to A_a A_b$:

$$\sum_p [m_p(g^R_{a\bar{m}p}g^L_{b\bar{p}n}+g^R_{b\bar{m}p}g^L_{a\bar{p}n})-m_m g^L_{a\bar{m}p}g^L_{b\bar{p}n}-m_n g^R_{b\bar{m}p}g^R_{a\bar{p}n}] + \sum_e{}' \left[g_{abe}\left[\frac{M_a^2-M_b^2+M_e^2}{2M_e^2}\right](m_n g^R_{\bar{e}\,\bar{m}n}-m_m g^L_{\bar{e}\,\bar{m}n})\right]=\tfrac{1}{2}\sum_k \eta_k g_{abk}g^L_{\bar{k}\bar{m}n}\ , \tag{3.4}$$

and an identical equation obtained by interchanging L and R in Eq. (3.4). It is of interest to specialize the above equation to the case of $n=m$ and $a=\bar{b}$. Taking Eq. (3.4) in this special case, subtracting the same equation with L and R interchanged, and using Eq. (3.3) to eliminate $\sum_e g_{a\bar{a}e}g^{L,R}_{\bar{e}\,\bar{n}n}$, we end up with

$$\sum_k \eta_k g_{a\bar{a}k}(g^R_{\bar{k}\bar{n}n}-g^L_{\bar{k}\bar{n}n})=\sum_p 2m_p(g^L_{\bar{a}\,\bar{n}p}g^R_{a\bar{p}n}-g^R_{\bar{a}\,\bar{n}p}g^L_{a\bar{p}n}+g^L_{a\bar{n}p}g^R_{\bar{a}\,\bar{p}n}-g^R_{a\bar{n}p}g^L_{\bar{a}\,\bar{p}n})\ . \tag{3.5}$$

Using Eq. (3.2), we see that for a CP-conserving Higgs sector the right-hand side of Eq. (3.5) vanishes. As a result, it follows that

$$\sum_k \eta_k g_{a\bar{a}k}(g^R_{\bar{k}\bar{n}n}-g^L_{\bar{k}\bar{n}n})=0 \quad (CP\text{-conserving case})\ . \tag{3.6}$$

This result is consistent with our observation in Sec. II that there is no tree-level coupling of a CP-odd neutral Higgs boson to vector-boson pairs.

Finally, we demand the cancellation of terms in the amplitudes for $FF\to A\phi$ which grow as E. (There are no terms which grow as E^2 in this case.) The resulting sum rules are

$\bar{f}_m f_n \to A_a\phi_i$:

$$\sum_e{}' \frac{1}{2M_e^2}g_{aei}(m_n g^R_{\bar{e}\,\bar{m}n}-m_m g^L_{\bar{e}\,\bar{m}n})-\sum_k \eta_k g_{aik}g^L_{\bar{k}\bar{m}n} =\sum(g^L_{i\bar{m}p}g^L_{a\bar{p}n}-g^R_{a\bar{m}p}g^L_{i\bar{p}n})\ , \tag{3.7}$$

and an identical equation obtained by interchanging L and R in Eq. (3.7).

As in Sec. II, the prime on the gauge-boson sums above indicates that only gauge bosons with nonzero mass are to be summed over. We will examine specific consequences of these general sum rules in Sec. V.

IV. SUM RULES FOR HIGGS-BOSON–GAUGE-BOSON COUPLINGS

In this section we apply the sum rules of Eqs. (2.4)–(2.6) to an SU(2)×U(1) gauge theory with arbitrary Higgs multiplets. We note that $g_{W^+W^-Z}=gc_W$ and $g_{W^+W^-\gamma}=gs_W=e$, with $c_W\equiv\cos\theta_W$ and $s_W\equiv\sin\theta_W$. In the results quoted in this section, we shall not assume the standard mass relation $m_Zc_W=m_W$ since this relation does not in general hold in models with Higgs bosons in triplet or higher representations. However, a phenomenologically viable model must approximately satisfy this relationship; thus, in practical applications of these sum rules to future data, making use of $m_Zc_W\simeq m_W$ is appropriate.

We first examine the $AA\to AA$ sum rules obtained from Eq. (2.4). By examining the various index choices, only two nontrivial sum rules are obtained:

$a=\bar{d}=W^+, b=c=W^-$:

$$g^2(4m_W^2-3c_W^2m_Z^2)=\sum_k \eta_k g^2_{W^+W^-\phi_k^0} -\sum_k \eta_k g_{W^+W^+\phi_k^{--}}g_{W^-W^-\phi_k^{++}}\ ; \tag{4.1}$$

$a=W^+, b=W^-, c=d=Z$:

$$g^2c_W^2m_Z^4/m_W^2=\sum_k \eta_k g_{W^+W^-\phi_k^0}g_{ZZ\phi_k^0} - \sum_k \eta_k g_{W^+Z\phi_k^-}g_{W^-Z\phi_k^+} \ . \tag{4.2}$$

Note that if CP is conserved then the sum over ϕ_k^0 runs only over CP-even, $\eta_k=+1$, neutral Higgs bosons. If one chooses one of the indices of Eq. (2.4) to be a photon, the sum rule immediately simplifies to $0=0$. This is a result of electromagnetic gauge invariance, which implies, for example, that $g_{W^+\phi_k^-\gamma}=0$.

