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Stability of the quasiregular singularities in Bell-Szekeres spacetime

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The behavior of geodesics and test minimally coupled scalar waves on the Bell-Szekeres spacetime is used to probe the nature of the quasiregular singularities present. Components of the stress-energy tensor in a parallel-propagated orthonormal frame diverge as does a stress-energy scalar at the quasiregular singularities. It is argued that this divergence makes the singularities unstable, converting them into scalar curvature singularities.

I. INTRODUCTION

Both colliding plane-wave spacetimes and the nature of spacetime singularities are interesting subjects in classical general relativity. Here we consider their overlap. In colliding plane-wave spacetimes the energy density of each wave focuses the other, leading to the formation of singularities.[1] The nature of the singularities formed has been the subject of discussion.[2–4] In most cases a scalar curvature singularity forms, but in at least one case, the Bell-Szekeres spacetime, the only singularities are quasiregular.[5] In this paper we will consider the stability of these quasiregular singularities.

Quasiregular singularities were first named by Ellis and Schmidt, who classified singularities in maximal, four-dimensional spacetimes into three basic types: scalar curvature, nonscalar curvature, and quasiregular.[6] The obstacle which bars the embedding of singular spacetimes into larger nonsingular spacetimes is obvious for those with scalar curvature singularities, where physical quantities such as energy density and tidal forces diverge for all observers who encounter the singularity. The physical significance of the other two types of singularity is less obvious. In the case of a nonscalar curvature singularity some, but not all, observers feel infinite tidal forces as they approach the singularity. It is still more curious that for a quasiregular singularity *no* observers see the physical quantities diverge, even though their world lines end at the singularity in a finite proper time.

Quasiregular singularities are the mildest type of true singularity, and they are also the least well understood.[6] By definition a singular point q is a C^k (or C^{k-}) *quasiregular* singularity ($k \geq 0$) if all components or derivatives of the Riemann tensor $R_{abcd;e_1 \cdots e_k}$ evaluated in an orthonormal (ON) frame parallel propagated (PP) along an incomplete geodesic ending at q are C^0 (or C^{0-}). In other words, the Riemann tensor components and derivatives tend to finite limits (or are bounded) in every PPON frame. On the other hand, a singular point q is a C^k (or C^{k-}) *curvature* singularity if some component or derivative is not bounded in this way. If all scalars in g_{ab}, the antisymmetric tensor η_{abcd}, and $R_{abcd;e_1 \cdots e_k}$ nevertheless tend to a finite limit (or are bounded) the singularity is *nonscalar* but if any scalar is unbounded, the point q is a *scalar* curvature singularity.

Quasiregular singularities have been found in "Taub-NUT-(Newman-Unti-Tamburino-)type" cosmologies,[7] cosmic-string models,[8–10] and in colliding plane-wave spacetimes.[4,5] One suspects from their strange properties that although they occur in exact solutions of Einstein's equations they may be unstable, so that the addition of generic matter or fields to quasiregular spacetimes may convert these mild singularities into a stronger form. We have previously studied the stability of singularities in Taub-NUT-type cosmologies using test scalar and electromagnetic fields.[7,11] We conjectured that if one introduces a test field whose stress-energy tensor evaluated in a PPON frame mimics the behavior of the Riemann tensor components which indicate a particular type of singularity (quasiregular, nonscalar curvature, or scalar curvature), then a complete nonlinear back-reaction calculation would show that this type of singularity actually occurs. For example, if a scalar quantity such as $T_{\mu\nu}T^{\mu\nu}$ constructed from a test field's stress-energy tensor diverges as a quasiregular singularity is approached, the conjecture is that a scalar curvature singularity will actually develop if the field is allowed to influence the geometry. Evidence for the conjecture was presented from a few known exact solutions, Taub-NUT-type cosmologies and Khan-Penrose spacetime. Evidence showed also that most test-field wave modes do in fact mimic scalar curvature singularities but that very special wave modes can mimic nonscalar or quasiregular singularities.[11] Therefore, if generic fields are added, one expects that the quasiregular singularities will be converted into scalar curvature singularities.

In this paper we extend our conjecture to include the quasiregular singularities in the Bell-Szekeres colliding impulsive plane electromagnetic wave spacetime by examining the behavior of test scalar fields. We show that the behavior of the fields and their stress-energy tensors is similar to their behavior in Khan-Penrose spacetime and the Taub-NUT-type spacetimes. In fact, as in those spacetimes, the quasiregular singularities appear unstable, and the conjecture predicts that they will turn into scalar curvature singularities if a complete back-reaction calculation is carried out.

