Soluble extensions of the Dirac oscillator with exact and broken supersymmetry

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We consider a large class of Dirac oscillator-type couplings that exhibit a three-dimensional hidden supersymmetry. A subclass of exactly soluble cases is determined by using the Infeld and Hull procedure. We find that the corresponding spectra possess a high degree of unphysical degeneracy, similar to the Dirac oscillator case. This difficulty is overcome by proposing a further generalization of this coupling, which breaks supersymmetry but retains exact solubility. We also discuss the covariance properties of the new coupling together with its Poincaré-invariant extension to the many-particle case.

The revival by Moshinsky and Szczepaniak[1] of a nonminimal coupling scheme for the one-particle Dirac equation, linear both in momenta and coordinates (referred to as the Dirac oscillator[2,3]), has sparked a number of investigations concerned with its covariance and *CPT* properties,[4] hidden supersymmetric character,[5,6] and the stability of the corresponding Dirac sea,[6] as well as with its generalization to two- and n-particle systems.[7,8]

The exact solubility of the Dirac oscillator, together with the appropriate mathematical behavior of its eigenstates (the many-particle states have definite mass and total angular momentum and are characterized by an irreducible representation of the Poincaré group[8,9]) suggest it may be used as a first approximation to describe the confinement potential for quarks in QCD.[4] On the mathematical side, such eigenfunctions could provide a convenient analytic basis to deal with more realistic interactions in a similar manner to the way nonrelativistic harmonic-oscillator eigenfunctions are used in nuclear physics.[10] However, in Refs. 7 and 8 the mass spectra of hadrons composed of nonstrange quarks interacting through Dirac oscillator potentials have been calculated, yielding the quite unpleasant result that the ground state of both mesons and baryons is infinitely degenerate, an undesirable feature already present in the single-particle case. A possible way of breaking such degeneracy was proposed some time ago by Cho.[2] Nevertheless, the question still arises if one can modify the Dirac oscillator Hamiltonian appropriately, in order to remove these unphysical features, retaining the simplicity of its solutions and the possibility of constructing a covariant generalization to relativistic many-particle systems. The purpose of this work is to report on two different exactly soluble generalizations of the Dirac oscillator coupling, including one that removes the infinite degeneracies which can be extended to a relativistic many-body Hamiltonian.

Let us consider the nonminimal substitution

$$\mathbf{p} \rightarrow \mathbf{p} - i\beta \mathbf{G} \tag{1}$$

in the free Dirac equation, where $\mathbf{G}$ is an arbitrary real vector operator and $\beta = \gamma^0$. Note that the Dirac oscillator corresponds to $\mathbf{G} = \mathbf{r}$. Using the standard representation of the Dirac matrices[11] the Hamiltonian is given by ($\hbar = c = m\omega = 1$)

$$E = H_D = \begin{bmatrix} m & \boldsymbol{\sigma}\cdot\mathbf{A}^\dagger \\ \boldsymbol{\sigma}\cdot\mathbf{A} & -m \end{bmatrix}, \tag{2}$$

where $\mathbf{A} \equiv \mathbf{p} - i\mathbf{G}$. In order to decouple the large and small components, ψ_1 and ψ_2 of the Dirac spinor ψ, we calculate H_D^2 and rewrite the eigenvalue equation as

$$(E^2 - m^2)\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\sigma}\cdot\mathbf{A}^\dagger)(\boldsymbol{\sigma}\cdot\mathbf{A}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{A}^\dagger) \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \equiv \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \tag{3}$$

In the above equation, H_+ and H_- are supersymmetric partners for any choice of $\mathbf{G}$, since we can define, following Refs. (3) and (6), the Hermitian supercharges

$$Q_1 = \begin{bmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{A}^\dagger \\ \boldsymbol{\sigma}\cdot\mathbf{A} & 0 \end{bmatrix}, \quad Q_2 = i\begin{bmatrix} 0 & -\boldsymbol{\sigma}\cdot\mathbf{A}^\dagger \\ \boldsymbol{\sigma}\cdot\mathbf{A} & 0 \end{bmatrix}, \tag{4}$$

