Vacuum-polarization effects in global monopole space-times

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The gravitational effect produced by a global monopole may be approximated by a solid deficit angle. As a consequence, the energy-momentum tensor of a quantum field will have a nonzero vacuum expectation value. Here we study this "vacuum-polarization effect" around the monopole. We find explicit expressions for both $\langle \phi^2 \rangle_{\rm ren}$ and $\langle T_{\mu\nu} \rangle_{\rm ren}$ for a massless scalar field. The back reaction of the quantum field on the monopole metric is also investigated.

I. INTRODUCTION

Phase transitions in the early Universe can give rise to the formation of macroscopic topological defects such as domain walls, strings, and monopoles.[1] These defects appear as the result of local or global symmetry breakings and many have important cosmological consequences. They can produce observational effects[1,2] and may also provide interesting mechanisms for galaxy formation.[1,3]

From the point of view of the gravitational effects, local monopoles are not particularly relevant, except for the fact that a huge quantity of them should be produced in the early Universe. Some adequate mechanism, such as inflation, must prevent their observation.[4] The main result concerning their gravitational field,[5] and this is the aspect we are going to study in detail, is that they generate a Schwarzschild metric. This is due to the fact that outside the monopole the energy of the gauge field is exactly compensated by the Higgs field, thus yielding a localized energy density.

It was not until recently[6] that the subject of global monopoles has been brought into play. In flat space-time they have a linearly divergent mass, but when one considers their formation in a cosmological scenario it can be argued that the cosmological horizon at the time of their formation ($t \sim t_{\rm GUT}$) can play the role of a natural cutoff.

With a linearly divergent mass it would be reasonable to expect that space-time would be highly curved near the global monopole, but by taking into account the gravitational field generated by one of these global monopoles Barriola and Vilenkin[6] have found that its gravitational effects can be described by a deficit solid angle plus a tiny mass localized at the origin, of the order $\eta_0/\sqrt{\lambda}$ (η_0 being the energy scale of symmetry breaking and λ the coupling constant for the potential producing this symmetry breaking). The Lagrangian of the Goldstone field is

$$L = -\tfrac{1}{2} g^{\mu\nu}\partial_\mu \phi^a \partial_\nu \phi^a - \frac{\lambda}{4}(\phi^a\phi^a - \eta_0^2)^2 , \tag{1.1}$$

where the monopole configuration has a global O(3) symmetry

$$\phi^a = \eta_0 f(r)\frac{x^a}{r}, \quad x^a x^a = r^2, \quad a = 1,2,3 . \tag{1.2}$$

The metric solution of the set of coupled Einstein plus scalar field equations can be written as[7]

$$ds^2 = -\beta(r)dt^2 + \alpha(r)dr^2 + r^2 d\Omega_2 , \quad d\Omega_2 = d\theta^2 + \sin^2\theta\, d\varphi^2 , \tag{1.3}$$

where

$$\alpha^{-1}(r) = 1 - \eta^2 - \frac{2GM_a}{r} , \tag{1.4a}$$

$$\beta(r) = 1 - \eta^2 - \frac{2GM_b}{r} , \tag{1.4b}$$

and $\eta = \sqrt{8\pi G}\,\eta_0$.

What seems to be remarkable of this solution in addition to the deficit solid angle η^2 is that both

$$M_a \simeq M_b \simeq -6\pi \frac{\eta_0}{\sqrt{\lambda}} \tag{1.5}$$

so the effective mass of the global monopole (although tiny) appears to be negative. In Ref. 7 the implications of this fact are discussed. However, as we have already stressed, the main effects outside the monopole core are produced by the deficit solid angle, and, if one is not explicitly concerned with the consequences of the existence of a negative mass, it can be neglected for the sake of simplicity. This happens to be the case when one studies the quantum effects in global monopole space-times as the particle production by the formation of global monopoles and the evaporation of mini black holes with one of these monopoles inside.[7,8]

In this paper we are going to study the vacuum-polarization effects in curved space-time of a global monopole described by the approximate metric

$$ds^2=-(1-\eta^2)dt^2+(1-\eta^2)^{-1}dr^2+r^2d\Omega_2 \ . \quad (1.6)$$

This metric, in spite of having constant coefficients g_{00} and g_{rr}, represents a curved space-time; thus, the stress-tensor of the quantum fields will have nonzero vacuum expectation values.

In Sec. II we will obtain the Euclidean propagator $G_E(x,x')$ for a massless quantum field on the monopole background. From it we will also obtain the renormalized mean value $\langle\phi^2\rangle_{\rm ren}$. Indeed it is due to the remarkable simple expression for the metric that we are able to obtain explicit formulas for this renormalized quantity. In Sec. III we will show that the results for $\langle\phi^2\rangle_{\rm ren}$ could have been anticipated by simple dimensional and symmetry considerations. Applying the same arguments to the energy-momentum tensor we find, up to a numerical constant, an analytic expression for $\langle T_{\mu\nu}\rangle_{\rm ren}$.

