PHYSICAL REVIEW D VOLUME 43, NUMBER 2 15 JANUARY 1991

Time in quantum gravity: An hypothesis

Carlo Rovelli
Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
and Dipartimento di Fisica, Universita' di Trento, Istituto Nazionale di Fisica Nucleare,
Sezione di Padova, Italy
(Received 27 September 1990)

A solution to the issue of time in quantum gravity is proposed. The hypothesis that time is not defined at the fundamental level (at the Planck scale) is considered. A natural extension of canonical Heisenberg-picture quantum mechanics is defined. It is shown that this extension is well defined and can be used to describe the "non-Schrödinger regime," in which a fundamental time variable is not defined. This conclusion rests on a detailed analysis of which quantities are the physical observables of the theory; a main technical result of the paper is the identification of a class of gauge-invariant observables that can describe the (observable) evolution in the absence of a fundamental definition of time. The choice of the scalar product and the interpretation of the wave function are carefully discussed. The physical interpretation of the extreme "no time" quantum gravitational physics is considered.

I. INTRODUCTION

A. A timeless problem

A major conceptual problem in quantum gravity[1] is the issue of what time is, and how it has to be treated in the formalism. The importance of this issue was recognized at the beginning of the history of quantum gravity,[2] but the problem is still unresolved and has recently received increasing attention[3]. The controversy about time in quantum gravity does not refer to a uniquely defined problem; the quarrel has been over the questions even more than over the answers.[4–17] In this paper we propose a point of view on the puzzle and a physical hypothesis for its solution.

The physical hypothesis that we put forward is the absence of a well-defined concept of time at the fundamental level.

We shall provide a precise mathematical form of this hypothesis. For the moment, we may illustrate it as follows. We suggest that at the Planck scale dynamical systems cannot be described as evolving in a universal time quantity t. More precisely, they cannot be described as Hamiltonian systems in the strict sense. Instead, evolution may only be defined with respect to physical clock variables.

As will be discussed, this revised concept of time is, in a sense, implicit in classical general relativity. In this paper, we show that quantum mechanics can be naturally extended in order to incorporate it. The idea is not new (actually, it is quite old: "Tempus item per se non est ..."[18]); it is implicit in several works (see for instance Refs. 2, 5, 7, 9, and 17). But to our knowledge it has never been defined and studied in detail.

The main assertion of this paper is that there is a natural extension of canonical Heisenberg-picture quantum mechanics, which remains well defined in the absence of a well-defined Schrödinger equation, and in the absence of a fundamental time. This extension is well-defined both in terms of the coherence of the formalism, and from the point of view of the viability of the standard probabilistic interpretation.

The key step that allows us to define this extension is a technical result concerning the observables of the theory. The result is that, even in the absence of a fundamental time and of an exact Schrödinger equation, there are gauge-invariant observables (commuting with the Hamiltonian constraint) which describe evolution with respect to physical clocks. These observables are self-adjoint operators on the space of the solutions of the Wheeler-DeWitt equation.

Thus, the solution we propose to the time issue in quantum gravity is the following. At the fundamental level, there is no absolute time in terms of which a Schrödinger equation could be defined. The fundamental theory is described by the extended Heisenberg-picture canonical quantum mechanics, equipped with the standard probabilistic interpretation. Evolution with respect to physical clocks is described by self-adjoint operators corresponding to the observables we mentioned.[12]

In the course of the paper, this picture will be motivated, detailed, and shown to be consistent. Of course, only a future complete quantum gravity theory can establish if it is also realistic.

The problem of time arises in the canonical formulation of the theory as follows. In quantum general relativity (as in any diffeomorphism-invariant quantum field theory), the Schrödinger equation is replaced by a Wheeler-DeWitt equation, in which the time coordinate has disappeared from the formalism. An accepted interpretation of this fact is that *physical* time has to be identified with one of the internal degrees of freedom of the theory itself (*internal time*).[4] Evolution "in time" is identified with evolution with respect to this internal time. We follow this interpretation. In this philosophy, it has been shown[5,6,13,14] that a Schrödinger equation may emerge from the Wheeler-DeWitt equation.

However, it is very likely that, for any choice of the internal time, only an *approximate* Schrödinger equation emerges. In other words, the evolution in the internal time is described by a Schrödinger equation only within some approximation. This situation is satisfactory as far as the connection between the theory and the world that we see is concerned. In fact, whatever experiment we may perform, we are always well inside this Schrödinger approximation. But a theory that makes sense only within an approximation is not a satisfactory theory. Thus, the following question is relevant. Does the theory make sense *beyond* the Schrödinger approximation?

If one attempts to take the theory seriously beyond the Schrödinger approximation, several difficulties arise. Just to mention one of them, if the Schrödinger equation is valid only to the first approximation, then the norm of the state is only approximately conserved. Can a probabilistic interpretation be maintained if the norm is not exactly conserved?

Different attitudes towards the physics of the Wheeler-DeWitt equation outside the Schrödinger approximation can be found in the literature. An illuminating discussion on the disagreements on the issue of time is given in Ref.19. Here, we quote some of these attitudes; the list and the references are exemplary only and are by no means exhaustive.

(a) The theory makes sense only if an "exact internal time" is found such that an exact (rather than approximate) Schrödinger equation holds.[20] In this case, there would not be any non-Schrödinger regime.

(b) The theory outside the Schrödinger regime requires modifications of the basic structure of quantum mechanics; for instance, we should use infinite norm states, or give up canonical (i.e., Hilbert space) quantum mechanics.[7]

(c) Because of these difficulties, a quantum-mechanical theory of the gravitational field does not make sense, and a radical revision of the basic ideas of quantum mechanics is needed for quantum gravity.[21]

(d) Standard quantum mechanics, suitably interpreted, can be used also for the non-Schrödinger regime.[2]

The solution we propose in this paper is more or less distinct from the ones listed. We suggest that the basic structure of canonical quantum mechanics, namely, the Hilbert space of states, self-adjoint operators representing observables, probabilistic interpretation, and wave function collapse, may still accommodate quantum gravitational physics. An exact internal time is not required, nor particularly relevant for the quantization. Rather, it is the concept of time itself that needs to be revised. The formalism of classical mechanics (as we will elucidate) is already capable of accommodating this revised concept of time. Canonical quantum mechanics, in turn, can be very naturally extended in order to incorporate this revised concept of time.

For clarity, let us say that here we do not address the problem of the existence of an exact internal time in general relativity. Instead, we assume, first, that a way to obtain an approximate description of the world as we see it (with time) can be extracted from the theory; second, that this description is valid only within the approximation. As far as the problem of the choice of the internal time is concerned, we refer to the literature,[5,8] and in particular to the recent work of Ashtekar on the definition of an internal time in the weak-field limit.[6] See also the works on the Machian cosmological time[14] and on the observables in general relativity.[16]

The discussion in the present paper is relevant not only for general relativity, but also for any generally covariant field theory. The same problems we treat here appear in the topological quantum field theories[22] and in any formulation of string theory that does not assume a fixed background metric on the target space.

The paper is organized as follows. In Sec. I B we introduce the basic physical hypothesis. In Sec. I C we motivate this hypothesis by discussing the concept of time in classical general relativity. In Sec. II A, we show that there is a formulation of classical mechanics which allows us to treat dynamical systems without making reference to universal time. In Sec. II B, we discuss the observables that describe the evolution with respect to clock time. In Sec. III A, the quantum mechanics of the systems without time is defined. In Sec. III B, two technical issues are investigated: the quantization procedure and the problem of choosing the scalar product. Sec. III C extends the results on the observables that describe evolution in clock time to the quantum domain. In Sec. IV A, the proposed solution to the time issue in quantum gravity is summarized. Section IV B contains a discussion of the difficulties of this solution and some speculations. Section IV C contains the conclusions.

There are three papers which are strictly related to the present one and complementary to it. In the first one,[15] a model with approximate Schrödinger equation and no absolute time is introduced. Its quantization is a concrete example of the ideas exhibited in the present paper. In the other two papers,[16] the problem of the observables of general relativity is studied, respectively, in the classical and quantum contexts. Gauge-invariant observables of the kind introduced in this paper are constructed.