We next examine the $AA\to A\phi$ sum rules obtained from Eq. (2.5). By examining the various index choices, four nontrivial sum rules are obtained (not including charge-conjugated versions of sum rules listed below):

$a=W^-, b=W^-, c=W^+, i=\phi_i^+$:

$$\tfrac{3}{2}gc_Wg_{ZW^-\phi_i^+}=\sum_k \eta_k g_{W^+\phi_i^+\phi_k^-}g_{W^-W^-\phi_k^{++}} - \sum_k \eta_k g_{W^-\phi_i^+\phi_k^0}g_{W^-W^+\phi_k^0} \ ; \tag{4.3}$$

$a=W^+, b=W^+, c=Z, i=\phi_i^{--}$:

$$gc_W\left[2-\frac{m_Z^2}{2m_W^2}\right]g_{W^+W^+\phi_i^{--}}=\sum_k \eta_k g_{Z\phi_i^{--}\phi_k^{++}}g_{W^+W^+\phi_k^{--}} - \sum_k \eta_k g_{W^+\phi_i^{--}\phi_k^+}g_{W^+Z\phi_k^-} ; \tag{4.4}$$

$a=W^+, b=W^-, c=Z, i=\phi_i^0$:

$$\tfrac{1}{2}gc_W\left[g_{ZZ\phi_i^0}-g_{W^+W^-\phi_i^0}\frac{m_Z^2}{m_W^2}\right]=\sum_k \eta_k g_{Z\phi_i^0\phi_k^0}g_{W^+W^-\phi_k^0} - \sum_k \eta_k g_{W^-\phi_i^0\phi_k^+}g_{W^+Z\phi_k^-} ; \tag{4.5}$$

and $a=Z, b=W^-, c=Z, i=\phi_i^+$:

$$gc_Wg_{W^-Z\phi_i^+}\left[1+\frac{m_Z^2}{2m_W^2}\right]=\sum_k \eta_k g_{W^-\phi_i^+\phi_k^0}g_{ZZ\phi_k^0} - \sum_k \eta_k g_{Z\phi_i^+\phi_k^-}g_{ZW^-\phi_k^+} \ . \tag{4.6}$$

Note that in both $AA\to AA$ and $AA\to A\phi$ sum rules, couplings to Higgs bosons with more than two units of charge do not appear. The $AA\to\phi\phi$ sum rules are more complex, and involve, for example, the quartic $AA\phi\phi$ couplings as well as the various trilinear Higgs-boson–vector-boson couplings. In addition, sum rules can be derived which involve the couplings of Higgs bosons of all possible integer charges. The detailed presentation of these sum rules is given in the Appendix.

V. SUM RULES FOR HIGGS-BOSON–FERMION COUPLINGS

In this section we apply the sum rules of Eqs. (3.4) and (3.7) to an SU(2)×U(1) gauge theory with arbitrary Higgs multiplets. In our analysis, we will suppress the complication of multiple quark generations, and will employ first family notation (u,d) when needed.[9] Our results will also apply to the Higgs-boson couplings to leptons if one replaces (u,d) with (ν,l^-) and uses the appropriate values for the fermion quantum numbers.

We first apply the $FF\to AA$ sum rules to an electroweak model with standard-model fermion and vector-boson content, but with an arbitrary Higgs sector. By checking all possible index combinations in Eq. (3.4) (and in the equation related to it by interchange of L and R), we find

$$\sum_k \eta_k g_{abk}g^L_{k\overline{m}n}=\sum_k \eta_k g_{abk}g^R_{k\overline{m}n} = \frac{-gg_aM_am_n}{2m_W}\delta_{mn}\delta_{a\overline{b}} \tag{5.1}$$

(no sum over a), where

$$g_a\equiv\begin{cases} g, & a=W^\pm \ ,\\ g/c_W, & a=Z \ .\end{cases} \tag{5.2}$$

This result is easy to understand. Since the left-hand side of Eq. (3.4) is independent of the Higgs representations of the model, one may choose to evaluate Eq. (3.4) in the case of the minimal Higgs structure of the standard model. That is, the bad high-energy behavior of the s-channel Higgs-boson-exchange contributions to $FF\to AA$ in a model with an arbitrary Higgs sector either must exactly cancel if $f_m\neq f_n$, or must match the corresponding contributions of s-channel neutral-Higgs-boson exchange in the minimal standard model (with one neutral Higgs scalar ϕ^0) if $f_m=f_n$. Indeed, we may easily reproduce Eq. (5.1) in this manner. Employing the notation introduced above, and using the index 0 to denote the ϕ^0, the vector-boson couplings to ϕ^0 are

$$g_{ab0}=g_aM_a\delta_{a\overline{b}} \ , \tag{5.3}$$

where g_a is defined in Eq. (5.2). The left- and right-handed fermion couplings to ϕ^0 are equal: $g^L_{0\overline{m}n}=g^R_{0\overline{m}n}\equiv g_{0\overline{m}n}$, with

$$g_{0\overline{m}n}=-\frac{gm_n}{2m_W}\delta_{mn} \ , \tag{5.4}$$

where m_n is the mass of the fermionic state labeled by index n. The following result is then obtained:

$$g_{ab0}g_{0\overline{m}n}=\sum_k \eta_k g_{abk}g^L_{k\overline{m}n}=\sum_k \eta_k g_{abk}g^R_{k\overline{m}n} \ , \tag{5.5}$$

which is precisely the sum rule obtained above. As a simple example, we have

$$\sum_k \eta_k g_{W^-Z\phi_k^+}g^{L,R}_{\phi_k^-u\overline{d}}=0 \ . \tag{5.6}$$

Although this result is trivial in multidoublet models, where there are no tree-level $W^-Z\phi^+$ couplings,[10] this sum rule imposes interesting constraints in models with more complicated Higgs sectors.