The global monopole is one of the few cases where classical geometry is simple enough to allow a computation of $\langle T_{\mu\nu}\rangle_{\rm ren}$ in closed form. In very few additional examples the program of one-loop renormalization can be completed. This makes global monopole spacetimes an interesting *arena* where quantum effects can be studied in detail. The spherical symmetry of the problem allows us to study different aspects from the usual example of Robertson-Walker spacetimes, where homogeneity and conformal flatness simplify in some sense the computations. We will devote most of Sec. IV to the study of the back-reaction problem. This problem for global monopoles seems to be not as exciting as for black holes, because the former do not have the event horizon that the latter possess. However, we have found other interesting features such as the appearance of a mass scale into the theory due to the renormalization process.

Throughout the paper we will study the regime $\eta \ll 1$ which seems to be the more probable ($\eta^2_{\rm GUT}\approx 10^{-5}$), but for the sake of completeness, in the Appendix we show an example with the coupling constant $\xi=1/8$ and arbitrary η^2 to have a glance at the strong gravitational field regime, where it is supposed that quantum effects play a more important role.

II. COMPUTATION OF $\langle\phi^2\rangle_{\rm ren}$

Let us consider a massless scalar field with arbitrary coupling to the curvature in the background of the global monopole. We will obtain in this section an integral expression for the renormalized value of $\langle\phi^2\rangle$ in the limit $\eta \ll 1$. In doing so, we will begin with the general case (η and ξ arbitrary). Then, for the sake of simplicity we will choose particular values of η and ξ.

The above-mentioned mean value can be computed from the Green's functions as

$$\langle\phi^2\rangle=\tfrac{1}{2}[G^{(1)}]=i[G_F]=[G_E] \ , \quad (2.1)$$

where the square brackets denote the coincidence limit and $G_E(x,x')$ is the Euclidean Green's function.

The Euclidean version of the monopole metric (1.6) is

$$ds^2=f^2d\tau^2+\frac{1}{f^2}dr^2+r^2d\Omega_2 \ , \quad (2.2)$$

where $f^2=(1-\eta^2)$. The equation for the propagator is

$$(-\Box+\xi R)G_E(x,x')=\frac{\delta^4(x,x')}{\sqrt{g}} \ , \quad (2.3)$$

where $R=2\eta^2/r^2$.

The Euclidean Green's function can be evaluated using Schwinger's representation

$$G_E(x,x')=\frac{1}{(-\Box+\xi R)}\frac{\delta^4(x,x')}{\sqrt{g}} =\int_0^\infty ds\,\exp[-s(\Box-\xi R)]\frac{\delta^4(x,x')}{\sqrt{g}} \ . \quad (2.4)$$

Now the eigenfunctions of the operator $(-\Box+\xi R)$ with eigenvalues $\lambda^2=p^2f^2+\omega^2/f^2$ are

$$f_\lambda(x)=\sqrt{p}\,\exp(-i\omega\tau)Y_{lm}(\theta,\varphi)\frac{J_{\nu_l}(pr)}{\sqrt{r}} \ , \quad (2.5)$$

where Y_{lm} are the spherical harmonics, J_{ν_l} are Bessel functions and $\nu_l^2=1/4+l(l+1)/f^2-2\xi(f^2-1)/f^2$. Thus, from the completeness relation

$$\frac{\delta^4(x,x')}{\sqrt{g}}=\sum_\lambda f_\lambda(x)f^*_\lambda(x') \quad (2.6)$$

and Eq. (2.4) we obtain

$$G_E(x,x')=\int_0^\infty ds\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_0^\infty dp\,p\,\exp[-i\omega(\tau-\tau')](rr')^{-1/2}\sum_{lm}Y_{lm}(\theta,\varphi)Y^*_{lm}(\theta',\varphi')J_{\nu_l}(pr)J_{\nu_l}(pr')\exp(-s\lambda^2) \ . \quad (2.7)$$

Using the addition theorem for spherical harmonics and setting $\tau=\tau',\theta=\theta',\varphi=\varphi'$, Eq. (2.7) reduces to

$$G_E(r,r')=\int_0^\infty ds\int_{-\infty}^{\infty}\frac{d\omega}{8\pi^2}\int_0^\infty dp\,p\,\exp\left[-s\left[\frac{\omega^2}{f^2}+p^2f^2\right]\right](rr')^{-1/2}\sum_l(2l+1)J_{\nu_l}(pr)J_{\nu_l}(pr') \ . \quad (2.8)$$

After integration over s, ω, and p we obtain

$$G_E(r,r')=\frac{1}{8\pi^2rr'}\sum_{l\geq 0}(2l+1)Q_{\nu_l-1/2}\left[\frac{r^2+r'^2}{2rr'}\right] \ . \quad (2.9)$$