B. Clocks and absolute time

Perception of the flow of time is probably an elementary experience. In Newtonian physics, as well as in standard quantum mechanics, it is assumed that this experience corresponds to the existence of an absolute quantity, the time. This quantity, namely, the time of Newtonian, Hamiltonain, or quantum mechanics, will be denoted t .

To measure t we use clocks. A clock is a system with a variable, for instance the position of a hand, which has a simple behavior in t. In this paper, we shall denote a clock variable (the position of the hand) as T ; we shall denote variables of different clocks as $T, T', T'', \ldots$. Good clocks may have, for instance, a linear behavior in t:

$$T(t) = \alpha t. \qquad (1)$$

It is an elementary-physics-course observation that we never really measure t; rather, we always measure T's. The value of a physical quantity Q, measured at a time t, is denoted $Q(t)$. Since time is determined by measuring a clock variable T, what is actually measured is not $Q(t)$ and $T(t)$, but only the combined quantity $Q(T)$. Thus, t does not ever appear in laboratory measurements.

The observation that we never reach t in the experiments, but we only reach $T, T', T'', \ldots$, is not a trivial observation. Since t cannot be observed, Eq.(1) can never be verified. We check clocks one against the other; namely, we measure $T(T')$, $T'(T'')$, ..., and so on. Correspondingly, the problem of constructing clocks has historically been, and still is, a delicate problem. Galileo used his pulse to measure the oscillation period of a pendulum and to discover that it was isochronous. A few years later doctors were using pendulums to measure the periods of people's pulses and to check whether they were isochronous.

Indeed, what we have is a large collection of clocks. The clocks agree one with the other within certain unavoidable experimental errors. Up to a certain approximation, they provide a reasonable standard, against which dynamical theories and new clocks can be checked. But anytime there is need of measuring time at smaller scales, experimentalists find themselves in the same situation as Galileo: the pulse as measure for the pendulum and the pendulum as measure for the pulse. From the experimental point of view, t can be defined only as the idealized extrapolation of the (concurring) value of a large ensemble of clock variables.

If t can never be reached experimentally, still it plays a major role in the conceptual framework of Newtonian and quantum mechanics. Indeed, Newton or Hamilton equations, as well as the Schrödinger equation, are grounded on the underlying assumption that there exists a t, in which the dynamics is defined.

There are many basic differences between the absolute time variable t and the clock variables $T, T', T'', \ldots$. Any realistic physical clock variable satisfies Eq.(1) only within some approximation. t is assumed to run from minus infinity to plus infinity; clock variables may vary within a bounded interval. In general, the agreement between the clock variable T and the assumed absolute t is taken for granted only down to a certain scale. Below that scale, higher-order physical effects, systematic or statistical errors and quantum fluctuations (Hartle has made a detailed study of the dynamics of clocks under various circumstances, concluding that ideal clocks capable of surviving quantum gravitational fluctuations do not exist[7]) jeopardize the performance of any clock: If I look carefully at the hand of my watch, I see that it proceeds in little jumps. These differences between t and T imply that, given a variable $Q(t)$ and a clock variable $T(t)$, in general it is not possible to describe the evolution in clock time $Q(T)$ in Hamiltonian form.

Given these observations, we may now state the basic physical idea of this paper. We put forward the hypothesis that the idealized absolute Hamiltonian, or Schrödinger, time t cannot be defined down to the Planck scale. At the Planck scale it is still possible to talk of the clock variables $T, T', T'', \ldots$, but it does not make sense to talk of the absolute time t.

More precisely, we suggest that the theoretical framework needed for understanding quantum gravity requires that one abandon the idea of the existence of the universal quantity t, of which the specific clock variables are approximations. Only the quantities $Q(T), Q(T'), \ldots$ are defined at the fundamental level. Since the evolution in the clock times does not admit a Hamiltonian description, similarly, we do not expect that a Schrödinger-equation description could be possible.

In the next section we motivate this hypothesis. In the following ones, we show that the theoretical instruments for handling the absence of t already exist in classical physics (Sec. II), and can be easily constructed in quantum physics (Sec. III).

C. Time in general relativity

The first adjustment of the idea of a universal time t follows from special relativity. In special relativity t is replaced by a class of related times: the Lorentz times of all the different Lorentz observers. Equivalently, the hypothesis of the existence of t is replaced by the hypothesis of the existence of the Minkowski manifold with its peculiar metric structure.

A much more radical and subtle modification of the concept of time is implicit in general relativity. In view of the quantization, and in particular in view of the fact that the Schrödinger equation requires t, the concept of time in classical general relativity has to be accurately considered.

As a preliminary step, let us consider the motion in an arbitrarily assigned gravitational field, namely in a given solution $g_{\mu\nu}$ of Einstein equations. (In this paper we assume a compact topology for the spacelike slices of spacetime.) Every object travelling along a world line l in $g_{\mu\nu}$ measures a time flow which is given by the proper time along l. Thus, there is a definition of a time quantity for every given solution $g_{\mu\nu}$ of Einstein equations and every given trajectory l in this solution. By itself, of course, the independent time coordinate x^0 [argument of $g_{\mu\nu}(\mathbf{x}, x^0)$] is not a physical time: physics, indeed, can be reformulated in terms of any reparametrization of x^0.

In quantum gravity, we are concerned not with the motion in given gravitational fields but with the dynamical evolution of the gravitational field itself. Einstein equations provide the evolution of the gravitational field in x^0; but x^0 is not a physical time. In which physical time is the evolution of the gravitational field given? As is well known, this question is far from trivial. We wish we were able to formulate (compact space) general relativity as a Hamiltonian system evolving in a physical time parameter t, but such a formulation has never been constructed.

Let us begin to study this question in physical terms. In nongravitational physics, the experimentalist has a clock and describes the evolution with respect to it. The clock is represented in the theory by the independent variable t. Now let us consider gravity. Assume the experimentalist has a clock and measures the evolution of the gravitational field with respect to this clock. To which variable of the theory does the clock correspond? The clock cannot be identified with the time coordinate x^0 for the following reason. The evolution of the gravitational field in the clock time is uniquely determined by the initial conditions, while the evolution of $g_{\mu\nu}$ in x^0, as given by Einstein equations, is underdetermined.

The solution, of course, is that the clock is a physical object; its motion and its rhythm are determined by its equations of motion. If we consider the equations of motion of the gravitational field *and* the clock, then the problem is not underdetermined. But the gravitational field enters the equations of motion of the clock (without a gravitational field the equations of motion of physical objects cannot even be written). The dynamics of the clock cannot be disentangled from the dynamics of gravity itself. [This is very different from nongravitational physics. In nongravitational physics, we can first solve the dynamics of the clock, and forget about it, and then study the dynamics of, say, the Maxwell field. One can, in fact, always assume that the interaction between the field and the clock (how the clock is affected by the field) can be made arbitrarily small.] In order to calculate the evolution measured by the experimentalist, we have to evolve the gravitational field *and* the clock variable together, then solve away $\mathbf{x}, x^0$, and obtain gravitational quantities as functions of the clock variable. (For a more detailed version of this discussion, see Ref. 16.) The conclusion is that, in order to predict the evolution in the physical clock time of the gravitational field, we have to consider the coupled gravity+clock dynamical system.

The same conclusion can be reached in a formal way as follows. In any theory in which there is gauge invariance, we must assume that only gauge-invariant quantities are observable.[23] Because of the general covariance of general relativity, gauge-invariant quantities must be independent of the coordinates $\mathbf{x}, x^0$. Let us focus on x^0. No quantity that depends upon x^0 can be gauge invariant. Indeed, it is possible to formulate general relativity without even referring to x^0, as in the Hamilton-Jacoby formulation.

It is not easy to construct (local) gauge-invariant quantities in general relativity. In principle, nothing forbids that observables in pure general relativity could be constructed by expressing certain gravitational degrees of freedom as functions of certain others. In practice, this has never been completely achieved theoretically, and seems hopeless experimentally. To our knowledge, the only way to construct gauge-invariant observables in a gravitational theory is to consider general relativity coupled with matter and to express the gravitational degrees of freedom as functions of the matter degrees of freedom. Gauge invariant quantities obtained in this way are constructed in Ref.16. In any case, we have to solve away x^0 and express certain degrees of freedom as functions of others. Among these others, we identify the physical-time degree of freedom.