which together with the operator $(H_D^2 - m^2)$ satisfy an $S(2)$ superalgebra

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}(H_D^2 - m^2), \tag{5a}$$

$$[Q_\alpha, (H_D^2 - m^2)] = 0, \tag{5b}$$

$$\alpha, \beta = 1, 2.$$

Such supersymmetric partners were previously considered also in Ref. 12 as the result of taking the square root of three-dimensional Hamiltonians. We remark that the supersymmetric character of the radial equation of the Dirac oscillator[5] is a manifestation of the full three-dimensional intrinsic supersymmetry of this coupling. The whole family of systems defined by the substitution (1) possesses at least an $S(2)$ superalgebra as a dynamical supersymmetry, which however does not guarantee exact solubility. In order to address this question we restrict our attention to the spherically symmetric case by choosing $\mathbf{G}=G(r)\hat{\mathbf{r}}$. It is enough to consider H_+, since supersymmetry directly relates its solutions to those of H_-. Using the standard decomposition

$$\psi_1=|N(l1/2)jm\rangle = \sum_{\mu,\sigma}\langle l\mu 1/2\sigma|jm\rangle[u_{Nl}(r)/r]\mathcal{Y}_{l\mu}(\Omega)\mathcal{X}_\sigma ,$$

the factorized form of H_+ translates into the radial Hamiltonian

$$H_+=\left[G(r)-\frac{\kappa}{r}-\frac{d}{dr}\right]\left[G(r)-\frac{\kappa}{r}+\frac{d}{dr}\right], \tag{6}$$

which acts upon the wave functions $u_{Nl}(r)$ with eigenvalues $\epsilon\equiv(E^2-m^2)$. Here $\kappa=(-1)^{j-l-1/2}(j+\frac{1}{2})$ are the eigenvalues of $(\boldsymbol{\sigma}\cdot\mathbf{L}+1)$. The form of the Hamiltonian (6) suggests the Infeld and Hull procedure to determine all possible functions $G(r)$ that provide exact solutions.[13] We find that only type-C and type-F factorizations are admissible, leading to the following choices for $G(r)$:

$$G_C(r)=\frac{\alpha}{r}+\beta r , \tag{7a}$$

$$G_F(r)=\frac{\alpha'}{r}+\beta' , \tag{7b}$$

with $\alpha,\alpha',\beta,\beta'$ being free parameters. The type-C case corresponds to a harmonic oscillator plus an additional centrifugal barrier, while the type-F case reduces to the Coulomb problem with an effective charge $e^2=2\beta'(\kappa-\alpha')$ also with a modification to the centrifugal potential. Both Hamiltonians can be solved directly by comparison with well-known results.[14] Here we write down the energy eigenvalues for the confining potential (7a), which could represent a good candidate for an improved version of the Dirac oscillator

$$\epsilon^C_{Nlj}=\begin{cases}2|\beta|(N+|j+1+\alpha|-j+\frac{1}{2})+2\beta(\alpha+j), & l=j+\frac{1}{2} , \quad (8a)\\ 2|\beta|(N+|j-\alpha|-j+\frac{3}{2})+2\beta(\alpha-j-1), & l=j-\frac{1}{2} . \quad (8b)\end{cases}$$

Unfortunately, we can show from Eq. (8) that the infinite spectrum degeneracies of the Dirac oscillator are not removed by the additional coupling. A similar problem arises for the Coulomb case (7b).

From these considerations, it is reasonable to surmise that the underlying supersymmetry is responsible for the high degeneracy exhibited by these systems. We are thus lead to consider supersymmetry-breaking interactions, imposing however the condition that the resulting Hamiltonian remains exactly soluble, in addition to retaining its covariance properties. These conditions turn out to be very restrictive. Nevertheless, we have succeeded in finding such an interaction. Going back for simplicity to the original Dirac oscillator, we propose a generalization of (2) given by

$$\tilde{H}_D=\begin{bmatrix}\lambda(\boldsymbol{\sigma}\cdot\mathbf{L}+1)+m & \boldsymbol{\sigma}\cdot(\mathbf{p}+i\mathbf{r})\\ \boldsymbol{\sigma}\cdot(\mathbf{p}-i\mathbf{r}) & \lambda(\boldsymbol{\sigma}\cdot\mathbf{L}+1)-m\end{bmatrix}, \tag{9}$$

where λ is an arbitrary real parameter. The supersymmetry-breaking spin-orbit operator $(\boldsymbol{\sigma}\cdot\mathbf{L}+1)$ has the remarkable property