It is convenient to use the following integral representation for the Legendre functions:[9]

$$Q_{\nu-1/2}(\cosh\rho)=\frac{1}{\sqrt{2}}\int_\rho^\infty dt\frac{\exp(-\nu t)}{\sqrt{\cosh t-\cosh\rho}} \tag{2.10}$$

so the Euclidean Green's function can be written as

$$G_E(r,r')=\frac{1}{8\sqrt{2}\pi^2 rr'}\int_\rho^\infty \frac{dt}{\sqrt{\cosh t-\cosh\rho}}\sum_{l\geq 0}(2l+1)\exp(-\nu_l t) , \tag{2.11}$$

where $\cosh\rho=(r^2+r'^2)/2rr'$.

For the particular value of $\xi=1/8$, ν_l is proportional to $l+1/2$ and the series in Eq. (2.11) can be easily evaluated. One obtains an integral representation for the propagator which, moreover, can be expressed in terms of hypergeometric functions for some values of f (the case $f=1/2$ is studied in the Appendix).

In what follows we will consider the more physical case of $1-f\ll 1$ (recall that $\eta^2_{\rm GUT}\approx 10^{-5}$). Regarding Eq. (2.11) we define

$$F(x,\eta)=\sum_{l\geq 0}(2l+1)x^{\nu_l(\eta)} , \tag{2.12}$$

where, to first order in η^2,

$$\nu_l=(l+1/2)(1+\eta^2/2)+\frac{(2\xi-\frac{1}{4})\eta^2}{2l+1} .$$

The Euclidean Green's function may then be written as

$$G_E(r,r')=\frac{1}{8\sqrt{2}\pi^2 rr'}\int_\rho^\infty \frac{dt}{\sqrt{\cosh t-\cosh\rho}}F(e^{-t},\eta) , \tag{2.13}$$

where

$$F(x,\eta)=x^{1/2}\frac{1+x}{(1-x)^2}\left[1+\frac{\eta^2\ln x}{2(1-x^2)}[\xi(1-x)^2+x]\right] . \tag{2.14}$$

We will now renormalize the propagator (2.13). We must subtract the Schwinger-DeWitt expansion[10]

$$G_{\rm SD}(x,x')=\frac{1}{16\pi^2}\left[\frac{2}{\sigma(x,x')} +(\xi-\tfrac{1}{6})R\ln[\tfrac{1}{2}\mu^2\sigma(x,x')]\right] , \tag{2.15}$$

where μ is an arbitrary scale and $\sigma(x,x')$ is one-half the square of the geodesic distance between x and x'. For the radial point splitting we have $\sigma(x,x')=(1/2f^2)(r-r')^2$.

To perform the subtraction it is useful to have an integral expression for the Schwinger-DeWitt expansion. We can obtain the representation from Eq. (2.15) by noting that the inverse of $\sigma(x,x')$ is proportional to the flat-space-time propagator and that the logarithmic term can be written in terms of $Q_0(\cosh\rho)$. We find

$$G_{\rm SD}(r,r')=\frac{1}{8\pi^2\sqrt{2}rr'}\int_\rho^\infty dt\frac{e^{-t/2}}{\sqrt{\cosh t-\cosh\rho}}\left[\frac{(1-\eta^2)(1+e^{-t})}{(1-e^{-t})^2}-2\eta^2(\xi-\tfrac{1}{6})\frac{r'}{r}\right] +\frac{\eta^2}{4\pi^2r^2}(\xi-\tfrac{1}{6})\ln[\mu(r+r')/2\sqrt{1-\eta^2}] . \tag{2.16}$$

The renormalized value of $\langle\phi^2\rangle$ is thus

$$\langle\phi^2\rangle_{\rm ren}=\lim_{r\to r'}[G_E(r,r')-G_{\rm SD}(r,r')] =-\frac{\eta^2(p-2\xi q)}{8\sqrt{2}\pi^2r^2}-\frac{\eta^2(\xi-\frac{1}{6})}{4\pi^2r^2}\ln\mu r , \tag{2.17}$$

where

$$p=\int_0^\infty dt\frac{e^{-t/2}}{\sqrt{\cosh t-1}}\left[\frac{1}{3}+\left[\frac{t}{\sinh t}-1\right]\frac{1+e^{-t}}{(1-e^{-t})^2}\right] , \tag{2.18}$$

$$q=\int_0^\infty dt\frac{e^{-t/2}}{\sqrt{\cosh t-1}}\left[1-\frac{t(1+e^{-t})}{1-e^{-2t}}\right] . \tag{2.19}$$

These integrals can be evaluated numerically. The results are

$$p=-0.39,\quad q=-1.41 . \tag{2.20}$$

Summarizing, we have computed the renormalized value of $\langle\phi^2\rangle$ for a massless field on the background of a global monopole. We have made the computations up to first order in η^2 and for arbitrary values of the constant ξ. The results presented in this section may be used to obtain the renormalized mean value of the energy-momentum tensor. Instead of doing this, in the next section we will show that using dimensional and symmetry considerations it is possible to determine $\langle T_{\mu\nu}\rangle_{\rm ren}$ up to a numerical constant.