The conclusion of both the physical and the formal discussions is that in general relativity, physical time has to be identified with one of the degrees of freedom of the theory itself (the "clock"). Such a definition of time is often referred to as internal time.

Internal times differ from a Hamiltonian time in many respects. First of all, the theory does not single out one or the other of these internal times. Second, none of the (proposed) internal times has all the features that characterize the t variable of Hamiltonian and quantum mechanics. For instance, reasonable internal-time variables may grow (in x^0) up to a maximum value and then decrease. More precisely, there is no proposed internal time such that the theory can be expressed as a well-defined Hamiltonian system evolving in this internal time. Third, by definition an internal time refers to specific physical variables, unlike the t quantity, which is supposed to be universal. Thus, general relativity treats time in a peculiar way, as compared to prerelativistic physics. The absolute quantity t has disappeared. In its place, there are different possible internal times, related to specific physical variables.

Now, the internal times can be identified with the clock variables $T, T', T'', \ldots$ discussed in the previous section. Thus, maybe quite surprisingly, general relativity does not provide the evolution in an absolute time $Q(t)$ and $T(t)$, but only the observable evolution $Q(T)$. More precisely, there exists a time quantity in the theory, which is x^0, but the evolutions $Q(x^0)$ and $T(x^0)$ are non-gauge-invariant, and therefore nonobservable: the absolute quantity t has been replaced by an arbitrary and unobservable gauge parameter x^0. The observation made in Sec. I B, that only $Q(T)$ can be observed, is incorporated in the formalism of general relativity.

As far as the classical theory is concerned, these fine distinctions are a bit superfluous. After all, once the metric has been calculated, a pseudo-Riemanian manifold, does not seem to be conceptually very different from a Minkowski space. (It is.[16]) However, the consequences of the above discussion are far reaching at the quantum level. As Wheeler first emphasized, as in the quantum theory the concept of trajectory disappears, in quantum gravity there is no pseudo-Riemanian manifold at all. More precisely, quantum observables are attached only to gauge-invariant quantities. Thus, there is no room in the quantum theory for $Q(x^0)$ and $T(x^0)$. Operators correspond only to gauge-invariant quantities. In the quantum domain, the absence of the absolute time t is not intuitively remedied by a picture of the pseudo-Riemanian manifold.

The only way out that we see is to completely abandon the idea of absolute time. Only the evolution with respect to clocks makes sense. In certain physical situations, or for particular solutions of Einstein equations (in particular, of course, for a flat solution), we may idealize these clocks in terms of t. At the Planck scale, we may not.

In standard quantum mechanics, the Schrödinger equation *requires* the existence of a t, which corresponds to the classical Hamiltonian time. To do quantum gravity, an alternative formulation of quantum mechanics is needed. This formulation should not require the idealized quantity t as part of the basic formalism; instead, it should be able to deal directly with $Q(T)$ quantities.

But before going to quantum mechanics, if we abandon the idea that time is one of the conceptual bedrocks of the theory, does it still make sense to do physics, calculate measurable quantities, and develop a consistent and satisfactory picture of an evolving universe?

II. CLASSICAL DYNAMICAL SYSTEMS WITHOUT TIME

A. Mechanics without time: presymplectic mechanics

The possibility of describing dynamical systems without Hamiltonian time has to be first explored in the context of classical mechanics.

Mechanics may be defined as the general theory of the evolution of physical systems in time. From this point of view, time is required for the very definition of the elementary mechanical concepts. For instance, the state of the system is defined at a given time. In such a conceptual framework, t is required.

However, there exists an alternative starting point for mechanics. This is provided by presymplectic mechanics. This formulation does not require the absolute time for defining the basic concepts of the theory.

We shall illustrate presymplectic mechanics by first showing that Hamiltonian mechanics admits a reformulation in terms of a presymplectic space, and then noticing that this reformulation does not require the variable that represents time to be specified, or even defined. Readers familiar with presymplectic mechanics may skip this presentation.

In presymplectic mechanics, which is an elegant generalization of standard Hamiltonian mechanics, a dynamical system is just defined by a presymplectic manifold (C,ω). Let (S,ω_S,H) be a Hamiltonian system: S is the phase space, ω_S is the symplectic form, and H is the Hamiltonian. Let q_i,p^i be canonical coordinates on S ($\omega_S = dp^i \wedge dq_i$). The dynamical system is completely described on the space $C = S \times R$, with coordinates q_i,p^i,t, by the presymplectic form

$$\omega = \omega_S - dH(p,q,t) \wedge dt. \tag{2}$$

The motions of the system are the integral lines of the null vector field of ω (orbits, or trajectories, of ω). We denote this vector field Y:

$$i_Y\omega \equiv Y^a\omega_{ab} = 0. \tag{3}$$

In the coordinates on C that we are considering, the variable t has a preferred role, as is clear from Eq.(2). This preferred role identifies t as the time variable. The presymplectic space, however, has a geometric, coordinate-independent meaning, like the phase space. In a different coordinate system on C (say p',q',t'), ω may have the same form as in Eq.(2), but with t substituted by a different variable, say t'.

$$\omega = \omega_S - dH'(p',q',t') \wedge dt'. \tag{4}$$

Thus, the presymplectic formulation may accommodate different time variables. For instance, it may accommodate the different Lorentz times of special relativity. The presymplectic formalism, indeed, provides us with the only way to write a relativistic dynamical system in canonical form without destroying manifest Lorentz covariance.

Time evolution is described in the presymplectic formulation in a peculiar way. Each orbit of ω represents a possible motion of the system. An orbit defines a correlation between two different variables of the system. For instance, every orbit defines a function $q_i(t)$. If t is our time variable, then this function describes the evolution of q_i in t. But the same orbit also defines the function $q_i(t')$. Thus, had we chosen t' as our time variable, the presymplectic formulation would equally well provide the evolution in t'.

A *state* of the system is defined as an orbit. Note that this definition of state does not refer to a particular choice of the time variable, nor to a particular moment of time. Rather, it represents, in a sense, the entire history of that particular state. Looking ahead at the quantum context, it is meaningful to refer to this kind of definition of state as a *Heisenberg state*.

The *observables* of the system are defined as the scalar functions Q on C that are constants along the trajectories (the orbits)

$$Y(Q) \equiv Y^a\partial_a Q = 0. \tag{5}$$

Functions such as $q_i(t)$, which describe time evolution, are also observables in the sense of Eq.(5). This statement may seem strange, but it will be carefully clarified in the next section.

Note that there is no observable corresponding to a generic variable q_i (unless q_i is a constant of the motion). The observable is the *function* $q_i(t)$. More precisely, there is one different observable for every real value of t.

As a simple example, consider the presymplectic description of a free relativistic particle. (x^0,x^a,p_a) are coordinates on C and

$$\omega = \sum_{a=1}^{3} dp_a \wedge dx^a - d\sqrt{\mathbf{p}^2+m^2} \wedge dx^0 \tag{6}$$

($\mu = 0,1,2,3;\ a = 1,2,3$ from now on). The well-known constants of motion $P_a = p_a$, $P_o = \sqrt{\mathbf{p}^2 + m^2}$ and $M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu$ are constant along the trajectories generated by ω. They are physical observables. The observables that describe the evolution in x^0 will be constructed in the next section.

As the example suggests, presymplectic systems often arise in theoretical physics in the form of constrained Hamiltonian systems with weakly vanishing canonical Hamiltonian. In the Lagrangian formalism, these systems can be described by a reparametrization-invariant action, in terms of a fictitious nonphysical parameter. Indeed, the constraint surface of these systems, equipped with the (degenerate) two-form induced by the unconstrained-phase-space symplectic two-form, is the presymplectic manifold. This presymplectic structure incorporates all the relevant information on the system. ω in Eq.(6), for instance, is the two-form on the constraint surface $K(x^\mu, p_\mu) = p^2 - m^2 = 0$ induced by the unconstrained phase-space symplectic form $dx^\mu \wedge dp_\mu$. In these systems, the constraint is often denoted the Hamiltonian constraint.