II. GEODESICS AND TEST SCALAR FIELDS

In the Bell-Szekeres spacetime,[12] δ-function electromagnetic waves propagate into an initially flat region. In the two-dimensional slice shown in Fig. 1, the planes $u=0$ and $v=0$ are δ-function waves. Region I is flat, but regions II, III, and IV are curved. Using Rosen coordinates, the double-null metrics are

$$\text{I:}\quad ds^2=2du\,dv-dx^2-dy^2\ , \tag{1a}$$

$$\text{II:}\quad ds^2=2du\,dv-[\cos(au)]^2(dx^2+dy^2)\ , \tag{1b}$$

$$\text{III:}\quad ds^2=2du\,dv-[\cos(bv)]^2(dx^2+dy^2)\ , \tag{1c}$$

$$\text{IV:}\quad ds^2=2du\,dv-[\cos(au-bv)]^2dx^2-[\cos(au+bv)]^2dy^2\ . \tag{1d}$$

Using the better-behaved Brinkmann coordinates, Clarke and Hayward[5] show that the lines L at $u=\pi/2a$, $v<0$ and at $v=\pi/2b$, $u<0$ are quasiregular singularities, that the only singularities in region IV are on the wave fronts $u=0$, $v=\pi/2b$ and $v=0$, $u=\pi/2a$, and that region IV may be extended. However, the presence of the singularities in region IV means that there is no unique extension; Clarke and Hayward[5] describe two natural extensions in their paper.

We are interested in the stability of the quasiregular singularities at L in regions II and III. To investigate their stability, we will describe geodesic and scalar wave behavior and calculate stress-energy tensors and scalars.

We limit our discussion of geodesics in region II to those that originate at the wave front and move toward the quasiregular singularity. Geodesics in region III are similar. Solutions of the geodesic equations in region II are

$$\begin{aligned} x&=x_0+(x_0'/a)\tan(au)\ ,\\ y&=y_0+(y_0'/a)\tan(au)\ ,\\ v&=v_0+[(x_0'^2+y_0'^2)/2a]\tan(au)+\tfrac{1}{2}\epsilon u\ , \end{aligned} \tag{2}$$

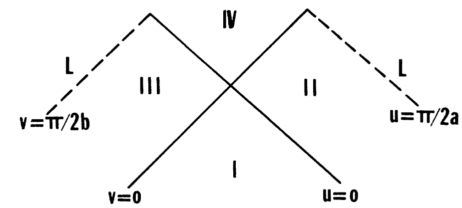

FIG. 1. Colliding impulsive electromagnetic waves. The surfaces $u=0$ and $v=0$ are δ-function plane electromagnetic waves. The dashed lines L are quasiregular singularities.

where $\epsilon=0,1$ depending on whether the geodesics are null or timelike, and where $x_0, y_0, v_0, x_0', y_0'$ are constants of integration. If either x_0' or y_0' is nonzero, then v becomes positive for some $u<\pi/2a$ and the light ray or particle passes out of region II into region IV.

Geodesics that remain in region II are those for which

$$x=x_0,\quad y=y_0,\quad v=v_0+\tfrac{1}{2}\epsilon u\ , \tag{3}$$

where $v_0<-\epsilon\pi/4a$. That is, only the small subset of geodesics with constant values of x and y that start from the wave front sufficiently early will reach $u=\pi/2a$ in region II.

To study the stability of these quasiregular singularities we will consider the behavior of a minimally coupled scalar field. Consider region II (one of the two regions bounded by a quasiregular singularity). The field equation for scalar waves is

$$\Box\Phi=g^{\lambda\kappa}\Phi_{,\lambda\kappa}+g^{-1/2}(g^{1/2}g^{\lambda\kappa})_{,\kappa}\Phi_{,\lambda}=0\ . \tag{4}$$

It is straightforward to solve (4) in region II. For simplicity, consider x,y-independent waves. A general solution is then

$$\Phi(u,v)=g(u)+\sec(au)f(v)\ . \tag{5}$$

The stress-energy tensor is $T_{\mu\nu}=(1/4\pi)(F_{\mu\alpha}F^{\alpha}_{\nu}-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})$. If we consider waves moving in the u direction toward the singularity, then $g(u)=0$ and $\Phi=\sec(au)f(v)$. In that case

$$T_{\mu\nu}=(1/4\pi)\mathrm{diag}(a^2\sec^2(au)\tan^2(au)f^2(v),\ \sec^2(au)[f'(v)]^2,\ a\tan(au)f(v)f'(v),\ a\tan(au)f(v)f'(v))\ . \tag{6}$$

Notice each component diverges as $au\to\pi/2$; that is, each component diverges as the singularity is approached.

Next consider the stress-energy tensor in a PPON frame:

$$T_{(ab)}=E^{\mu}_{(a)}E^{\nu}_{(b)}T_{\mu\nu}\ . \tag{7}$$

For consistency we will consider only orthonormal frames carried by geodesic observers with no x or y motion. Such frame vectors are given in the Appendix.