III. DIMENSIONAL ARGUMENTS AND THE EVALUATION OF $T_{\mu\nu}$

The metric (1.6) which we are using to represent the global monopole does not contain dimensional parameters. As a consequence, since we are working with natural units ($\hbar=c=1$), the mean values $\langle\phi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$ can only depend on the radial coordinate r, the renormalization scale μ and the Newton constant G. However, the Newton constant cannot appear at the one-loop approximation (thus it will enter linearly in the right-hand side of semiclassical Einstein equations). Simple dimensional considerations then give

$$\langle\phi^2\rangle_{\rm ren}=\frac{F(\mu r)}{r^2}\ , \tag{3.1}$$

$$\langle T_\mu^\nu\rangle_{\rm ren}=\frac{G_\mu^\nu(\mu r)}{r^4}\ , \tag{3.2}$$

where the functions $F(\mu r)$ and $G_\mu^\nu(\mu r)$ depend also on the dimensionless quantities η and ξ.

Keeping in mind the renormalization process of the previous section, it is possible to deduce the form of these functions. In fact, to obtain $\langle\phi^2\rangle_{\rm ren}$ it was necessary to subtract from the exact μ-independent propagator the Schwinger-DeWitt expansion Eq. (2.15). Then

$$\langle\phi^2\rangle_{\rm ren}(\mu)-\langle\phi^2\rangle_{\rm ren}(\mu')=-\frac{1}{8\pi^2}(\xi-\tfrac{1}{6})R\ \ln\frac{\mu}{\mu'} \tag{3.3}$$

and the function $F(\mu r)$ must satisfy

$$F(\mu r)-F(\mu' r)=-\frac{1}{4\pi^2}(\xi-\tfrac{1}{6})\eta^2\ln\frac{\mu}{\mu'}\ . \tag{3.4}$$

The general solution to this equation is

$$F(\mu r)=\frac{1}{4\pi^2}[a-(\xi-\tfrac{1}{6})\eta^2\ln\mu r]\ , \tag{3.5}$$

where a depends only on ξ and η^2.

The dimensional arguments then lead to

$$\langle\phi^2\rangle_{\rm ren}=\frac{1}{4\pi^2r^2}[a(\xi,\eta^2)-(\xi-\tfrac{1}{6})\eta^2\ln\mu r] \tag{3.6}$$

which is, of course, the general form of the results we found in the previous section and in the Appendix. The exact calculations we made there allowed us to compute $a(\xi,\eta^2)$ in two cases: arbitrary ξ and small η^2 [cf. Eqs. (2.17)–(2.20)] and $\xi=\frac{1}{8}$ and arbitrary η^2 [cf. Eq. (A2)].

Let us now consider the energy-momentum tensor. It is well known that, under a change of the renormalization scale[11]

$$\langle T_\mu^\nu\rangle_{\rm ren}(\mu)-\langle T_\mu^\nu\rangle_{\rm ren}(\mu')=\frac{1}{16\pi^2}\ln\frac{\mu}{\mu'}[\tfrac{1}{180}(3\,{}^{(2)}H_\mu^\nu-{}^{(1)}H_\mu^\nu)+\tfrac{1}{2}(\xi-\tfrac{1}{6})^2\,{}^{(1)}H_\mu^\nu)], \tag{3.7}$$

where

$${}^{(1)}H_\mu^\nu=2R_{;\mu}^{\ ;\nu}-2RR_\mu^\nu+\tfrac{1}{2}g_\mu^\nu(R^2-4\Box R)\ ,$$

$${}^{(2)}H_\mu^\nu=R_{;\mu}^{\ ;\nu}-\Box R_\mu^\nu-2R_\rho^\alpha R_{\mu\alpha}{}^{\nu\rho}+\tfrac{1}{2}g_\mu^\nu(R_{\alpha\rho}R^{\alpha\rho}-\Box R)\ .$$

The right-hand side in Eq. (3.7) is a conserved tensor. It is also trace free for $\xi=1/6$. Following the same steps as before we find

$$\langle T_\mu^\nu\rangle_{\rm ren}=\frac{1}{16\pi^2r^4}[A_\mu^\nu(\xi,\eta^2)+B_\mu^\nu(\xi,\eta^2)\ln\mu r]\ , \tag{3.8}$$

where the A_μ^ν are in principle arbitrary numbers and

$$B_\mu^\nu=\tfrac{1}{90}\eta^2\left[1-\frac{\eta^2}{2}\right]{\rm diag}(1,1,-1,-1)+\eta^2(8-7\eta^2)(\xi-\tfrac{1}{6})^2{\rm diag}\left[\frac{-4+5\eta^2}{8-7\eta^2},1,-1,-1\right]\ . \tag{3.9}$$

