Gaussian reference fluid and interpretation of quantum geometrodynamics

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The Wheeler-DeWitt equation of vacuum geometrodynamics is turned into a Schrödinger equation by imposing the normal Gaussian coordinate conditions with Lagrange multipliers and then restoring the coordinate invariance of the action by parametrization. This procedure corresponds to coupling the gravitational field to a reference fluid. The source appearing in the Einstein law of gravitation has the structure of a heat-conducting dust. When one imposes only the Gaussian time condition but not the Gaussian frame conditions, the heat flow vanishes and the dust becomes incoherent. The canonical description of the fluid uses the Gaussian coordinates and their conjugate momenta as the fluid variables. The energy density and the momentum density of the fluid turn out to be homogeneous linear functions of such momenta. This feature guarantees that the Dirac constraint quantization of the gravitational field coupled to the Gaussian reference fluid leads to a functional Schrödinger equation in Gaussian time. Such an equation possesses the standard positive-definite conserved norm. For a heat-conducting fluid, the states depend on the metric induced on a given hypersurface; for an incoherent dust, they depend only on geometry. It seems natural to interpret the integrand of the norm integral as the probability density for the metric (or the geometry) to have a definite value on a hypersurface specified by the Gaussian clock. Such an interpretation fails because the reference fluid is realistic only if its energy-momentum tensor satisfies the familiar energy conditions. In the canonical theory, the energy conditions become additional constraints on the induced metric and its conjugate momentum. For a heat-conducting dust, the total system of constraints is not first class and cannot be implemented in quantum theory. As a result, the Gaussian coordinates are not determined by physical properties of a realistic material system and the probability density for the metric loses thereby its operational significance. For an incoherent dust, the energy conditions and the dynamical constraints are first class and can be carried over into quantum theory. However, because the geometry operator considered as a multiplication operator does not commute with the energy conditions, the integrand of the norm integral still does not yield the probability density. The interpretation of the Schrödinger geometrodynamics remains viable, but it requires a rather complicated procedure for identifying the fundamental observables. All our considerations admit generalization to other coordinate conditions and other covariant field theories.

I. INTRODUCTION

Geometrodynamics views the Einstein theory of gravitation as a Hamiltonian dynamical system. The intrinsic geometry and extrinsic curvature of a spacelike hypersurface in an Einstein spacetime are limited by a system of constraints which also generate the evolution of these quantities under the deformation of the hypersurface. Nothing in the structure of the constraints helps us to distinguish the true dynamical degrees of freedom from the quantities which determine the hypersurface. This causes severe problems in the interpretation of quantum geometrodynamics where one is expected to describe an experimental arrangement which would measure a given dynamical variable at a given instant of time.

An underlying reason for these difficulties seems to be that in a generally covariant theory there is no *a priori* way of recognizing spacetime events or, in the canonical theory, of recognizing instants of time and points of space. As a consequence, one cannot measure the metric, but only geometry. Geometry is the metric modulo diffeomorphisms. The invariance under diffeomorphisms leads to the constraints. The constraints tell us that not all components of the metric are dynamically significant, but do not tell us how to separate those which are significant from those which are not.

Long before people discovered how to treat gravity as a dynamical system and became preoccupied with its quantization, they invented a conceptual device for recognizing events: the notion of a reference fluid.[1] The particles of the fluid identify space points, and clocks carried by them identify instants of time. These fix the reference frame and the time foliation. The way in which the fluid moves and the clocks tick is encoded in the coordinate conditions. These are statements about the metric which hold in the coordinate system of the fluid and are violated in any other system.

The reference fluid is traditionally considered as a tenuous material system whose back reaction on geometry can be neglected. There is just enough matter to tell us where we are, but not enough of it to disturb the geometry. Such a view of the reference fluid has its drawbacks. Because the spacetime geometry is unaffected by the fluid, gravity is subject to the same old constraints and Hamilton equations. The coordinate conditions break the diffeomorphism invariance too late, after its unwanted consequences are already felt. Second, the reference fluid eludes a canonical description: The coordinate conditions tell us how the fluid moves only indirectly, by telling us how the metric looks in the comoving system. The fluid is not described by its own canonical variables, and its motion is not derived from a Hamiltonian. Gravity is a dynamical system, but the reference fluid is not. Such a standpoint makes it difficult to incorporate the reference fluid as a true physical system into quantum theory.

This physical picture of the reference fluid is reflected in the technical procedure by which the coordinate conditions are taken into account. First, the Einstein equations are derived from the action which is invariant under spacetime diffeomorphisms and does not mention the reference fluid. Second, the coordinate conditions are imposed. They do not change the Einstein equations, but merely specify the spacetime platform from which one observes the gravitational field. In the canonical theory, this procedure corresponds to taking the original system of first-class constraints on the geometric variables and supplementing it by four additional constraints which express the coordinate conditions.[2] The enlarged system of constraints is no longer first class, and one is thus able to eliminate four variables and their conjugate momenta from the theory, either directly or by using the Dirac brackets. These are the variables fixed by the coordinate conditions. The resulting Hamiltonian yields the dynamics of the remaining variables in the coordinate system carried by the fluid.