Then the nonzero components of $T_{(ab)}$ are

$$\begin{aligned} T_{(00)}=T_{(33)}&=a^2\sec^2(au)\tan^2(au)f^2(v)+\frac{\sec^2(au)[f'(v)]^2}{4}\ ,\\ T_{(03)}=T_{(30)}&=a^2\sec^2(au)\tan^2(au)f^2(v)-\frac{\sec^2(au)[f'(v)]^2}{4}\ ,\\ T_{(11)}=T_{(22)}&=a\sec^2(au)\tan^2(au)f(v)f'(v)\ . \end{aligned} \tag{8}$$

These also diverge badly as $au\rightarrow\pi/2$. By our conjecture this indicates that the quasiregular singularity will be converted to a curvature singularity if a complete back-reaction calculation is carried out.

Finally consider a scalar in the stress-energy tensor $T_{\mu\nu}T^{\mu\nu}$. It takes the form

$$T_{\mu\nu}T^{\mu\nu}=\frac{a^2}{4\pi^2}\sec^4(au)\tan^2(au)f^2(v)[f'(v)]^2 \tag{9}$$

which also diverges as the singularity is approached. By our conjecture this indicates that the curvature singularity formed will be a scalar curvature singularity.

Both $T_{\mu\nu}T^{\mu\nu}$ and $T_{(ab)}$ diverge as $au\rightarrow\pi/2$ for x,y-independent scalar waves. The Bell-Szekeres spacetime plus such fields therefore "mimics" the behavior of a spacetime which reacts to the presence of the fields by converting the quasiregular singularity into a scalar curvature singularity. Although no back-reaction calculation has been carried out, we expect these fields will convert the quasiregular singularities at $au=\pi/2$ in region II and at $av=\pi/2$ in region III into scalar curvature singularities. There is no proof that this conversion takes place. However, in the few cases where a back-reaction calculation on a quasisingular spacetime has been made, the mimicking of scalar curvature, nonscalar curvature, and quasiregular singularities by the behavior of test fields is a completely reliable guide.

Thus, as in Khan-Penrose spacetime the mimicking conjecture predicts that the quasiregular singularities of Bell-Szekeres spacetime are unstable. In each case it is predicted that the quasiregular singularities will be converted to scalar curvature singularities if generic waves are added. In the Khan-Penrose case we were able to demonstrate[13] the validity of the conjecture by comparing with an exact solution discovered by Chandrasekhar and Xanthopoulos.[14] Here no check of the conjecture is possible because as yet there are no known exact solutions with which to compare.

Chandrasekhar and Xanthopoulos have carried out a linear perturbation analysis of the Bell-Szekeres spacetime.[15] They find in region II that u-dependent perturbations of the metric exhibit strong divergences at the quasiregular singularities. Although their analysis is linear, and therefore not a full back-reaction calculation, their results are clearly consistent with the mimicking conjecture.

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APPENDIX: PARALLEL-PROPAGATED ORTHONORMAL FRAME VECTORS

Using the geodesic equations, it is straightforward to derive frame vectors in region II which satisfy the parallel propagation condition $E^{\mu}_{(a);\nu}E^{\nu}_{(0)}=0$, and the orthogonality condition $E_{(a)\mu}E^{\mu}_{(b)}=\eta_{(ab)}$. The frame vectors are

$$E^{\mu}_{(0)}=\begin{bmatrix} 1\\ \frac{1}{2}\left[1+\frac{x_0'^2+y_0'^2}{\cos^2(au)}\right]\\ x_0'/\cos^2(au)\\ y_0'/\cos^2(au)\end{bmatrix},\quad E^{\mu}_{(1)}=\begin{bmatrix}0\\ x_0'/\cos(au)\\ 1/\cos(au)\\ 0\end{bmatrix},$$

$$E^{\mu}_{(2)}=\begin{bmatrix}0\\ y_0'/\cos(au)\\ 0\\ 1/\cos(au)\end{bmatrix},\quad E^{\mu}_{(3)}=\begin{bmatrix} 1\\ \frac{1}{2}\left[\frac{x_0'^2+y_0'^2}{\cos^2(au)}-1\right]\\ x_0'/\cos^2(au)\\ y_0'/\cos^2(au)\end{bmatrix}.$$

In the special case of no x or y motion, the frame vectors are simply

$$E^{\mu}_{(0)}=\begin{bmatrix}1\\ \frac{1}{2}\\ 0\\ 0\end{bmatrix},\quad E^{\mu}_{(1)}=\begin{bmatrix}0\\ 0\\ \sec(au)\\ 0\end{bmatrix},$$

$$E^{\mu}_{(2)}=\begin{bmatrix}0\\ 0\\ 0\\ \sec(au)\end{bmatrix},\quad E^{\mu}_{(3)}=\begin{bmatrix}1\\ -\frac{1}{2}\\ 0\\ 0\end{bmatrix}.$$

The x,y-independent geodesics which carry these frames all hit the quasiregular singularity.

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