It is possible to find some restrictions on the tensor A_μ^ν. In fact, the spherical symmetry of the problem implies that the only possible nonzero components are A_t^t, A_r^r, $A_t^r=A_r^t$, $A_\theta^\theta=A_\varphi^\varphi$. Moreover, the complete energy-momentum tensor must be conserved, that is, $\langle T_\mu^\nu\rangle_{{\rm ren};\nu}=0$, $\mu=0,1,2,3$. From these four equations to be satisfied we must have that

$$A_r^t=0,\quad A_r^r+A_\theta^\theta-\tfrac{1}{2}B_r^r=0\ . \tag{3.10}$$

Using these relations we can write all the nonzero components of the tensor A_μ^ν in terms of one of them (say, A_r^r) and the trace of the complete energy-momentum tensor. Defining T through $\langle T_\mu^\mu\rangle_{\rm ren}=T/16\pi^2r^4$ we obtain

$$A_\mu^\nu={\rm diag}(T+A_r^r-B_r^r,A_r^r,-A_r^r+\tfrac{1}{2}B_r^r,-A_r^r+\tfrac{1}{2}B_r^r)\ , \tag{3.11}$$

where B_r^r is given in Eq. (3.9) and A_r^r and T are functions of ξ and η^2.

For the particular value $\xi=1/6$, T is given by the trace anomaly. We have

$$\langle T_\mu^\mu\rangle_{\rm ren}=\frac{T}{16\pi^2r^4}=\frac{1}{2880\pi^2}(R_{abcd}R^{abcd}-R_{ab}R^{ab}+\Box R)=\frac{\eta^2}{770\pi^2r^4}\left[1-\frac{\eta^2}{2}\right] \tag{3.12}$$

so in this case the complete energy-momentum tensor can be written in terms of the single component $A_r^r(\xi=\frac{1}{6},\eta^2)$. (The numerical value of A_r^r may be computed using our results of Sec. II, by constructing $\langle T_r^r\rangle_{\rm ren}$ from the exact propagator. This is a straightforward but long calculation that we will not attempt to do here.) In the next section, we will use the general form of the energy-momentum tensor derived from the dimensional considerations to investigate the back reaction of the quantum fields on the metric of the global monopole.

Finally, we would like to remark that the arguments we used here to obtain the general form of the energy-momentum tensor have been previously applied to cosmic strings[12,13] and black holes.[14] In both cases the renormalization scale μ does not enter into the play since for that particular metrics $R_{\mu\nu}=0$ and ${}^{(1)}H_{\mu\nu}={}^{(2)}H_{\mu\nu}=0$ [see Eqs. (3.3) and (3.7)]. In the case of cosmic strings, there being no dimensional parameters at all, one simply has

$$\langle\phi^2\rangle_{\rm ren}=\frac{a}{r^2},\quad \langle T_\mu^\nu\rangle_{\rm ren}=\frac{A_\mu^\nu}{r^4}\ .$$

The black-hole case is more complicated because there is a dimensional quantity which is the mass of the black hole. This inhibits one to apply the dimensional arguments. Moreover, the component $\langle T_r^t \rangle_{\rm ren}$ which is zero for the global monopole may be nonzero for the black hole, being responsible for Hawking radiation.

IV. BACK-REACTION CORRECTIONS

As an application of the previous results, we will discuss here the effect of vacuum polarization on the metric of the monopole. The semiclassical approach to the back-reaction problem consists of solving the Einstein equations for a classical background ($g_{\mu\nu}$ is a c number), using as a source the energy-momentum tensor of the classical matter present plus the $\langle T_{\mu\nu} \rangle_{\rm ren}$ corresponding to the quantum fields[15,16]

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}+\epsilon_1^{(1)}H_{\mu\nu}+\epsilon_2^{(2)}H_{\mu\nu} =8\pi G[T_{\mu\nu}^{\rm clas}+\langle T_{\mu\nu}(\phi,g_{\mu\nu})\rangle_{\rm ren}] \; . \tag{4.1}$$

The fourth-order corrections appearing in the left-hand side of the Einstein equations must be necessarily included in order to renormalize the theory. The coefficients ϵ_i are arbitrary (they should be fixed experimentally) and must depend on the renormalization scale μ since the theory must be independent of this scale. From Eqs. (3.7) and (4.1) we get

$$\mu\frac{d\epsilon_1}{d\mu}=\frac{G}{4\pi}[-\tfrac{1}{90}+(\xi-\tfrac{1}{6})^2] \; ,$$
$$\mu\frac{d\epsilon_2}{d\mu}=\frac{G}{120\pi}$$

which are the renormalization-group equations for ϵ_1 and ϵ_2. In what follows we will assume that at a given scale $\mu=\mu_0$ both coefficients ϵ_1 and ϵ_2 do vanish.