We next examine the $FF\to A\phi$ sum rules given in Eq. (3.7). Again, we will specialize to an electroweak model with standard-model fermion and vector-boson content, but with an arbitrary Higgs sector. We examine the basic index choices of interest. (Charge-conjugated versions of the sum rules presented below can be easily obtained.)

$m=n, a=Z, i=\phi_i^0$:

$$\frac{gT_n^{3L}}{c_W}\left[\frac{m_n g_{ZZ\phi_i^0}}{2m_Z^2}+g^L_{\phi_i^0\bar f_n f_n}\right]=\sum_k \eta_k g_{Z\phi_i^0\phi_k^0} g^L_{\phi_k^0\bar f_n f_n} ,$$
$$\frac{-gT_n^{3L}}{c_W}\left[\frac{m_n g_{ZZ\phi_i^0}}{2m_Z^2}+g^R_{\phi_i^0\bar f_n f_n}\right]=\sum_k \eta_k g_{Z\phi_i^0\phi_k^0} g^R_{\phi_k^0\bar f_n f_n} , \quad (5.7)$$

where m_n is the mass of f_n and $T_n^{3L}=\pm\frac{1}{2}$ for up-type (down-type) fermions, respectively. In deriving Eq. (5.7), we have used $g^R_{Z\bar n n}-g^L_{Z\bar n n}=gT_n^{3L}/c_W$. If we add the two equations above, we obtain a particularly simple form:

$$\frac{gT_n^{3L}}{c_W}g^P_{\phi_i^0\bar f_n f_n}=-\sum_k \eta_k g_{Z\phi_i^0\phi_k^0} g^S_{\phi_k^0\bar f_n f_n} , \quad (5.8)$$

which relates the pseudoscalar (P) and scalar (S) Higgs-fermion couplings. In particular, if CP is conserved, this sum rule is trivial if ϕ_i^0 is a CP-even scalar. If ϕ_i^0 is CP-odd, then we can set $\eta_k=1$ and the sum runs only over CP-even Higgs scalars.

$m=n=u, a=W^+, i=\phi_i^-$:

$$\frac{gm_u}{4m_Z^2 c_W}g_{ZW^+\phi_i^-}+\sum_k \eta_k g_{W^+\phi_i^-\phi_k^0} g^L_{\phi_k^0\bar u u}=0 , \quad (5.9)$$

$$\frac{-gm_u}{4m_Z^2 c_W}g_{ZW^+\phi_i^-}+\sum_k \eta_k g_{W^+\phi_i^-\phi_k^0} g^R_{\phi_k^0\bar u u}=\sqrt{\tfrac{1}{2}}g g^R_{\phi_i^-\bar d u} , \quad (5.10)$$

where we have used $g^L_{W\bar f f'}=-g/\sqrt{2}$ and $g^R_{W\bar f f'}=0$, as are appropriate in the standard SU(2)×U(1) model. In a model with only Higgs doublets (and singlets), $g_{ZW^+\phi_i^-}=0$, and the above sum rules take on a particularly simple form.

$m=n=d, a=W^+, i=\phi_i^-$:

This case yields two sum rules which are analogous to the ones just presented. The first is obtained from Eq. (5.9) by replacing m_u with m_d and $g^L_{\phi_k^0\bar u u}$ with $g^R_{\phi_k^0\bar d d}$. The second is obtained from Eq. (5.10) by replacing m_u with m_d, $g^R_{\phi_k^0\bar u u}$ with $g^L_{\phi_k^0\bar d d}$, and $g^R_{\phi_i^-\bar d u}$ with $-g^L_{\phi_i^-\bar d u}$.

$m\neq n, a=Z, i=\phi_i^-$:

$$\frac{gm_d}{2\sqrt{2}m_W^2}g_{ZW^+\phi_i^-}=\sum_{k\neq i}\eta_k g_{Z\phi_i^-\phi_k^+} g^L_{\phi_k^-\bar d u} , \quad (5.11)$$

$$\frac{gm_u}{2\sqrt{2}m_W^2}g_{ZW^+\phi_i^-}=-\sum_{k\neq i}\eta_k g_{Z\phi_i^-\phi_k^+} g^R_{\phi_k^-\bar d u} . \quad (5.12)$$

In deriving the above equations, we have used Eq. (A2) of the Appendix. Note that these equations are trivial for the two-Higgs-doublet model which possesses only one physical charged Higgs pair (i.e., the equations reduce to 0=0), but are nontrivial for models with more extended Higgs sectors.

$m\neq n, a=W^-, i=\phi_i^0$:

$$\frac{gm_d}{2\sqrt{2}m_W^2}g_{W^+W^-\phi_i^0}=-\frac{g}{\sqrt{2}}g^L_{\phi_i^0\bar d d}+\sum_k \eta_k g_{W^-\phi_i^0\phi_k^+} g^L_{\phi_k^-\bar d u} , \quad (5.13)$$

$$\frac{gm_u}{2\sqrt{2}m_W^2}g_{W^+W^-\phi_i^0}=-\frac{g}{\sqrt{2}}g^R_{\phi_i^0\bar u u}-\sum_k \eta_k g_{W^-\phi_i^0\phi_k^+} g^R_{\phi_k^-\bar d u} . \quad (5.14)$$

Note that if CP is conserved, the left-hand sides of the two sum rules above vanish for a CP-odd scalar $\phi_i^0\equiv A^0$, and one obtains two rather simple sum rules for $g^L_{A^0\bar d d}$ and $g^R_{A^0\bar u u}$.