In the next section we will show that the observables $q(t)$ satisfy

$$i_X\omega = -dH, \quad \frac{d}{dt}q(t) = X(q) = \{q(t), H\} \tag{7}$$

(where we have used also the more familiar Poisson-brackets notation). Note that time evolution can be described as the existence of a particular structure on the set of the observables: there exists a class of observables of the form $q(t)$ such that the t evolution is generated by a Hamiltonian function H [in the sense of Eq.(7)]. We call this structure on the set of the observables a *time structure*. Equivalently, a time structure is a splitting of ω as in Eq.(2).

In conclusion, in presymplectic mechanics the states of the system are represented by the orbits of ω, and do not evolve. Observables are represented by scalar functions on C, constant along the orbits. Note that the definitions of state and of observable do not make any reference to the concept of time. This is why presymplectic dynamics may describe different times for the same system.

What interests us here is a possibility offered by the presymplectic formalism which is more radical than the possibility of accommodating different times. Since presymplectic mechanics does not require the existence of a time t for the definition of the basic mechanical concepts, it can also describe systems in which there is no Hamiltonian time t at all.

More precisely, there are presymplectic dynamical systems (C, ω) which are *not* Hamiltonian systems. [In this paper we use "Hamiltonian system" in the strict sense, which does not include the cases (which are in fact presymplectic systems) in which the Hamiltonian is weakly vanishing.] These systems do not admit a time structure. (If a presymplectic system does not admit a time structure, there cannot be a corresponding Hamiltonian system, because the Hamiltonian of the Hamiltonian system defines a time structure on the corresponding presymplectic system.) For instance, the trajectories of ω may be closed. We define a presymplectic dynamical system that has no corresponding Hamiltonian formulation: ***dynamical system without time.***

A simple example of a presymplectic dynamical system without time is given by the constraint surface C of the constraint

$$K = q_1^2 + q_2^2 + p_1^2 + p_2^2 - M \sim 0. \tag{8}$$

Since C is compact, it does not admit a time structure, because a time structure implies the decomposition $C = \Sigma \times R$, where R is the real line. Thus, there is no Hamiltonian system corresponding to this presymplectic system.

In the dynamical systems without time, one can still talk of states (the orbits of ω), of observables, and also of evolution. In fact, the orbits still determine a functional relation between different variables. One variable, say q_1 in the model (8), can be interpreted as a clock variable. Every orbit defines the evolution $Q(q_1)$ for any other variable Q as a function of q_1. But q_1 does not have the properties that characterize the Hamiltonian time t, so the evolution in q_1 cannot be described as a Hamiltonian evolution. For instance, there are values of q_1 which are not reached by certain trajectories. This cannot happen in a Hamiltonian system.

The physical interest of the systems with no time is that they can be interpreted as systems describing the evolution with respect to physical clocks, as opposed to the evolution with respect to the absolute time t. Dynamical systems of this kind arise in theoretical physics. Examples are given by certain cosmological models,[2] by any topological field theory,[22] by the Barbour-Bertotti model,[13] and many others.

The example par excellence, however, is of course general relativity (on a compact space). The Arnowitt-Deser-Misner (ADM) constraint surface equipped with the two-form induced by the symplectic two-form of the ADM phase space is the presymplectic manifold. General relativity can do without an absolute Hamiltonian time because as a dynamical system, general relativity is not a Hamiltonian system but a presymplectic system.

We do not regard the lack of a (natural) Hamiltonian description of general relativity as a failing of the formalism. Instead, we take it as a profound indication that the absolute time t is not a physical quantity, and that only evolution with respect to clocks is observable.

The suggestion of this paper is that this indication has to be taken seriously. If so, it has to be extended also to quantum mechanics.

B. The description of evolution: constants of the motion that evolve

In the previous section, a problem was left pending. The general definition of an observable, namely Eq.(5), seems to be in contradiction with the statement that the evolution of a variable q_i as a function of another variable t is observable. In this section, we show that this contradiction does not exist. The analysis will be rather technical; but it is on a technical point concerning the existence of these observables that the hypothesis we are proposing relies. (We believe that many difficulties in the canonical quantization of parametrized systems follow from the confusion on this point.)

We assume for simplicity that the presymplectic system is defined by a (Hamiltonian) constraint K on a phase space q_n, p_n, $n = 1, ..., N$. We single out a variable on the phase space, say q_1, which we use as a clock variable. For the moment, we assume that q_1 is a Hamiltonian time.

The definition as (5) of observable, is equivalent to the requirement that an observable Q is given by a function $Q(q_n, p_n)$, which has vanishing Poisson brackets with the constraint K. The question we address is how can such quantities, constant along the trajectories generated by K, describe the evolution? The answer is that there do exist observables that satisfy Eq.(5) *and* represent the evolution in q_1. We now define these observables.

Let us focus on another variable, say q_i, $i = 2, ..., N$. We want the observable that represents the evolution of q_i as a function of q_1. As we mentioned, this will not be a single observable, but rather a one-parameter family of observables, each one representing the value of q_i at a different value, say t, of the clock variable q_1. Let us call these observables $Q_i(t)$. There is one observable $Q(t)$ for every real number t; $Q(t)$ is an observable, namely, a function on C (or the restriction to C of a function on the phase space):

$$Q_i(t) = Q_i(t; q_n, p_n). \tag{9}$$

The t dependence of $Q_i(t)$ should not be confused with its dependence on C. $Q_i(t)$ must be constant on the orbits *for every* t in order to be observable. And it should be equal to the function $q_i(q_1)$ *in any point of* C in order to describe what we want it to describe. The key point is that the two requirements are not contradictory. In fact, we define the observable $Q_i(t)$ for every real number t, as follows.

$Q_i(t)$ is constant along each trajectory, and on each trajectory it has the numerical value equal to the value of the variable q_1 in the point P where that trajectory intersects $q_1 = t$.

Equivalently, $Q_i(t)$ is defined by the two equations

$$\{Q_i(t; q_n, p_n), K(q_n, p_n)\} = 0, \tag{10}$$

$$Q_i(t; t, q_2, q_3, ..., p_n) = q_i. \tag{11}$$

The first equation implies that $Q_i(t)$ is observable: namely, constant along the trajectories. The second determines the value of $Q_i(t)$ on any trajectory. This value is the numerical value obtained by looking for the point $q_1 = t$ along that trajectory and reading out q_i in that point. At any point P that lies in a trajectory l, the function $Q_i(t)$ is equal to the function $q_i(q_1)$ determined by l. We denote the observables defined by equations such as Eqs.(10) and (11) *evolving constants of the motion* or *evolving constants.* (The fact that observables of this kind can describe evolution is discussed, for instance, in Refs. 17 and 24.)

Consider the presymplectic system that represents the relativistic particle, defined in the preceding section. The observables $X^a(t)$ that describe the evolution of x^a as a function of x^0 are

$$X^a(t; x^\mu, p_\mu) = x^a - \frac{p^a}{\sqrt{\mathbf{p}^2 + m^2}}(x^0 - t). \tag{12}$$

It is easy to check that Eqs.(10) and (11) are satisfied. The second one is immediate. The first,

$$\{X^a(t; x^\mu, p_\mu), p^2 - m^2\} = 0, \tag{13}$$

follows directly from the fact that $X^a(t)$ can be expressed as a function of the well-known constants of the motion P_μ, $M^{\mu,\nu}$:

$$X^a(t; x^\mu, p_\mu) = \frac{P_a}{P_0}t + \frac{M^{a0}}{P_0}. \tag{14}$$

The last equation shows that in any point of any trajectory, $X^a(t)$, seen as a function of t, is equal to the function $x^a(x^0)$ determined by that trajectory. Thus, $X^a(t)$ is a function on the phase space which is constant along the trajectories generated by K, but describes the evolution of x^a in x^0.