$$\{\boldsymbol{\sigma}\cdot\boldsymbol{\pi},(\boldsymbol{\sigma}\cdot\mathbf{L}+1)\}=0 , \tag{10}$$

valid for any vector operator $\boldsymbol{\pi}$ which satisfies the condition $\boldsymbol{\pi}\cdot\mathbf{L}=0$. This property leads to a block-diagonal structure for $\tilde{H}^2_D$, in spite of the fact that supersymmetry is broken.[6] In this case the corresponding 2×2 block-diagonal Hamiltonians are

$$\tilde{H}_+=p^2+r^2+[2\lambda(\lambda+2m)-4]\mathbf{L}\cdot\mathbf{S} +\lambda^2L^2+\lambda(\lambda+2m)-3 , \tag{11a}$$

$$\tilde{H}_-=p^2+r^2+[2\lambda(\lambda-2m)+4]\mathbf{L}\cdot\mathbf{S} +\lambda^2L^2+\lambda(\lambda-2m)+3 , \tag{11b}$$

following the notation of Eq. (3).

These Hamiltonians include a λ-dependent spin-orbit coupling, together with an interaction term proportional to L^2, which constitute the main modifications with respect to the original Dirac oscillator ($\lambda=0$). The kets $|N(l1/2)jm\rangle$ are still eigenvectors of $\tilde{H}_+$ and $\tilde{H}_-$ with new eigenvalues

$$\tilde{\epsilon}^+_{Nlj}=\begin{cases}[2(N+j)+3]+(j+\frac{1}{2})[\lambda^2(j+\frac{1}{2})-2\lambda m], & l=j+\frac{1}{2} ,\\ [2(N-j)+1]+(j+\frac{1}{2})[\lambda^2(j+\frac{1}{2})+2\lambda m], & l=j-\frac{1}{2} ,\end{cases} \tag{12a}$$

$$\tilde{\epsilon}^-_{Nlj}=\begin{cases}[2(N-j)+3]+(j+\frac{1}{2})[\lambda^2(j+\frac{1}{2})+2\lambda m], & l=j+\frac{1}{2} ,\\ [2(N+j)+5]+(j+\frac{1}{2})[\lambda^2(j+\frac{1}{2})-2\lambda m)], & l=j-\frac{1}{2} .\end{cases} \tag{12b}$$

From expressions (12) we readily verify that the degeneracy of the system is completely removed by the λ-dependent contributions to the energy, except for particular choices of this parameter. Observe that the full eigenspinor ψ of the Hamiltonian (9) can be directly obtained from a linearly coupled system.

We now introduce the covariant Dirac operator which induces Hamiltonian (9), together with its generalization to the multiparticle case. A suggestive way of rewriting (9) is

$$\tilde{H}_D=\boldsymbol{\alpha}\cdot(\mathbf{p}-i\beta\mathbf{r}+\lambda\gamma_5\mathbf{J})-\frac{\lambda}{2}+\beta m \tag{13}$$

which emphasizes the fact that new nonminimal interactions can be introduced through additional vectors in the system, such as $\gamma_5\mathbf{J}$, where $\mathbf{J}=\mathbf{L}+\mathbf{S}$. The above Hamiltonian is obtained, in the standard manner, from the covariant Dirac operator

$$\gamma^\mu[p_\mu-i\bar{x}_\mu(\gamma^\alpha u_\alpha)+(\lambda/2)(\gamma_5\epsilon_{\mu\nu\rho\sigma}u^\nu J^{\rho\sigma}-u_\mu)]+m \ , \tag{14}$$

where $\epsilon_{0123}=+1$ and $J^{\mu\nu}=x^\mu p^\nu-x^\nu p^\mu+(i/4)[\gamma^\mu,\gamma^\nu]$ are the generators of Lorentz transformations. We are using the metric tensor $g_{\mu\nu}=\text{diag}(-1,1,1,1)$. We have also introduced the timelike vector $u_\alpha=(1,\mathbf{0})$ in the frame $p_\alpha=(-E,\mathbf{p})$ where the calculation is performed.[4] Here $\bar{x}_\alpha=x_\alpha+u_\alpha(x^\beta u_\beta)$ and it can be verified that the term proportional to $\gamma_\mu\bar{x}^\mu$ in (14) reduces to the Pauli interaction $\sigma_{\mu\nu}F^{\mu\nu}$ with the electromagnetic tensor considered in Refs. 4 and 5.