This completes our enumeration of the Higgs-boson-coupling sum rules in the SU(2)×U(1) gauge theory.

VI. APPLICATIONS

In practical applications of the sum rules, important simplifications result in the case where the Higgs sector is CP conserving, in which case, as noted in Sec. II, the Hermiticity phases η_i defined earlier are ± 1 in the convention where the g's are taken to be real. For instance, if we recall that $\eta_{\phi_i^0}=+1$ if $g_{W^+W^-\phi_i^0}$ and $g_{ZZ\phi_i^0}$ are nonzero, it is obvious that the first term on the right-hand side in Eq. (4.1) is positive definite. Similarly, by noting that $g_{W^-Z\phi_k^+}=\eta_{\phi_k^+}g_{W^+Z\phi_k^-}$ and $g_{W^-W^-\phi_k^{++}}=\eta_{\phi_k^{++}}g_{W^+W^+\phi_k^{--}}$, it is apparent that the second sums on the right-hand side of Eqs. (4.1) and (4.2) are positive definite. However, not all terms in the sum rules have a definite sign. For instance, the first term of Eq. (4.2) does not have a well-determined sign, since the W^+W^- and ZZ couplings of a neutral Higgs boson could differ by an overall sign.[11] Another important simplification results from the observation (noted in Sec. II) that $g_{Z\phi_i^0\phi_j^0}\neq 0$ requires $\eta_{\phi_i^0}\eta_{\phi_j^0}=-1$. It then follows that for a CP-even neutral Higgs boson ($\eta_{\phi_i^0}=+1$), the first term on the right-hand side of Eq. (4.5) is always zero.

If we do not assume that CP is conserved in the Higgs sector, then one can obtain unitarity constraints on the size of various CP-violating phases. A number of very interesting sum rules of this type have recently been derived by Weinberg.[12] However, for the sake of simplicity, we shall continue to assume here that CP is conserved. In almost all phenomenological applications involving the direct production of Higgs bosons, this should be a very good approximation. Moreover, for the remainder of this section, we will also assume that the Higgs sector is such that $\rho=m_W^2/(c_W^2 m_Z^2)=1$ (this latter relation being experimentally verified at the 1% level). Indeed, if one simply sets $m_W=c_W m_Z$ in the sum rules, one is guaranteed that any Higgs sector respecting these sum rules must satisfy $\rho=1$, and consequences thereof.

As a first illustration, we prove that (when $\rho=1$) doubly charged Higgs bosons with $W^\pm W^\pm$ couplings can be present (implying triplet or higher Higgs representations) if and only if there is one or more singly charged Higgs boson(s) with nonzero $W^\pm Z$ coupling(s). (This result was originally proved in Ref. 13 by another technique.) It will be useful to define

$$g_{ZZ\phi_i^0}\equiv\frac{gm_Z\lambda_{ZZ\phi_i^0}}{c_W},\quad g_{W^+W^-\phi_i^0}\equiv gm_W\lambda_{W^+W^-\phi_i^0}\ . \tag{6.1}$$

First, we note that if $g_{W^+Z\phi_k^-}=0$ for all ϕ_k^-, then Eqs. (4.2) and (4.5) imply that $\sum_i\lambda^2_{W^+W^-\phi_i^0}=1$ and Eq. (4.1) then requires that $\sum_k g^2_{W^+W^+\phi_k^{--}}=0$, implying that all such couplings must vanish. Working in the reverse direction, we note that if there are no nonzero $W^+W^+\phi^{--}$ couplings, then Eq. (4.1) can be rewritten as $\sum_i\lambda^2_{W^+W^-\phi_i^0}=1$. We next multiply Eq. (4.5) by $g_{W^+W^-\phi_i^0}$ and sum over i, and use Eq. (4.3) (neglecting the doubly charged Higgs term) to simplify the result. Eliminating $\sum_k g^2_{ZW^-\phi_k^+}$ from the latter result using Eq. (4.2) we obtain

$$\sum_i(-\lambda^2_{W^+W^-\phi_i^0}+4\lambda_{W^+W^-\phi_i^0}\lambda_{ZZ\phi_i^0})=3\ . \tag{6.2}$$

Combining $\sum_i\lambda^2_{W^+W^-\phi_i^0}=1$ with Eq. (6.2), we obtain $\sum_i\lambda_{W^+W^-\phi_i^0}\lambda_{ZZ\phi_i^0}=1$. If this is substituted into Eq. (4.2) we find that $\sum_k g^2_{ZW^-\phi_k^+}=0$ is required, implying that all WZ couplings to singly charged Higgs boson must be zero. Equation (4.5) then requires that $\lambda_{W^+W^-\phi_i^0}=\lambda_{ZZ\phi_i^0}$ for all i. That is, in models with $\rho=1$ and no tree-level $ZW^\pm\phi^\mp$ couplings, we have

$$\sum_i g^2_{W^+W^-\phi_i^0}=g^2m_W^2,\quad \sum_i g^2_{ZZ\phi_i^0}=\frac{g^2m_Z^2}{c_W^2}\ . \tag{6.3}$$

Of course, all these results are those expected for a model in which only neutral members of doublet (and, possibly, singlet) Higgs fields acquire nonzero vacuum expectation values.