Evolving constants can be constructed for any function $q = q(q_n, p_n)$, and for different choices of the time variable $q_T = q_T(q_n, p_n)$. The evolving observable $Q(T)$ which gives the evolution of q in the clock time q_T is defined by the two equations

$$\{Q(T), K\} = 0, \tag{15}$$

$$Q(q_T) = q, \tag{16}$$

where the dependence upon the coordinates is not indicated. The explicit form of these observables is obtained by solving the dynamics generated by the constraint (geometrically, this amounts to constructing the orbits) and inverting the solutions of the equations of motion. From the defining equations, we get

$$\frac{\partial Q(T)}{\partial T}\{q_T, K\} = \{q, K\}, \tag{17}$$

which can be used to propagate $Q(T)$ in T.

In a generic dynamical system, one is not able to solve the dynamics exactly and to construct the evolving constant in explicit form. However, the evolving constants are always defined by Eqs.(15) and (16). In the presymplectic theory, in a sense dynamics has been reduced to kinematics. If we know the expression for any observable $Q(t)$ for every t, then we know everything about the system, and the dynamics is solved. This is analogous to what happens in Heisenberg mechanics. If we know every Heisenberg observable $\hat{Q}(t)$ for every t, the dynamics is solved. Kinematics becomes nontrivial: it is nontrivial to construct the observables.

If the dynamical system admits a time structure and q_t is a good Hamiltonian time, then $K = p_t + H$, where the Hamiltonian H does not depend on the momentum p_t, conjugate to q_t. In this case, Eq.(17) becomes

$$\frac{\partial Q(t)}{\partial t} = \{q, H\}. \tag{18}$$

Since H does not depend on p_t, the commutator of q with H is the same as the commutator of $Q(t)$. Therefore,

$$\frac{\partial Q(t)}{\partial t} = \{Q(t), H\}. \tag{19}$$

This is the Hamilton equation of motion. Thus, in a Hamiltonian system the evolving constants are nothing more than the usual observables, seen as functions of t.

In a system without time, it is still possible to define evolving constants analogous to the ones just defined. These are the observables that describe the evolution in the clock time q_T. The evolving constants of the system without time Eq.(8) have been constructed in Ref.15. In the general case, the Hamilton equation of motion (19) does not hold. The Hamilton equation is replaced by Eq.(15), which is more general .

To summarize, the definitions of state and observable in the Hamiltonian formalism require the existence of a time t, which is absolute and fixed once and for all. By contrast, the definition of the formal structure of the dynamical theory in the presymplectic formalism does not require t. Evolution with respect to a dynamical variable q_T, chosen as a clock, is described by a particular class of observables $Q(T)$. The basic equation that $Q(T)$ satisfies is Eq.(15). If the system admits a Hamiltonian formulation in the time variable q_t, this equation reduces to the Hamilton evolution equation.

III. QUANTUM MECHANICS WITHOUT TIME

A. Extending quantum mechanics

Can the hypothesis of the absence of the absolute time be incorporated in quantum mechanics? Is there an existing formulation of quantum mechanics which does not require t to exist? Is there a form of quantum mechanics that extends Schrödinger-equation quantum mechanics, in the same sense in which the presymplectic mechanics extends the Hamilton-equation mechanics?

The answer is almost. To define the Schrödinger picture a time variable t is needed. A Schrödinger quantum state is defined as the state of the system at time t, precisely as a point of the phase space represents a state of the classical system at a time t. However, in the Heisenberg picture t is not required to define the basic concepts of the theory. Because of that, it is possible to define an extremely natural extension of the Heisenberg picture which may deal with a system in which there is no Hamiltonian time t.

The Heisenberg states are often introduced as the states at $t = 0$. But they can also be interpreted, in a more fundamental way, as a global (time unrelated) characterization of the state. These states are the quantum analog of the trajectories in the presymplectic formalism. The interpretation of the Heisenberg states as states representing the entire history of the system has been stressed by Dirac.[25] If the system admits a Schrödinger picture, we may represent the state space of the Heisenberg picture in terms of the space of the Schrödinger states at $t = 0$. This is the analog of labeling the presymplectic trajectories by means of their q_i coordinates at $q_1 = 0$. (The opportunity of using Heisenberg states was vigorously advocated by Dirac in the first edition of his celebrated book on quantum mechanics.[25]. In Sec.I.3 Dirac argues that special relativity *forces* us to use Heisenberg states. His physical definition of the Heisenberg observable "at a given time" (Sec.II.9) is precisely the one we use here. It is interesting to notice that in later editions Dirac shifted from the Heisenberg definition of state (which he calls the relativistic one) to the Schrödinger one (which he calls the nonrelativistic one). He does that in order to gain in simplicity (after all, he is doing nonrelativistic quantum mechanics), but he complains (in the preface) that "it seems a pity" to give up on the relativistic notion. At the end of his life, Dirac returned to advocate Heisenberg states, and in 1981 he gave a talk in Erice (Sicily) using a single transparency, on which there was written only "$i\hbar\, dA/dt = [A, H]$: Heisenberg mechanics is the good mechanics.")

Similarly, Heisenberg observables correspond to the presymplectic observables. There is no Heisenberg observable corresponding to the variable q; rather, there is a one-parameter set of observables $\hat{Q}(t)$ corresponding to the values of q at $q_t = t$.

When the system admits a classical Hamiltonian formulation, there is a Hamiltonian operator $\hat{H}$, and the Heisenberg observables are related by

$$\hat{Q}(t+t') = e^{i\hbar t'\hat{H}}\ \hat{Q}(t)\ \ e^{-i\hbar t'\hat{H}}, \tag{20}$$

Equation (20) is the quantum realization of the time structure. In differential form, it becomes

$$i\hbar\ \partial_t\hat{Q}(t) = [\hat{Q}(t), \hat{H}]. \tag{21}$$

It is extremely important to emphasize that this equation is the quantum version of the Hamilton equation of motion (19), and it is also the Schrödinger equation as it looks in the Heisenberg picture.

Now we arrive at our main point. In the classical systems without time (in the technical sense defined in Sec.II A) the Hamilton equation (19) does not hold. It is reasonable to expect that in their quantum physics the corresponding Eq.(21) would not hold either. In those systems, an evolving constant $Q(T)$ would correspond to a quantum operator $\hat{Q}(T)$ which does not satisfy Eq.(20). The key point is that this fact does not disturb the Heisenberg picture at all. The Heisenberg picture remains well defined also if the relation between

$\hat{Q}(T)$ observables at different T's is not given by Eq.(20).

What may go wrong in these systems is that the set of all the observables $\hat{Q}_i(T)$ at a fixed T may not form a complete set. If so, a state is not characterized by its projection on the eigenstates of a family of $\hat{Q}_i(T)$ for a fixed T. This means that the outcome of the measurements of $\hat{Q}_i(T)$ for a fixed T does not uniquely characterize the state. Namely, one cannot define a Schrödinger picture. (See Ref.15 for a concrete example in which all that happens.)

Suppose a definition of the Hilbert space $\mathcal{H}$ of the Heisenberg states is given. Suppose the definition of Heisenberg operators $\hat{Q}$ as self-adjoint operators on $\mathcal{H}$ is also given. And suppose that among these operators there are also evolving constants $\hat{O}_i(T)$. Then, we can run the entire standard machinery of the probabilistic interpretation of quantum mechanics: the outcome of the measurement of a quantity Q on a state ψ is an eigenvalue q of the $\hat{Q}$ operator; the probability of getting q is the modulus square of the projection of ψ on the q eigenvector, and so on. All this also makes sense if the operators $\hat{O}_i(T)$ do not satisfy Eq.(21). (The equation they satisfy will be studied in Sec.III C.)

The Heisenberg states are the quantum version of the presymplectic states: they represent "histories" of the system. The Heisenberg operators $\hat{O}$ correspond directly to the presymplectic observables O. Among these, there are the quantum evolving constants $\hat{O}_i(T)$, corresponding to the classical evolving constants $O_i(T)$. If the classical system admits also a Hamiltonian formulation, then we have Eq.(20), and we may define a Schrödinger picture. If it does not (it is a system without time), everything still makes sense. But the Schrödinger picture cannot be defined.