A complete solution of the back-reaction problem is a very difficult task since it involves a self-consistent calculation of $\langle T_{\mu\nu} \rangle_{\rm ren}$. Instead of this, we will follow an approximate procedure and use as a source of Eq. (4.1) the renormalized energy-momentum tensor we computed in the previous section. In this way we can obtain the first quantum correction to the monopole metric Eq. (1.6).

A static spherically symmetric metric in the Schwarzschild gauge reads

$$ds^2=-\beta(r)dt^2+\alpha(r)dr^2+r^2d\Omega_2 \; . \tag{4.2}$$

Plugging (4.2) into (4.1), it is easy to obtain[17,7]

$$\alpha^{-1}(r)=1-\eta^2-\frac{2M_aG}{r}+\frac{8\pi G}{r}\int_\infty^r r'^2\langle T_t^t\rangle_{\rm ren}dr' \; , \tag{4.3}$$

$$\beta(r)=\alpha^{-1}(r)\exp\left[8\pi G\int_\infty^r\langle (T_r^r-T_t^t)\rangle_{\rm ren}r'\alpha(r')dr'\right] \; , \tag{4.4}$$

where we have reinserted the mass of the monopole[7]

$$M_a(r)=M+\frac{\eta^2}{2r}+O(r^{-3}) \tag{4.5}$$

which, as we already stressed, is negative for $r>r_{\rm core}$. Equations (4.3) and (4.4) are valid outside the monopole core.

Our results of the preceding section for the $\langle T_{\mu\nu} \rangle_{\rm ren}$ can be summarized by

$$\langle T_\mu^\nu\rangle_{\rm ren}=\frac{1}{16\pi^2r^4}(A_\mu^\nu+B_\mu^\nu\ln\mu_0 r) \; , \tag{4.6}$$

where A_μ^ν and B_μ^ν are functions of ξ and η^2 which go to zero as η^2 for small η.

We call

$$\frac{G}{2\pi}A_t^t=A\,,\quad \frac{G}{2\pi}B_t^t=B\,,\quad \frac{G}{2\pi}(A_r^r-A_t^t)=C \; ,$$
$$\frac{G}{2\pi}(B_r^r-B_t^t)=D \; . \tag{4.7}$$

Then, expression (4.3) can be integrated

$$\alpha^{-1}(r)=1-\eta^2-\frac{2GM}{r}-\frac{\eta^2G+A+B}{r^2}-\frac{B}{r^2}\ln\mu_0 r \; . \tag{4.8}$$

We must point out that the expression (4.8) for the metric coefficient $\alpha(r)$ is to be taken valid only up to r^{-3}th order. Not only because the r dependence of $M_a(r)$ shown in Eq. (4.5), but due to the fact that we have computed the $\langle T_\mu^\nu \rangle_{\rm ren}$ [Eq. (4.6)] for a quantum field in the background (1.6), i.e., neglecting the effects of the mass of the monopole.

Similarly, from (4.4) one can obtain

$$\beta(r)=1-\eta^2-\frac{2GM}{r}-\frac{\eta^2G+A+B+C/2+D/4}{r^2} -\frac{D/2+B}{r^2}\ln\mu_0 r \tag{4.9}$$

valid up to r^{-3}th order. In the particular case of small η and $\xi=\frac{1}{6}$ the above equations become [see Eq. (3.9)]

$$\alpha^{-1}(r)\simeq\beta(r)\simeq 1-\eta^2-\frac{2GM}{r}-\frac{G\eta^2}{180\pi r^2}\ln\mu r \; , \tag{4.10}$$

where we have included into the scale μ the r^{-2} term.

This corrected metric (4.8) and (4.9), will be a good approximation to the actual physical problem as long as the first-order quantum perturbations $\approx\eta^2G/r^2$, be small compared to one: i.e.,

$$r\gg\sqrt{G}\,\eta=r_h \; . \tag{4.11}$$

For comparison, the radius of the core of the monopole, within which is located most of the mass M, is given by[7]

$$r_{\rm core}\approx\frac{1}{\eta_0\sqrt{\lambda}} \tag{4.12}$$

which is always much larger than r_h for typical values of η.

Thus, the metric given by (4.8) and (4.9), is a good approximation for the exterior of the "core" of the global monopole and quantum polarization effects are not large anywhere outside the core. Of course, this perturbation will become more and more important as we approach values of η near one.

To understand better the nature of the one-loop corrections to the metric we will examine how the deficit solid angle varies when one includes these corrections (see Hiscock's paper[13] on vacuum polarization near cosmic strings).