As a second example of the utility of the sum rules, we imagine that the Z factories at the SLAC Linear Collider (SLC) and CERN LEP discover a single neutral Higgs boson ϕ^0 with substantial $ZZ\phi^0$ coupling of magnitude gm_Z/c_W as given by the standard model. We will not consider sum rules which contain quartic couplings, as these will surely be the last to be tested experimentally; i.e., we focus only on the sum rules that can be expressed in terms of trilinear Higgs-boson–vector-boson couplings.

Even if $|g_{ZZ\phi^0}|=gm_Z/c_W$, it is clear from the sum rules that there are many remaining options. Equations (4.1) and (4.2) imply that a non-standard-model value and/or sign for $g_{W^+W^-\phi^0}$ is possible so long as there are neutral and charged Higgs bosons with appropriate couplings present. Indeed, in order to make any further statements, additional experimental information is required. Let us thus imagine that the $W^+W^-\phi^0$ coupling is also determined to have standard-model strength ($|g_{W^+W^-\phi^0}|=gm_W$), perhaps by measuring production cross sections at a hadron collider via WW fusion. While it would certainly be tempting to conclude that the minimal standard model is correct, this need not be the case. The sum rules of Eqs. (4.1) and (4.2) reduce to

$$0=\sum_{\phi_k^0\neq\phi^0}g^2_{W^+W^-\phi_k^0}-\sum_k g^2_{W^+W^+\phi_k^{--}} \tag{6.4}$$

and

$$(1-\epsilon)g^2m_Z^2=\sum_{\phi_k^0\neq\phi^0}g_{W^+W^-\phi_k^0}g_{ZZ\phi_k^0}-\sum_k g^2_{W^+Z\phi_k^-}\ , \tag{6.5}$$

where we have defined $\epsilon=\mathrm{sgn}(g_{W^+W^-\phi^0}/g_{ZZ\phi^0})$. These equations make it clear that it is crucial to determine whether or not there are any singly charged or doubly charged Higgs bosons with $W^\pm Z$ or $W^\pm W^\pm$ couplings, respectively. We have already stressed that, if all $W^+W^+\phi^{--}$ couplings are absent, then all $W^+Z\phi^-$ couplings must also be absent (assuming $\rho=1$), and vice versa. In this case, Eq. (6.4) implies that $g_{W^+W^-\phi_k^0}=0$ for any $\phi_k^0\neq\phi^0$, and Eq. (6.5) then implies that $\epsilon=1$, i.e., that $g_{ZZ\phi^0}$ has the same sign as $g_{W^+W^-\phi^0}$, as in the standard model. However, this scenario is only the simplest of many possibilities.

Let us give an explicit example of just how perverse nature could be in satisfying the sum rules in a manner consistent with the scenario just outlined. We consider an example from the triplet model with a custodial SU(2) symmetry constructed in Refs. 14 and 15 and explored in greater depth in Ref. 13. In that model the physical Higgs bosons are the $H_5^{++,--},H_5^{+,-},H_5^0$ [belonging to a fiveplet under the custodial SU(2) symmetry], the $H_3^{+,-},H_3^0$ (belonging to a triplet), and two singlet Higgs bosons H_1^0 and $H_1^{0\prime}$. If the doublet-triplet mixing angle of the model (denoted by θ_H) is chosen so that $s_H\equiv\sin\theta_H=\sqrt{\frac{3}{8}}$ then the $H_1^{0\prime}$ has standard-model couplings (both in sign and magnitude). Nevertheless, discovery of the $H_1^{0\prime}$ clearly would not imply that it is the standard-model Higgs boson. Indeed, in this specific nonminimal model, Eqs. (6.4) and (6.5) are satisfied due to the cancellation of the various terms on the right-hand side. Explicitly, we have the following additional couplings (in units of gm_W):[8,13]

$$\begin{aligned}&H_5^{++}W^-W^-:\ \sqrt{\tfrac{3}{4}},\quad H_5^0ZZ:\ -\sqrt{\tfrac{1}{2}}c_W^{-2}\ ,\\&H_5^+W^-Z:\ -\sqrt{\tfrac{3}{8}}c_W^{-1},\quad H_1^0W^-W^+:\ \sqrt{\tfrac{5}{8}}\ ,\\&H_5^0W^-W^+:\ \sqrt{\tfrac{1}{8}},\quad H_1^0ZZ:\ \sqrt{\tfrac{5}{8}}c_W^{-2}\ .\end{aligned} \tag{6.6}$$

It is easily verified that these couplings give zero when inserted on the right-hand sides of Eqs. (6.4) and (6.5).

Of course, such perversely fine-tuned scenarios are not likely to arise in nature, in which case the first-discovered neutral Higgs boson will not have exactly standard-model-like W^+W^- and ZZ couplings if there is a nonminimal Higgs sector. The sum rules will then provide a guide as to what remains to be searched for. For example, if a neutral Higgs boson or a set of neutral Higgs bosons are found such that $\sum_k g^2_{W^+W^-\phi_k^0} > g^2 m_W^2$, then Eq. (4.1) would imply that doubly charged Higgs bosons with WW couplings must be present. This is what would occur in the triplet model just discussed. Conversely, if a number of Higgs bosons are discovered with couplings that satisfy the coupling constant sum rules given in this paper, this will provide a strong indication that all the Higgs bosons connected to the gauge sector have been found.