In this paper, we shall advocate an alternative way of handling the coordinate conditions which corresponds to viewing the reference fluid as a material system coupled to gravity. We impose the coordinate conditions *before* varying the action, by adjoining them to the action with a set of Lagrange multipliers. The additional terms in the action introduce a source into the Einstein law of gravitation and break the diffeomorphism invariance of the theory. The source describes the gravitational effect of the reference fluid. The loss of invariance suspends the original constraints. As a result, the metric in the comoving frame of the fluid becomes measurable.

At this stage, our approach has two defects. The loss of invariance, while beneficial to our quest for observables, prevents us from working in any other coordinate system than that associated with the fluid. Second, we are still lacking the canonical variables whose change would describe the motion of the fluid. Both of these defects are rectified by parametrizing the action.[3] Parametrization restores the invariance of the action by expressing it in arbitrary coordinates while adjoining the privileged coordinates stipulated by the coordinate conditions to the metric field variables. The privileged coordinates play the role of the fluid variables, and their variation yields the equations of motion of the fluid.

In the canonical formalism, the fluid variables become complemented by conjugate momenta. Canonical coordinates of the fluid identify the space frame and instants of time; the canonical momenta tell us how the frame moves and the clocks tick. Gravity acts on the fluid, and the fluid produces a gravitational field. Because the parametrized action is again diffeomorphism invariant, the dynamics of the gravitational field coupled to the coordinate fluid is generated by the familiar super-Hamiltonian and supermomentum constraints. However, these constraints now live in an extended phase space of the metric variables supplemented by the fluid variables. They are first-class constraints, and there are no other constraints in the theory.

The fluid variables mark the spacetime events. In their role of canonical coordinates, they determine an embedding in the encompassing spacetime. On such an embedding, the metric and extrinsic curvature can be freely specified and become thus classical observables. In the Dirac constraint quantization, the classical constraints are turned into restrictions on the physical states of the system. They yield a functional Schrödinger equation according to which the state of the metric field evolves from one embedding to another. The resulting formalism offers a clean framework for interpreting quantum geometrodynamics: The fluid variables enable us to recognize the instants of time and points of space. As a result, not only the geometry, but the metric itself becomes measurable. The conceptual problems associated with vacuum geometrodynamics seem to disappear when gravity is coupled to the reference fluid.

The idea of using phenomenological fluids to interpret quantum gravity is quite old. In the covariant approach, DeWitt coupled the gravitational field to an elastic medium carrying mechanical clocks.[4] From these objects, he constructed idealized apparatuses which, in the spirit of the Bohr-Rosenfeld analysis of measurability in quantum electrodynamics,[5] were able to detect appropriate projections of the quantized Riemann curvature tensor. A little later, DeWitt used the same device for interpreting the canonical minisuperspace quantization of the Friedmann universe.[6] He introduced a cloud of clocks into the model and studied their correlation with the radius of the Universe.

An interest in minisuperspace quantization surged when Misner[7] and his school[8] conducted a systematic study of homogeneous anisotropic cosmologies. In interpreting their results, they did not introduce matter, but focused attention on the gravitational variables. In this manner, the difficulties associated with interpreting the solutions of the vacuum Wheeler-DeWitt equation became apparent.[9] In an attempt to circumvent these difficulties, Lund[10] introduced the perfect fluid as a source of spherically symmetric inhomogeneous Tolman-Bondi models. Because of the symmetry, the fluid flows without rotation. The velocity potential of the flow can serve as a clock. The super-Hamiltonian constraint is linear in the momentum conjugate to the velocity potential and the Dirac quantization of the model thus leads to a Schrödinger equation.

Neither DeWitt nor Lund attempted to generalize their respective interpretations of minisuperspace quantum models to full quantum geometrodynamics. DeWitt's approach was recently rediscovered by Rovelli.[11] Rovelli used a square-root Hamiltonian for the clocks. He argued that when the clock's momentum is large in comparison with its mass, the Dirac constraint quantization approximately leads to a Schrödinger equation. In this argument, Rovelli disregarded the positivity of the square-root Hamiltonian which, through the necessity of using the absolute value, spoils the Schrödinger nature of the constraint. Unruh and Wald[12] also advocated the Schrödinger approach to geometrodynamics. They noted that the suspension of the constraints is equivalent to the introduction of a fluid source. They did not explore the structure and canonical description of such a source any further, possibly because their aim was to construct a purely geometric theory. Instead, they studied a time variable brought in by the unimodular coordinate condition.[13]