Quantum mechanics may be synthesized in axiomatic form (see, for instance, Ref.26). A set of axioms refers to the definition of state and observable, to the identification of the expectation value of the measurement with the mean value of the operator and to the collapse of the wave function. One of the axioms (Postulate P3 in Ref.26) refers to time evolution; let us call it the time axiom. In the Heisenberg picture, the time axiom requires that all the observables depend on the time variable t, and that an operator $\hat{H}$ exists such that Eq.(21) holds.

In the Heisenberg picture, the time axiom can be dropped without compromising the other axioms or the probabilistic interpretation of the theory. Thus, we may formulate our basic proposal on the quantization of the classical systems without time.

(1) We define the structure given by the axioms of Heisenberg picture quantum mechanics, excluding the time axiom, as quantum mechanics without time.

(2) We suggest that the quantum physics of the presymplectic systems that do not have a Hamiltonian version is governed by quantum mechanics without time, as defined in 1.

General relativity is one of these systems; thus, we suggest, nonperturbative quantum gravity is to be constructed in the framework of quantum mechanics without time.

A quantum system without time is constructed in Ref.15. It quantizes the system (8). The Hilbert space is defined, and the operators that represent the evolution of q_2 as a function of the clock time q_1 are constructed. (More precisely, the corresponding self-adjoint projection operators are defined.) We urge the reader to refer to that paper for a concrete implementation of the general theory discussed here.

In a quantum system without time, there may be an approximation within which the Schrödinger equation (21) holds. This is the way we expect that standard quantum mechanics may be recovered. If there is an approximate Schrödinger equation, the fact that the Schrödinger norm is only approximately conserved is just a consequence of the approximation and does not disturb the full theory. The quantum states of the model defined by Eq.(8) admit[15] a representation of the form

$$\psi(q_1, q_2). \tag{22}$$

They satisfy an *approximate* Schrödinger equation

$$-i\hbar\frac{\partial}{\partial q_1}\psi(q_1, q_2) = \hat{H}\psi(q_1, q_2) + \text{small terms}. \tag{23}$$

The norm

$$||\psi||(q_1) = \int dq_2 |\psi(q_1, q_2)|^2, \tag{24}$$

obtained by fixing the internal time q_1 and integrating on the remaining variables, is not conserved in q_1. But this norm is *not* the one defined by the correct scalar product of the theory. The fact that this norm is not conserved in q_1 does not contradict the probabilistic interpretation. It is as harmless as the fact that the integral in q_2 of the modulus square of the wave function of a two-dimensional harmonic oscillator depends on q_1.

Finally, let us discuss the wave-function collapse. The measurement of the quantity Q at the clock time T is accompanied by the projection of the state on an eigenstate of the operator $\hat{Q}(T)$. The Heisenberg states get projected at any measurement. The information that the measurement is performed at the clock time T is contained in the fact that the eigenstates of $\hat{Q}(T)$, on which the state gets projected, depend on T. The question "when" the projection occurs is meaningless since the state does not evolve.

But there seems to be a problem here. Projectors do not commute. Even if it is meaningless to say when the projections occur, nevertheless, the order in which they occur is not meaningless. But, unlike the Hamiltonian time t, a clock time T may (classically) increase and then decrease along a trajectory. Thus, in general T does not define an ordering relation. How do we know the *order* in which to perform the wave-function projections? If we replace the well-behaved t by the ill-behaved T, how do we know how to order the collapses?

In order to answer this question, we should notice that

t and the ordering of the collapses are not necessarily related. This fact was emphasized by Dirac in Ref.25 and is clearly discussed by Hartle in Ref.7. The following example shows that time and collapse ordering may be unrelated. The formalism of quantum mechanics allows a sequence of measurements not ordered in the time in which the system evolves. We can measure $B(t)$ and *then* $A(t')$ with $t' < t$. The wave function is projected twice: *First* on the eigenstate of the $\hat{B}(t)$ operator and *then* on the eigenstate of the $\hat{A}(t')$ operator. This sequence of projections describes the conditional probability of being detected at $A(t')$ for a particle that will be detected at $B(t)$. This probability is well defined in terms of frequency. One may think that both measurements have been performed many times, and we are requested to calculate the distibution of the $A(t')$ outcomes knowing the $B(t)$ outcomes.

The example suggests that the ordering of the collapses is not determined by t. Rather, the ordering depends on the *question* that we want to formulate. The ordering is usually related to t only because we are more interested in calculating the future than the past. If we want the probability that A has the value a at t', given that B had or will have the value b at t, we first have to project on the b eigenstate of $\hat{B}(t)$ and then on the a eigenstate of $\hat{A}(t')$, irrespective of which comes first between t and t'. Since the ordering of the projections is not determined by the natural ordering defined by t, we may replace t with a clock variable T that does not define any ordering. The issue deserves a more accurate analysis; yet the previous discussion suggests that the collapse of the wave function should not cause problems of interpretation in quantum mechanics without time.

B. Quantization procedures and physical Hilbert structure

In this section we consider certain problems that emerge in constructing the Heisenberg theory $(\mathcal{H}, \hat{O}_i)$ starting from a given presymplectic system (C, ω) without time.

We discuss two quantization procedures. The first one is the standard one. The second one is more abstract and difficult to apply, but it is more complete.

We assume, from now on, that the presymplectic system is defined by constrained Hamiltonian systems with Hamiltonian constraint $K(q_n, p_n)$ and weakly vanishing canonical Hamiltonian. A well-known procedure for the quantization of any constrained system is the following.[23] One begins by quantizing the unconstrained phase space. Let $\mathcal{H}$ be the state space and $\hat{q}_n, \hat{p}_n$ be the resulting quantum state space and operators. The physical state subspace $\mathcal{H}_{\text{ph}}$ is extracted by solving the constraint equations

$$\hat{K}\ \psi = 0 \tag{25}$$

on $\mathcal{H}$. The observables

$$\hat{Q} = Q(\hat{q}_n, \hat{p}_n) \tag{26}$$

have to be well defined on $\mathcal{H}_{\text{ph}}$. If so, they must send $\mathcal{H}_{\text{ph}}$ in itself. In order for this to be true, $\hat{Q}$ must commute (on $\mathcal{H}_{\text{ph}}$) with $\hat{K}$:

$$[\hat{Q}, \hat{K}] = 0. \tag{27}$$

It follows that the corresponding classical observable must have vanishing Poisson brackets with the constraint

$$\{Q(q_n, p_n), K(q_n, p_n)\} = 0. \tag{28}$$

Note that these observables are the ones which are constant along the orbits of the presymplectic system, and therefore they are precisely the presymplectic observables, as defined in Sec. II A. The standard treatment of the constrained systems agrees with the basic rules for the presymplectic systems.

If the procedure can be completed, it provides the $(\mathcal{H}_{\text{ph}}, \hat{O}_i)$ structure. The $\hat{O}_i(T)$ observables that quantize the classical evolving constants will be discussed in the next section.

In this program there are technical difficulties such as solving the constraint equations and finding and ordering the physical observables. There is also a general problem: in general, the physical states that solve the constraint equation have infinite norm in the natural Hilbert structure of $\mathcal{H}$ (this happens when zero is in the continuum spectrum of the constraint operator, which is the usual case). Indeed, one has to define a new physical scalar product on the space of the solutions.

We discuss this problem here because this difficulty is sometimes taken as a proof that the quantum theory cannot be defined in the absence of time. Indeed, let us suppose there is a Hamiltonian time variable t among the arguments of the unconstrained wave function. Then, there is a canonical way of finding the physical scalar product. One defines the Schrödinger picture, and the fixed t formulation provides a physical scalar product, given by the L_2 structure in the rest of the variables. The evolution in t is unitary and thus the definition does not depend on the particular t chosen.

Now, does this mean that if the Hamiltonian time t does not exist, then the scalar product cannot be defined? The answer is no: in the general case, we do not have this simple prescription for constructing the physical scalar product; but this does not mean that the physical scalar product cannot be defined. It just means that we cannot use the time structure as a hint for its construction.

To find the physical scalar product is always a problem for *any* constrained system in Dirac quantization. For instance, the same problem appears in non-Abelian Yang-Mills theories, where it has far reaching consequences. The problem is not related to the absence of time.