The deficit solid angle may be defined as

$$\Delta\Omega = 4\pi - S/R^2 , \tag{4.13}$$

where S is the surface of a sphere of proper radius R centered at the center of the monopole.

For the zeroth-order metric (1.6) the deficit solid angle is η^2. To compute the first-order corrections it is convenient to transform to proper radial coordinate R, in such a way that the metric (4.2) becomes

$$ds^2 = -\tilde{\beta}(R)dt^2 + dR^2 + R^2(1+\Delta\Omega)d\Omega_2 . \tag{4.14}$$

So, the deficit solid angle due to the vacuum polarization for $r \gg r_{\text{core}}$ is given by

$$\Delta\Omega_q = -\eta^2 - \frac{\tilde{A} + B \ln[(1-\eta^2)^{1/2}\mu_0 R]}{(1-\eta^2)R^2} , \tag{4.15}$$

where $\tilde{A} = \eta^2 G + A + B$.

The deficit solid angle is now a function of R. If the vacuum energy density is negative ($\tilde{A} < 0, B < 0$), the deficit solid angle increases as the monopole is approached, while for positive vacuum energy density, the deficit solid angle decreases as the monopole is approached.

Still, when one considers the effects of the tiny mass of the monopole, one also obtains corrections to the deficit solid angle given by

$$\Delta\Omega_M = -\eta^2 - \frac{2GM}{(1-\eta^2)^{1/2}R} \ln[-(1-\eta^2)^{1/2}R/GM] . \tag{4.16}$$

Because of the fact that $M < 0$, for large R we will have an increase of the deficit solid angle with respect to the zeroth order.

If many fields are present, it is possible for the deficit solid angle to differ significantly for the zeroth-order value η^2. Such a perturbation is still small enough to the linearized corrections to be a good approximation to the actual metric. However, the number of fields, N, to significantly increase or decrease $\Delta\Omega$ is unrealistically large even compared to $N \simeq 10^2$ for typical grand unified theories.

V. CONCLUSIONS

In this paper we studied the vacuum-polarization effects around a global monopole. The exterior metric of the monopole can be approximately described by a curved space with a deficit solid angle [Eq. (1.6)]. On this background metric, we have found closed expressions for $\langle \phi^2 \rangle_{\text{ren}}$ and $\langle T_{\mu\nu} \rangle_{\text{ren}}$ for a free, massless scalar field.

We made some direct calculations for the renormalized mean value of ϕ^2 finding the normal modes of the Klein-Gordon equation and resumming them to build up the propagator. As the monopole metric does not contain any dimensional parameter, we showed that these results could have also been obtained (up to a numerical constant) through simple symmetry considerations and dimensional analysis. Using the same kind of arguments, we have found the general form of the renormalized mean value of the energy-momentum tensor.

As usual for massless free fields on curved spaces, the renormalization process induces a dimensional scale into the theory. Even for a conformally coupled field ($\xi = \frac{1}{6}$), we have found that the renormalized energy-momentum tensor does depend on this scale, since the background curved space of the monopole is not conformal to a static Einstein space. This is the salient feature of the above calculations, that it is not present in other metrics usually studied in the literature such as Robertson-Walker, black-hole, and local cosmic-string spacetimes. For the string, the exterior metric describes a space with a deficit angle. Unlike the monopole, this space is locally flat, and the polarization effects are due to the nontrivial topology. The cosmic-string metric does not contain any dimensional parameter and, since it is locally flat, no dimensional scale is induced by the renormalization process. As a consequence, the dimensional analysis is simpler and one finds that[13] $\langle T_{\mu\nu} \rangle_{\text{ren}} \sim r^{-4}$, where r is the distance to the string. For the monopole, as we have mentioned, we found an additional term proportional to $\ln(\mu r)/r^4$, where μ is the scale induced by the renormalization program.

We have also considered the back reaction of the quantum field on the original metric of the monopole. We obtained (perhaps as expected) that for $r > r_{\text{core}}$ the corrections are small thus allowing the one-loop approach to be valid around the monopole. We have pointed out that the corrections would become larger if N, the number of quantum fields, is large or if η, the scale of symmetry breaking, approaches one.

A more complete study of the back reaction should include the quantum fluctuations of the Goldstone field ϕ^a which is responsible for the formation of the monopole. However, we can argue that the effect of these fluctuations is smaller than the effect produced by the massless fields. Indeed, the fluctuations of the Goldstone fields around the classical value have a mass of order $m \sim r_{\text{core}}^{-1}$. For massive fields, $\langle T_\mu^\nu \rangle_{\text{ren}}^{(m)}$ may be approximated by its Schwinger-DeWitt expansion[16]

$$\langle T_\mu^\nu \rangle_{\text{ren}}^{(m)} \sim \frac{\eta^2}{m^2 r^6} \sim \left[\frac{r_{\text{core}}}{r} \right]^2 \langle T_\mu^\nu \rangle_{\text{ren}}^{(m=0)} .$$

We see then that, for $r \gg r_{\text{core}}$ the leading quantum correction outside the monopole is given by the massless fields.