While the use of phenomenological fluids in quantum gravity is old, their association with coordinate conditions is a new feature of our approach. It stems from the problem of representing spacetime diffeomorphisms in canonical gravity.[14] Isham and Kuchař[15] resolved this problem by breaking the invariance of general relativity by the Gaussian coordinate conditions and restoring it again by parametrization. This procedure leads to the modification of the constraints by terms which are linear in the momenta conjugate to the Gaussian coordinates. Hartle[16] discussed the Schrödinger equation obtained by imposing the new constraints as restriction on the physical states. In the end he dismissed the new terms as devoid of physical reality. Halliwell and Hartle[17] related the new form of the constraints to the sum-over-histories approach to quantum gravity. Rovelli[11] raised the question whether the Isham-Kuchař procedure is related to his fluid of clocks. This paper elucidates such a relationship and, by identifying the newly introduced variables with the velocity potentials of the Gaussian reference fluid, resolves the issue of their physical realizability.

The methods introduced in this paper are applicable to the reference fluid associated with any coordinate conditions.[18] However, the technical discussion is limited to the Gaussian coordinate conditions. The present work is part of a broader program of studying the role of coordinate conditions in canonical quantization of generally covariant systems.[19] The analysis of harmonic coordinate condition has already been completed and will be presented separately.[20] The general procedure has also been applied to the unimodular coordinate condition of Unruh and Wald.[12,13] We concluded that the associated time variable is inappropriate for specifying a spacelike hypersurface and thus fails to provide a satisfactory interpretation of quantum geometrodynamics.[21] We have used the same approach for introducing the conformal, harmonic, and light-cone gauges in canonical theory of a bosonic string.[22] Similar techniques have been applied to the canonical formalism of (induced) two-dimensional gravity.[23]

The rest of this paper is entirely devoted to the study of the Gaussian reference fluid. We follow the general algorithm of adjoining the Gaussian coordinate conditions to the action by a set of four Lagrange multipliers and then restoring the invariance of the action by parametrization. It turns out that the term "reference fluid" eminently fits the physical interpretation of the field equations: The energy-momentum tensor appearing in the Einstein law of gravitation has the structure of a heat-conducting dust. The Lagrange multipliers which enforce the Gaussian coordinate conditions attain the meaning of the mass density and heat flow. When we impose only the Gaussian time condition, but not the Gaussian frame condition, the heat flow vanishes and the fluid becomes an ordinary incoherent dust. At this stage, an important point arises which plays a pivotal role in the interpretation of quantum geometrodynamics: The dust can be real only if its energy-momentum tensor satisfies the familiar energy conditions. It turns out that for the heat-conducting dust, the strong, dominant, and weak energy conditions are all the same. They are expressed by a single inequality involving the mass and heat multipliers and the metric tensor. If the dust conducts heat, the energy conditions may be satisfied by the initial data, but violated in the dynamical evolution.

The canonical description of the Gaussian reference fluid introduces the momenta conjugate to the Gaussian coordinates. The mass and heat multipliers are linear homogeneous functions of such momenta, and so are the energy and momentum densities of the reference fluid. The importance of this feature cannot be overemphasized. It underlies the Isham-Kuchař procedure for representing spacetime diffeomorphisms by canonical transformations, and it ensures that the constraints lead to the Schrödinger equation in the quantum theory.

The presence of the fluid variables makes it possible to simplify the general canonical formalism by requiring either the frame to be Gaussian or the foliation to be Gaussian, or both of these at once. This leads to three versions of reduced canonical formalisms which are directly applicable to quantum theory. The energy conditions in the canonical formalism can be expressed as an additional set of constraints on the metric and its conjugate momentum. For a heat-conducting dust, the dynamical constraints and energy conditions fail to be first class. The total system of constraints turns out to be first class only when the heat flow vanishes and the dust becomes incoherent.

The Dirac constraint quantization of the gravitational field coupled to the Gaussian reference fluid leads to a functional Schrödinger equation. In the reduced canonical formalism adopted to the Gaussian frame, the state becomes a functional of the *metric* induced on a given hypersurface. For an incoherent dust, the Gaussian frame coordinates are missing, the reduction cannot be accomplished, and the state depends only on *geometry*. These two conceptual frameworks thus lead to different notions of what variables are physically observable. By reducing the states to the Gaussian foliation, the functional Schrödinger equation reduces to an ordinary Schrödinger equation.