We next turn to some consequences of the sum rules involving Higgs-fermion couplings. Suppose that [in an $SU(2)\times U(1)$ gauge theory] a charged Higgs boson is discovered with tree-level couplings to $f\bar{f}$. If this charged Higgs boson is also found to have a nonzero W^+Z coupling of tree-level magnitude, then by Eq. (5.6) there must be other charged Higgs boson which also couple to both W^+Z and $f\bar{f}$. Alternatively, when one or more charged Higgs boson(s) is(are) present, Eq. (5.6) can be satisfied if those Higgs bosons with quark couplings have no W^+Z coupling and vice versa. This is the case in the triplet model with custodial $SU(2)$ symmetry discussed just above. There, the $H_5^+W^-Z$ coupling is nonzero, but the H_5^+ has no fermion couplings, whereas the H_3^+, which does have fermion couplings, has no coupling to W^+Z.

Many additional applications of the sum rules of Sec. V are also easily derived. Let us consider an $SU(2)\times U(1)$ gauge theory with one family of quarks and leptons, with standard-model gauge-boson–fermion couplings. Suppose that a neutral Higgs boson ϕ_i^0 is discovered which *fails* to obey the relation

$$m_f g_{W^+W^-\phi_i^0} + 2m_W^2 g^{R,L}_{\phi_i^0 \bar{f} f} = 0 \ , \tag{6.7}$$

which is predicted to hold in the standard model (with minimal Higgs structure). Then, Eqs. (5.13) and (5.14) immediately imply that one or more charged Higgs bosons with nonzero $W^+\phi^-\phi_i^0$ couplings *and* appropriate $f\bar{f}$ couplings must exist. For example, consider once again the triplet model described earlier. As shown in Ref. 13 (see also Ref. 8), H_5^0 has a nonzero W^+W^- coupling but no $f\bar{f}$ couplings. Equations (5.13)–(5.14) are satisfied by virtue of the fact that the H_3^+ has both fermion couplings and a nonzero coupling to $W^+H_5^0$. Conversely, H_3^0 has nonzero $f\bar{f}$ couplings but does not couple to W^+W^-. This implies that charged Higgs bosons must exist which couple to fermion pairs. Indeed, in the model, it is the H_3^+ that has $g_{H_3^- \bar{d} u} \neq 0$ and $g_{W^- H_3^+ H_3^0} \neq 0$, and which thereby allows the sum rules of Eqs. (5.13) and (5.14) to be satisfied.

VII. CONCLUSIONS

In this paper we have explicitly exhibited sum rules for Higgs-boson couplings to gauge bosons, scalar bosons, and fermions which arise as a result of the requirement of cancellation of (potentially) bad high-energy behavior of the $2\to 2$ tree-level scattering amplitudes. After presenting sum rules which are valid in any spontaneously broken gauge theory with arbitrary gauge group and matter field content, we examined the consequences for the $SU(2)\times U(1)$ model, with standard fermion content, but with an arbitrary Higgs sector. We focused on the predictions of models with a CP-conserving sector which satisfied the constraint of $\rho \equiv m_W^2/m_Z^2 \cos^2\theta_W = 1$. Extensions of our analysis are straightforward. For example, one could examine electroweak models with extended gauge groups as well as extended Higgs sectors. One would then be able to explicitly write out the more complicated sum rules involving the coupling of Higgs bosons and fermions to the new gauge bosons.

The Higgs-boson sum rules should play an important role in elucidating the properties of the scalar sector of the electroweak gauge theory. Even if it happens that a neutral Higgs boson is discovered whose couplings to W^+W^- and ZZ are equal in magnitude to those predicted by the standard model, experimentalists will have additional work to do. First, singly charged Higgs bosons with $W^\pm Z$ couplings and doubly charged Higgs bosons with $W^\pm W^\pm$ couplings must be searched for and either eliminated as a possibility or found. Once the full Higgs-boson spectrum and its couplings to vector-boson pairs are explored, it will be necessary to verify that the vector-boson–Higgs-boson–Higgs-boson and fermion–fermion–Higgs-boson coupling sum rules are satisfied. Only then can we begin to become confident that the structure of the Higgs sector is correctly determined and is consistent. All of the examples described in this paper make it clear that the sum rules allow one to determine a great deal about the Higgs sector if even a single (relatively light) neutral Higgs boson is discovered in the near future, and its couplings to fermion and vector-boson pairs can be determined with reasonable accuracy. If one can experimentally verify that these couplings are not those predicted by the standard model (with minimal Higgs content), the sum rules discussed in this paper will provide an important guide as to which additional Higgs bosons (and with what couplings) still await discovery.

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APPENDIX: THE $AA \to \phi\phi$ UNITARITY SUM RULES

As noted in Sec. III, the $AA \to \phi\phi$ sum rules are rather complex, and involve the quartic $AA\phi\phi$ couplings as well as the $AA\phi$ and $A\phi\phi$ trilinear couplings. Furthermore, unlike the sum rules considered in Sec. III, the sum rules presented here can involve couplings to Higgs bosons of arbitarary (integer) charge. Our strategy will be to present sum rules involving neutral, singly, and doubly charged Higgs bosons. We will then indicate how to generalize to higher charges.