Our next step is to generalize the Dirac operator (14) to the case of $n\geq 2$ identical particles. We observe that in this situation we have a natural realization for the vector u_α given by $u_\alpha=-P_\alpha/(-P^\beta P_\beta)^{1/2}$, where $P_\alpha=\sum_{s=1}^n p_{\alpha s}$ is the momentum of the center of mass. Following Refs. (15) and (9), the n-particle free Dirac equation is written as

$$\sum_{s=1}^{n}\Gamma_s[\gamma_s^\mu(n^{-1}P_\mu+p'_{\mu s})+m]\psi=0 \ , \tag{15}$$

where $p'_{\mu s}=p_{\mu s}-n^{-1}P_\mu$ is the relative momentum of the particle s with respect to the center of mass and the scalar matrix Γ_s is defined by

$$\Gamma_s=(\gamma_s^\alpha u_\alpha)^{-1}\Gamma,\quad \Gamma=\prod_{s=1}^{n}(\gamma_s^\alpha u_\alpha) \ . \tag{16}$$

Here γ_s^μ is the direct product of $(n-1)$ 4×4 unit matrices and one γ^μ matrix in position s. As shown in Ref. 9, the covariant operator leading to the n-particle Dirac oscillator Hamiltonian is obtained from (15) by making the substitution

$$p'_{\alpha s}\to p'_{\alpha s}-ix'_{\alpha s}\Gamma \ , \tag{17}$$

where $x'_{\alpha s}$ is the relative coordinate of particle s with respect to the center of mass. The corresponding extension for the Hamiltonian (13) is achieved through the replacement

$$p'_{\alpha s}\to p'_{\alpha s}-ix'_{\alpha s}\Gamma+\frac{\lambda}{2}(\gamma_{5s}\epsilon_{\alpha\nu\rho\sigma}u^\nu J_s'^{\rho\sigma}-u_\alpha) \tag{18}$$

in the free Dirac operator (15), where $J_s'^{\mu\nu}=x_s'^\mu p_s'^\nu-x_s'^\nu p_s'^\mu+(i/4)[\gamma_s^\mu,\gamma_s^\nu]$. One may doubt whether to replace γ_5 in (14) by γ_{5s}, as done in Eq. (18), or to consider $\gamma_5\to\Gamma_5=\prod_{s=1}^n\gamma_{5s}$ in analogy with the pure Dirac oscillator term. The last possibility can be rejected since the Lorentz transformation $\Gamma_5\to(\det\Lambda)^n\Gamma_5$ would imply that the corresponding term behaves as a vector for odd values of n only. The multiparticle Dirac operator constructed according to (18) commutes with the generators of the Poincaré group P_μ, $J^{\mu\nu}=\sum_{s=1}^n J_s'^{\mu\nu}$ because by construction this operator is a Lorentz scalar and is translationally invariant due to the use of relative coordinates. Thus, the many-particle states will indeed be characterized by irreducible representations of the Poincaré group and could be used as an appropriate basis for relativistic systems. Finally in the center-of-mass system $P^0=M$, $\mathbf{p}=\mathbf{0}$, the mass operator is

$$M=\sum_{s=1}^{n}[\boldsymbol{\alpha}_s\cdot(\mathbf{p}'_s-i\mathbf{x}'_s\Gamma)+\lambda(\boldsymbol{\Sigma}_s\cdot\mathbf{L}'_s+1)+m\beta_s] \ , \tag{19}$$

where $\boldsymbol{\Sigma}_s$ is the spin operator of particle s.

We have proposed an n-particle Poincaré-invariant interaction that is a supersymmetry-breaking generalization of the Dirac oscillator coupling, which also leads to confining potentials of the linear type. The one-particle case is exactly soluble and the resulting spectrum is nondegenerate. These features are essential for a realistic application of the n-particle solutions to the spectroscopy of mesons[7] $(n=2)$ and baryons[8] $(n=3)$ and could lead to a fully relativistic basis for nuclear excitations involving subnucleonic degrees of freedom, a subject of considerable current interest.[16] After the completion of this work we became aware of Ref. 17 where an alternative way of breaking the infinite degeneracy is proposed.

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