In order to fix the physical scalar product, namely, to add a Hilbert structure on the linear space $\mathcal{H}_{\text{ph}}$, the conditions that the scalar product has to satisfy must be considered. Assume that the linear structure of the physical state space has been worked out (up, maybe, to completion issues). We have the linear space of the solutions of the constraint equations and a complete set of linear operators $\hat{O}_i$ on this space. How do we choose the scalar product? There is a key condition on the choice. Namely, the operators $\hat{O}_i$ have to be self-adjoint. (The notion of self-adjointness depends on the scalar product.) This is a highly nontrivial condition on the scalar product. There are several examples that show that this requirement determines the Hilbert structure.[27] Thus, there is a precise rule for fixing the scalar product. This rule works also for the quantum systems without time.

There exists an alternative to Dirac quantization, which overcomes most of these difficulties;[28] in particular, it overcomes the difficulties of choosing the physical scalar product. This alternative quantization prescription consists in directly quantizing the presymplectic system, rather than going through the quantization of the unconstrained system.

The main result of Refs. 28 and 29 is that one can quantize the presymplectic system by looking for an operator realization of a closed and complete algebra of presymplectic observables. This can be obtained by finding a transitive group G of automorphisms of the (C,ω) structure and choosing a unitary representation $\mathcal{U}$ of G. This quantization procedure makes use only of the geometric structure of the presymplectic space (C,ω). Again, no specification of the time variable is needed in order to complete the quantization procedure. The problems of ordering, solving the constraints, and picking the scalar product are bypassed. The difficulty, of course, is to find G.

An example in which group-theoretical methods provide a quantization of a system, without any reference to a choice of time, is the strong-coupling limit of general relativity.[30] An example of a system without time that can be quantized with group-theoretical methods is given in Ref.13.

C. Quantum evolving constants

A basic claim of this paper is that the quantum observables must satisfy Eq.(27) and must correspond to classical quantities that satisfy Eq.(28).

In ordinary constrained systems (with nonvanishing canonical Hamiltonian), this result is the standard requirement of gauge invariance. In the systems we are considering (with vanishing canonical Hamiltonian) this result has often been rejected with the motivation that, if we restrict ourselves to the observables that satisfy Eq.(28), then we cannot describe evolution. According to such a view, the Hamiltonian constraint generates dynamics and not gauges, and therefore, if we want to describe quantities that evolve in time, we must have quantities that do not commute with K. Then, of course, lots of troubles follow because an operator that does not commute with $\hat{K}$ is not well defined on the space of physical states, and the entire structure of quantum mechanics collapses.

Such a viewpoint is wrong because it relies on the wrong assumption that there is no way to describe time evolution in terms of observables that commute with K. As we showed, the evolution is well described by observables that commute with K. These are the evolving constants, introduced in Sec. II B. Thus, there is not any reason for rejecting Eqs. (27) and (28). (A relevant example of presymplectic system is given by a system in which the integral orbits of ω have dimension greater than 1. In this case the presymplectic formalism does not distinguish between the dimensions of the orbits related to the evolution and the ones related to a gauge invariance. This distinction is not needed in order to establish a physical interpretation: in any case observables are functions on the space C constant along the orbits.)

If $Q(T;q_n,p_n)$ is the observable defined by Eqs. (15) and (16), the corresponding quantum operator is defined by the corresponding quantum equations

$$[\hat{Q}(T),\hat{K}]=0, \tag{29}$$

$$\hat{Q}(\hat{q}_T)=\hat{q} \tag{30}$$

(the second equation can be more easily defined in a representation in which q_T is diagonal). The first is the fundamental equation that every observable must satisfy. The second defines the specific observable. As in the classical case, dynamics and kinematics are interrelated. If the explicit form of $Q(T;q_n,p_n)$ is known, then $\hat{Q}(T)$ can be defined by choosing an ordering in

$$\hat{Q}=Q(T;\hat{q}_n,\hat{p}_n), \tag{31}$$

such that Eq.(29) is satisfied. Another nontrivial condition on the ordering follows from the fact that there should exist a scalar product such that all the quantum observables are self-adjoint for every T.

The quantum analog of Eq.(17) becomes

$$\frac{\partial\hat{Q}(T)}{\partial T}[\hat{q}_T,\hat{K}]=[\hat{q},\hat{C}]. \tag{32}$$

This equation can be integrated to evolve $\hat{Q}_i(T)$ in T.

If the presymplectic system corresponds to a Hamiltonian system, and q_t is a Hamiltonian time, then there is a Hamiltonian, and, in parallel with Eq.(18), Eq.(32) reduces to

$$i\hbar\frac{\partial\hat{Q}(t)}{\partial t}=[\hat{Q}(t),\hat{H}]. \tag{33}$$

This is the standard Schrödinger equation, written in the Heisenberg picture. Thus, as was anticipated in Sec. III A, when a presymplectic system corresponds to a Hamiltonian system, the quantum evolving constants are nothing but the standard Heisenberg observables, and the fundamental equation (29) is nothing but the Heisenberg picture version of the Schrödinger equation.

In conclusion, in the general case the basic equation for every observable is Eq.(27). A particular class of observables is given by the the constants evolving in a clock time $T=q_T$; these are defined by Eq.(29). If the clock time happens to be a good Hamiltonian time, then the basic equation (29) reduces to the Schrödinger equation for the evolving constants.

Evolution can be described also in the absence of a Hamiltonian time, and in the context of the basic canonical quantum-mechanics formalism: Hilbert space, finite norm states, self-adjoint operators.

IV. PERSPECTIVES

A. Quantum gravity

We may now come back to gravity, put together the different results, and put forward in a more precise form the solution of the time issue that we propose.

In general relativity the notion of an absolute time t is absent. In place of it, internal time variables T that represent physical clocks can be identified. We believe that the choice of one of these clock times and its identification with the absolute time t of Hamilton mechanics is not only very difficult, but also *irrelevant* and contrary to the basic physical ideas of general relativity.

Quantum mechanics admits a natural extension, quantum mechanics without time, which can deal with systems in which an absolute Hamiltonian time t is not defined. Quantum gravity, we propose, is described by this quantum theory without time.

In principle, the theory can be constructed as follows. The space H_{ph} of the solutions of the Wheeler-DeWitt equation is considered.[31] A class of quantum operators that commute with the Wheeler-DeWitt constraint is constructed. Among these, there should be observables that express the evolution with respect to a clock variable. (Certain observables of this kind have been constructed.[16]) The scalar product is then chosen on H_{ph} in such a way that the observables are self-adjoint.

If this program can be completed, then the fundamental theory is complete. In principle, any outcome of any measurement can be computed, in terms of mean values of the self-adjoint operators on the physical states.

We expect that the theory admits a choice of internal time T such that the evolution of the observables in T is given by a Schrödinger equation in T within a certain approximation. Within this approximation, T behaves like an absolute time t, and we are in the familiar regime of standard quantum mechanics. But we also expect that the Schrödinger approximation breaks down in the general case (at the Planck scale?). At this fundamental level, time is not defined.

It is worth adding here a comment about quantum cosmology in order to avoid confusion. In a quantum theory of the entire Universe specific problems arise. There is the problem of maintaining the probabilistic interpretation if a single copy of the system is available; and the issue of the definition of a quantum theory in which the observer is part of the system.

Since gravity is the basic instrument for cosmology, the study of the Universe as a whole and the study of general relativity are often related. However, the specific problems of quantum cosmology (such as the one just mentioned) and the specific problems of quantum gravity (such as the time issue) are logically independent. To be convinced of this independence, consider that nothing prevents us from studying the quantum cosmology of a flat universe (with, say, just Yang-Mills fields). Alternatively, we can study the gravitational field and consider the observer and the other fields as external (of course, external in the dynamical sense, not in the space-time sense). In the first case we have a quantum cosmology with no gravity; in the second case we have a quantum gravity which is not quantum cosmology.