We begin with the W^+W^- sector. For this sector, it will be useful to define the subsidiary quantity

$$f_{\phi_i^Q\phi_j^{-Q}} \equiv c_W g_{Z\phi_i^Q\phi_j^{-Q}} + s_W g_{\gamma\phi_i^Q\phi_j^{-Q}} , \tag{A1}$$

where $g_{\gamma\phi_i^Q\phi_j^{-Q}} = -e\eta_i Q\delta_{ij}$. In the special case of $i=j$,

$$g_{Z\phi_i^Q\phi_i^{-Q}} = -\frac{g}{2c_W}(1-2s_W^2)\eta_i Q , \tag{A2}$$

so that $f_{\phi_i^Q\phi_i^{-Q}} = -\frac{1}{2}\eta_i gQ$. When $i \neq j$, only the Z term contributes in Eq. (A1). By examining the various index choices, the following sum rules are obtained:

$$a=W^+,b=W^-,i=\phi_i^0,j=\phi_j^0 : \quad \sum_k \eta_k g_{W^+\phi_i^0\phi_k^-} g_{W^-\phi_k^+\phi_j^0} + \frac{1}{4m_W^2} g_{W^+W^-\phi_i^0} g_{W^+W^-\phi_j^0} - \tfrac{1}{2} g c_W g_{Z\phi_i^0\phi_j^0} = \tfrac{1}{2} g_{W^+W^-\phi_i^0\phi_j^0} ; \tag{A3}$$

$$a=W^+,b=W^-,i=\phi_i^+,j=\phi_j^- : \quad \sum_k \eta_k g_{W^+\phi_i^+\phi_k^{--}} g_{W^-\phi_k^{++}\phi_j^-} - \tfrac{1}{2} g f_{\phi_i^+\phi_j^-} = \tfrac{1}{2} g_{W^+W^-\phi_i^+\phi_j^-} ; \tag{A4}$$

$a=W^+,b=W^-,i=\phi_j^-,j=\phi_i^+$:

$$\sum_k \eta_k g_{W^+\phi_j^-\phi_k^0} g_{W^-\phi_k^0\phi_i^+} - \sum_k \eta_k g_{W^+\phi_i^+\phi_k^{--}} g_{W^-\phi_k^{++}\phi_j^-} + \frac{1}{4m_Z^2} g_{W^+Z\phi_j^-} g_{ZW^-\phi_i^+} = -g f_{\phi_i^+\phi_j^-} , \tag{A5}$$

where we have used Eq. (A4) to substitute for $g_{W^+W^-\phi_i^+\phi_j^-}$;

$$a=W^+,b=W^-,i=\phi_i^{++},j=\phi_j^{--} : \quad \sum_k \eta_k g_{W^+\phi_i^{++}\phi_k^{---}} g_{W^-\phi_k^{+++}\phi_j^{--}} - \tfrac{1}{2} g f_{\phi_i^{++}\phi_j^{--}} = \tfrac{1}{2} g_{W^+W^-\phi_i^{++}\phi_j^{--}} , \tag{A6}$$

where the first term is not present if there are no triply charged Higgs bosons;

$a=W^+,b=W^-,i=\phi_j^{--},j=\phi_i^{++}$:

$$\sum_k \eta_k g_{W^+\phi_j^{--}\phi_k^+} g_{W^-\phi_k^-\phi_i^{++}} - \sum_k \eta_k g_{W^+\phi_i^{++}\phi_k^{---}} g_{W^-\phi_k^{+++}\phi_j^{--}} + \frac{1}{4m_W^2} g_{W^+W^+\phi_j^{--}} g_{W^-W^-\phi_i^{++}} = -g f_{\phi_i^{++}\phi_j^{--}} , \tag{A7}$$

where we have used Eq. (A6) to substitute for $g_{W^+W^-\phi_i^{++}\phi_j^{--}}$. Equation (A6) generalizes in obvious fashion to higher charge states, while the generalization of Eq. (A7) for $Q \geq 3$ is

$$\sum_k \eta_k g_{W^+\phi_j^{-Q}\phi_k^{(Q-1)}} g_{W^-\phi_k^{-(Q-1)}\phi_i^Q} - \sum_k \eta_k g_{W^+\phi_i^Q\phi_k^{-(Q+1)}} g_{W^-\phi_k^{(Q+1)}\phi_j^{-Q}} = -g f_{\phi_i^Q\phi_j^{-Q}} . \tag{A8}$$

Let us now turn to the W^+W^+ channel. Sum rules for the W^-W^- channel are obtained by charge conjugation of the ones given below:

$$a=W^+,b=W^+,i=\phi_i^-,j=\phi_j^- : \quad \sum_k \eta_k g_{W^+\phi_i^-\phi_k^0} g_{W^+\phi_k^0\phi_j^-} + \frac{1}{4m_Z^2} g_{W^+Z\phi_i^-} g_{W^+Z\phi_j^-} = \tfrac{1}{2} g_{W^+W^+\phi_i^-\phi_j^-} ; \tag{A9}$$

$$a=W^+,b=W^+,i=\phi_i^{--},j=\phi_j^0 : \quad \sum_k \eta_k g_{W^+\phi_i^{--}\phi_k^+} g_{W^+\phi_k^-\phi_j^0} + \frac{1}{4m_W^2} g_{W^+W^+\phi_i^{--}} g_{W^+W^-\phi_j^0} = \tfrac{1}{2} g_{W^+W^+\phi_i^{--}\phi_j^0} ; \tag{A10}$$

and finally, for $Q \geq 3$, we have the generalization of Eq. (A10),

$$\sum_k \eta_k g_{W^+\phi_i^{-Q}\phi_k^{(Q-1)}} g_{W^+\phi_k^{-(Q-1)}\phi_j^{(Q-2)}} = \tfrac{1}{2} g_{W^+W^+\phi_i^{-Q}\phi_j^{(Q-2)}} . \tag{A11}$$