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APPENDIX

The value $\xi=1/8$ simplifies notably the computations

$$\nu_l^2=\tfrac{1}{4}+\frac{l(l+1)}{f^2}-\frac{2\xi(f^2-1)}{f^2}=\frac{(l+\frac{1}{2})^2}{f^2}\Longrightarrow\nu_l=\frac{l+\frac{1}{2}}{f}\ .$$

This gives, as in the case of cosmic strings,[18] Bessel functions of order just proportional to the order appearing in the Minkowski space. Then, Eq. (2.11) can be recasted as an integral containing a geometric series which when summed up gives

$$G_E(r,r')=\frac{1}{16\sqrt{2}\pi^2 rr'}\times\int_\rho^\infty dt\frac{\cosh\left[\frac{t}{2f}\right]}{\sqrt{\cosh t-\cosh\rho}\,\sinh^2\left[\frac{t}{2f}\right]} \quad (A1)$$

that is an exact integral expression for the propagator. Using Eq. (2.16),

$$\langle\phi^2\rangle_{\rm ren}(r)=\frac{1}{32\pi^2r^2}\int_0^\infty\frac{dt}{\sinh t/2}\left[\frac{\cosh t/2f}{\sinh^2 t/2f}-f^2\frac{\cosh t/2}{\sinh^2 t/2}+\frac{f^2-1}{6}e^{-t/2}\right]-\frac{1}{96\pi^2r^2}(f^2-1)\ln(\mu r/f)=\frac{1}{32\pi^2r^2}[I(f)-\tfrac{1}{3}(f^2-1)\ln\mu r]\ . \quad (A2)$$

A numerical evaluation of the above integral gives the exact value of $\langle\phi^2\rangle_{\rm ren}$ for $\xi=\frac{1}{8}$ and arbitrary deficit angle. The numerical result can be fitted by the polynomial

$$I(f)=0.117\,973\,7(1-f^2)+0.046\,302(1-f^2)^2+0.0728(1-f^2)^3 \quad (A3)$$

within 1% error for $0<f\le 1$. This result can be compared with Eq. (2.17) valid for small η^2. Note that the approximation is quite good for $\eta^2<0.2$. We can obtain analytic expressions for some particular values of f. For example, let us take $f=\frac{1}{2}$, which represents a case of strong gravitational field. After the change of variables $x=\cosh t$ in Eq. (A1) we find[9]

$$G_E(r,r')=\frac{\sqrt{rr'}}{16\pi(r+r')(r-r')^2}F\left[-\tfrac{1}{2},\tfrac{1}{2},1,\frac{4rr'}{(r+r')^2}\right], \quad (A4)$$

where $F(a,b,c,x)$ is an hypergeometric function.

To obtain the renormalized propagator we must subtract from (A4)

$$G_{\rm SD}=(r,r')=\frac{1}{16\pi^2}\left[\frac{2}{\sigma(r,r')}+(\xi-\tfrac{1}{6})R\,\ln(\tfrac{1}{2}\mu^2\sigma)\right],$$

where $\sigma(r,r')=2(r-r')^2$, $\xi=\frac{1}{8}$, and $R=3/2r^2$. Thus,

$$\langle\phi^2\rangle_{\rm ren}(r)=\lim_{r\to r'}\frac{1}{16\pi(r+r')(r-r')^2}F\left[-\tfrac{1}{2},\tfrac{1}{2},1,\frac{4rr'}{(r+r')^2}\right]-\frac{1}{8\pi^2}\left[\frac{1}{2(r-r')^2}-\frac{1}{32r^2}\ln[\mu^2(r-r')^2]\right]. \quad (A5)$$

Now, from the asymptotic behavior of the hypergeometric functions,[9] we obtain

$$G_E(r,r')\approx\frac{\sqrt{rr'}}{16\pi(r+r')(r-r')^2}\left\{\frac{2}{\pi}-\frac{(r-r')^2}{2\pi(r+r')^2}\left[2\ln\left[\frac{r-r'}{r+r'}\right]+\gamma\right]\right\}+O(r-r')\ , \quad (A6)$$

where $\gamma=\psi(1/2)+\psi(3.2)-\psi(1)-\psi(2)$. Finally, the renormalized propagator is

$$\langle\phi^2\rangle_{\rm ren}(r)=\frac{1}{128\pi^2r^2}[0.59+\ln(\mu r)]\ . \quad (A7)$$

Of course, this result agrees with the numerical calculation Eq. (A3) for $f=\frac{1}{2}$.

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