It has been repeatedly suggested that the solution of the problems of quantum gravity require one to consider the whole Universe (see for instance Ref.7), and therefore that quantum gravity and quantum cosmology should be studied together. This is a very interesting possibility. But it is just a possibility. In this paper, we have adopted the opposite philosophy. We considered the time issue in quantum gravity and we neglected any cosmological question.

One can question quantum mechanics on the grounds that for the Universe the interpretation of probability is problematic or on the grounds that the theory should include the observer. These problems are not connected with the gravitational field. On the contrary, the absence of a time variable and of a Schrödinger equation, which is a characteristic feature of the gravitational dynamics, does not spoil the standard interpretation of quantum mechanics.

Then, there is no problem in the interpretation of the wave function in quantum gravity. Quantum mechanics (without the time axiom) provides a precise and well-defined scheme of interpretation. Most of the confusion on issues concerning interpretation is generated by asking non-gauge-invariant questions. Very often in the quantization of models, too little attention is put on the key requirement of gauge invariance of the observables [Eq.(27)], and on the requirement (on the scalar product) that the gauge-invariant observable must be self-adjoint.

In particular, the popular interpretation of $|\psi[g]|^2$, as the probability of measuring the three-geometry g, is wrong. $|\psi[g]|^2$ is not a gauge-invariant quantity. Only quantities of the kind $\langle\psi|\hat{Q}|\psi\rangle$, where $\hat{Q}$ commutes with the Wheeler-DeWitt constraint, have physical meaning.

B. Problems and comments

We do not think that the proposed solution of the time issue is clear and complete. Both at the technical and at the conceptual level, there are points that remain open or unclear.

Physically, not just any variable can be used as a clock. We relaxed the requirement that a clock variable T must be a Hamiltonian time. However, we did not provide any alternative definition of a "time" variable. What does characterize the physical variables T that can be used as clocks? To our view, this is an open question.

A related technical question is the compatibility of the two equations (10) and (11) that define the evolving observables. It is clear from the geometrical picture that if $\{q_T, K\} = 0$, Eq.(17) cannot be integrated. In the general case, the functional relation that an orbit defines between two variables is only implicit. $Q(T)$ may be multivalued. Physically, there is nothing wrong with that, but the definitions should be adjusted. If an orbit intersects $q_T = T$ in M points P_m, we may introduce M observables $Q^{(m)}$ such that

$$O^{(m)}(q_T) = q(P_m), \tag{34}$$

and so on. In other words, Eq.(16) should be replaced by a weaker equation. The details should be worked out.

$Q(T)$ may be defined only on a bounded interval. This interval may depend on the orbit. Thus, there are values of T for which $Q(T)$ is defined only in certain regions of C. Outside these regions, $Q(T)$ may become complex. In the quantum domain, this implies that the operator $\hat{Q}_i(T)$ may have complex eigenvalues and therefore is not self-adjoint.

A way out is provided in Ref.15. The idea is to use the projection operators on the eigenstates of $\hat{Q}(T)$ [$\hat{Q}(T)$ is still symmetric] corresponding to real eigenvalues instead of $\hat{Q}(T)$ itself. The projectors are self-adjoint and correspond to the basic yes/no experimental observations. The present paper is more conceptual than technical and, for the purpose of clarity, we did not use the projector formalism. But we think the projector formalism is very likely to be the correct formalism in the general case.

Finally, assuming that the hypothesis we are presenting is realistic, developing a physical intuition of the systems with no time is a nontrivial problem. Simple models[15,17,32] may help. In the model in Ref.15, the clock time fails to be a good time because of global properties of the orbits. Locally, the system behaves as a Hamiltonian system; but on the entire orbit one has to patch different times. The topology of the orbit is closed and there is no way to map R smoothly onto an orbit.

An interesting relation between global properties of the orbits and small-scale measurements emerges in this model. If one measures a variable in a quasiclassical state with a high precision, the state is severely affected by the collapse. The collapse excites components of the wave function, which correspond to classical trajectories in which the clock variable reverses its direction right away.

It is tempting to speculate that this behavior could be general. Suppose that in gravity we take the growing radius R of the Universe as a clock variable. Since the Universe may recollapse, there are orbits for which R "goes back." One can speculate that a measurement which implies a Planck-scale precision may project the state of the gravitational field on components of the wave function which correspond to a universe that would immediately start to recontract. Thus, Planck-scale measurements may destroy the unitarity of the evolution in R.

Apart from these (wild) speculations, the example suggests two ways in which a realistic gravitational system without time may be concretely considered. One is to recall that it is difficult to imagine that in general relativity there could be a good clock that may run forever on *any* solution of Einstein equations (recall that most solutions develop singularities). The second (maybe related) one is to think that nontime behavior may appear at very short time intervals. More precisely, there may be physical reasons for which there are no good clocks that resolve time below the Planck time.

C. Conclusions

In this paper, we propose a solution to the problem of time in quantum gravity. We make the hypothesis that the concept of absolute time t, as used in Hamiltonian mechanics as well as in Schrödinger quantum mechanics, is not relevant in a fundamental description of quantum gravity.

This time has to be replaced by arbitrary clock times T in terms of which the dynamics may not be of the Schrödinger form. The motivation for this hypothesis is that in general relativity there is no observable absolute time. A Hamiltonian formulation of gravity in the strict sense (choice of a clock time T and the identification of T with the Hamiltonian time) is contrary to the basic physical ideas of general relativity and *irrelevant* for the quantization.

An extension of quantum mechanics, which does not need t, is required in order to incorporate in quantum mechanics the physical ideas of general relativity. This extension (quantum mechanics without time) is defined in a very natural way, just by dropping the time axiom from the Heisenberg picture.

In a quantum-mechanical system without time, the Schrödinger equation (which in the Heisenberg picture is $\dot{\hat{Q}} = i\hbar[\hat{Q}, \hat{H}]$) is replaced by the equation $[\hat{Q}, \hat{K}] = 0$. There are observables $\hat{Q}(T)$ (evolving constants) that describe the evolution with respect to a physical clock variable. In spite of the fact that evolution in T may be nonunitary, the probabilistic interpretation is viable. Unitary evolution and the Schrödinger equation may be recovered within an approximation.

As far as the problem of time is concerned, in a quantum theory of gravity there is no need to give up the probabilistic interpretation of the wave function, Hilbert space, finite norm states, and self-adjoint operators corresponding to observables. The notion of absolute time is not necessary, and we think that the difficulties in dealing with a theory without time are only psychological. We suggest that, in looking for a quantum gravity theory, "time" should simply be forgotten.

Our proposal is in a sense conservative and in a sense radical. It is conservative since we keep as much of general relativity and as much of standard quantum mechanics as possible. Our philosophy is that general relativity and quantum mechanics summarize our basic knowledge of the world and that we should not change them, unless forced by experiments or by a requirement of internal consistency. It is radical because we assume that at the fundamental level time is not defined. Thus, a radical revision of a familiar concept is required. However, this modification of the concept of time (with respect to the time of Hamilton mechanics) is forced by general relativity itself and is implicit in its formalism. The proposed extension of quantum mechanics is nothing but the insertion of this revised concept of time in the basic structure of the quantum formalism. The fact that this can be done so naturally is, for us, a good sign.

Of course, the proposed solution of the issue of time is only an hypothesis. In order to verify this hypothesis, a nonperturbative quantum gravitational theory has to be constructed. In spite of the recent progress in this direction,[31] it is well known that there are major technical difficulties in the actual construction of a canonical quantum theory of gravity. In this paper we have tried to resolve an *a priori* difficulty which could have undermined canonical quantization. We have shown that a conceptual framework, in which time in quantum gravity is not a problem for the canonical theory, does exist. Whether or not nature chooses this conceptual framework is an open question.

ACKNOWLEDGMENTS

I am particularly grateful to Abhay Ashtekar, Chris Isham, Karel Kuchar, and Lee Smolin for many interesting discussions. I also thank Daniele Amati, Bruno Bertotti, Bryce DeWitt, Al Janis, Jim Hartle, Carlos Kozameh, Ted Newman, Raphael Sorkin, and Marco Toller for useful suggestions and interesting exchanges of ideas. This work was supported by the NSF Grant No. PHY-9012099.

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