We turn next to the W^+Z initial state; W^-Z sum rules are obtianed by charge conjugation:

$$a=W^+,b=Z,i=\phi_i^-,j=\phi_j^0 : \quad \sum_k \eta_k g_{W^+\phi_i^-\phi_k^0} g_{Z\phi_k^0\phi_j^0} + \frac{1}{4m_Z^2} g_{W^+Z\phi_i^-} g_{ZZ\phi_j^0} + \tfrac{1}{2} g c_W g_{W^+\phi_i^-\phi_j^0} = \tfrac{1}{2} g_{W^+Z\phi_i^-\phi_j^0} ; \tag{A12}$$

$$a=W^+,b=Z,i=\phi_j^0,j=\phi_i^- : \quad \sum_k \eta_k g_{W^+\phi_j^0\phi_k^-} g_{Z\phi_k^+\phi_i^-} - \sum_k \eta_k g_{W^+\phi_i^-\phi_k^0} g_{Z\phi_k^0\phi_j^0}$$

$$+ \frac{1}{4m_W^2} g_{W^+W^-\phi_j^0} g_{W^+Z\phi_i^-} - \frac{1}{4m_Z^2} g_{W^+Z\phi_i^-} g_{ZZ\phi_j^0} - g c_W g_{W^+\phi_i^-\phi_j^0} = 0 , \tag{A13}$$

where we used Eq. (A12) to eliminate the quartic coupling in writing Eq. (A13).

$a=W^+, b=Z, i=\phi_i^{--}, j=\phi_j^+$:

$$\sum_k \eta_k g_{W^+\phi_i^{--}\phi_k^+} g_{Z\phi_k^-\phi_j^+} + \frac{1}{4m_W^2} g_{W^+W^+\phi_i^{--}} g_{W^-Z\phi_j^+} + \tfrac{1}{2} g c_W g_{W^+\phi_i^{--}\phi_j^+} = \tfrac{1}{2} g_{W^+Z\phi_i^{--}\phi_j^+} \; ; \tag{A14}$$

$a=W^+, b=Z, i=\phi_j^+, j=\phi_i^{--}$:

$$\sum_k \eta_k g_{W^+\phi_j^+\phi_k^-} g_{Z\phi_k^{++}\phi_i^{--}} - \sum_k \eta_k g_{W^+\phi_i^{--}\phi_k^+} g_{Z\phi_k^-\phi_j^+} - \frac{1}{4m_W^2} g_{W^+W^+\phi_i^{--}} g_{W^-Z\phi_j^+} - g c_W g_{W^+\phi_i^{--}\phi_j^+} = 0 \; , \tag{A15}$$

where we have, in the usual manner, substituted for the quartic coupling in Eq. (A15) using Eq. (A14). Generalization to higher charges ($Q \geq 3$) is obvious.

$a=W^+, b=Z, i=\phi_i^{-Q}, j=\phi_j^{(Q-1)}$:

$$\sum_k \eta_k g_{W^+\phi_i^{-Q}\phi_k^{(Q-1)}} g_{Z\phi_k^{-(Q-1)}\phi_j^{(Q-1)}} + \tfrac{1}{2} g c_W g_{W^+\phi_i^{-Q}\phi_j^{(Q-1)}} = \tfrac{1}{2} g_{W^+Z\phi_i^{-Q}\phi_j^{(Q-1)}} \; ; \tag{A16}$$

$a=W^+, b=Z, i=\phi_j^{(Q-1)}, j=\phi_i^{-Q}$:

$$\sum_k \eta_k g_{W^+\phi_j^{(Q-1)}\phi_k^{-Q}} g_{Z\phi_k^{Q}\phi_i^{-Q}} - \sum_k \eta_k g_{W^+\phi_i^{-Q}\phi_k^{(Q-1)}} g_{Z\phi_k^{-(Q-1)}\phi_j^{(Q-1)}} - g c_W g_{W^+\phi_i^{-Q}\phi_j^{(Q-1)}} = 0 \; . \tag{A17}$$

Finally, we can detail the ZZ initial state relations using just three equations:

$a=Z, b=Z, i=\phi_i^0, j=\phi_j^0$:

$$\sum_k \eta_k g_{Z\phi_i^0\phi_k^0} g_{Z\phi_k^0\phi_j^0} + \frac{1}{4m_Z^2} g_{ZZ\phi_i^0} g_{ZZ\phi_j^0} = \tfrac{1}{2} g_{ZZ\phi_i^0\phi_j^0} \; ; \tag{A18}$$

$a=Z, b=Z, i=\phi_i^+, j=\phi_j^-$:

$$\sum_k \eta_k g_{Z\phi_i^+\phi_k^-} g_{Z\phi_k^+\phi_j^-} + \frac{1}{4m_W^2} g_{ZW^-\phi_i^+} g_{W^+Z\phi_j^-} = \tfrac{1}{2} g_{ZZ\phi_i^+\phi_j^-} \; ; \tag{A19}$$

the higher charge, $Q \geq 2$ generalization of this last equation is

$$\sum_k \eta_k g_{Z\phi_i^Q\phi_k^{-Q}} g_{Z\phi_k^Q\phi_j^{-Q}} = \tfrac{1}{2} g_{ZZ\phi_i^Q\phi_j^{-Q}} \; . \tag{A20}$$

This completes our explicit enumeration of the $AA \to \phi\phi$ unitarity sum rules.

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