The functional Schrödinger equation possesses the standard positive-definite conserved norm. One would like to interpret the integrand of the norm integral as the probability density for the metric (or the geometry) to have a definite value on a hypersurface specified by the Gaussian clock. For a heat-conducting dust, such an interpretation fails because the energy conditions cannot be implemented in quantum theory. This means that the Gaussian coordinates cannot be identified by observing physical properties of a realistic material system, and the interpretation of the probability density for a *metric* loses thereby its operational significance. For incoherent Gaussian dust, the energy conditions *can* be carried over into quantum theory. The integrand of the norm integral still cannot be interpreted as the probability density for the geometry, but the reason is more subtle: When the geometry operator is implemented as a multiplication operator, it throws the state out of the subspace on which the energy conditions are satisfied. The interpretation of the Schrödinger equation is viable, but it requires a rather complicated procedure for finding the fundamental observables and correct expressions for their probability densities. There is no such thing as a free interpretation of quantum geometrodynamics.

II. GAUSSIAN COORDINATE CONDITIONS AND THE ACTION PRINCIPLES

Gaussian coordinate conditions are the simplest proposal ever made on how to break the coordinate freedom of the general theory of relativity:[24] In the normal Gaussian coordinates $X^K=(T,X^k)$, four components of the spacetime metric γ^{KL} are fixed by four algebraic relations, namely,

$$\gamma^{00}+1=0 , \tag{2.1}$$

and

$$\gamma^{0k}=0 . \tag{2.2}$$

At least locally, the spacetime can be foliated by the leaves of a constant Gaussian time T, and the worldlines $X^k=$const form a congruence which defines the Gaussian frame of reference. The *time condition* (2.1) ensures that the normal proper time separation between two neighboring leaves of the Gaussian time foliation is everywhere the same and equal to dT. The *frame condition* (2.2) means that the congruence of the Gaussian frame is orthogonal to the Gaussian time foliation. One can impose the time condition (2.1) without fixing the frame (2.2), or one can impose both conditions at once. These two alternatives provide two different conceptual frameworks for canonical gravity.

Note that the frame condition (2.2) fixes the Gaussian frame of reference, but it leaves the labeling of its points arbitrary. Indeed, Eq. (2.2) still holds after the relabeling

$$X^k \rightarrow X^{k'}=X^{k'}(X^l) . \tag{2.3}$$

The formalism we are going to build is covariant under the transformations (2.3).

Our goal is to incorporate the Gaussian conditions into the action principles of general relativity. In an arbitrary system of coordinates, the Einstein equations *in vacuo* follow from the Hilbert action[25]

$$S^G[\gamma_{KL}]=\int_{\mathcal{M}} d^4X|\gamma|^{1/2}R[\gamma_{KL}] , \tag{2.4}$$

by variation of the spacetime metric $\gamma_{KL}(X)$. An equivalent canonical form of the action is obtained by the Dirac-ADM (Arnowitt-Deser-Misner) rearrangement:[26] First, the spacetime metric is replaced by the lapse N, the shift N^a, and the spatial metric g_{ab}. Second, the momentum p^{ab} canonically conjugate to g_{ab} is introduced by the Legendre dual transformation. The Hamiltonian $H^G{}_N+H^G{}_{\mathbf{N}}$ of the canonical action

$$S^G[g_{ab},N,N^a]=\int_{\mathbb{R}} dt\left[\int_\Sigma d^3x\, p^{ab}(x)\dot{g}_{ab}(x) - H^G{}_N-H^G{}_{\mathbf{N}}\right] \tag{2.5}$$

is obtained by smearing the gravitational super-Hamiltonian

$$H^G=g^{-1/2}(p_{ab}p^{ab}-\tfrac{1}{2}p^2)-g^{1/2}R[g_{ab}] , \tag{2.6}$$

and supermomentum[27]

$$H^G{}_a=-2p_{a|b}^{b} , \tag{2.7}$$

by the lapse and the shift multipliers:

$$H^G{}_N=\int_\Sigma d^3x\, N(x)H^G(x) ,\qquad H^G{}_{\mathbf{N}}=\int_\Sigma d^3x\, N^a(x)H^G{}_a(x) . \tag{2.8}$$

The variation of N and N^a leads to the constraints

$$H^G(x)=0=H^G{}_a(x) , \tag{2.9}$$

and the variation of g_{ab},p^{ab} leads to the Hamilton equations of motion

$$\dot{g}_{ab}(x)=\{g_{ab}(x),H^G{}_N+H^G{}_{\mathbf{N}}\} ,\qquad \dot{p}^{ab}(x)=\{p^{ab}(x),H^G{}_N+H^G{}_{\mathbf{N}}\} . \tag{2.10}$$

The foliation in the Dirac-ADM action is arbitrary. In particular, we can require that it be Gaussian, i.e., that $t=T$. The time condition (2.1) then takes the form

$$N=1 . \tag{2.11}$$

If we also require that the points on each hypersurface be labeled by spatial Gaussian coordinates X^k, i.e., that $x^k=X^k$, the frame condition (2.2) gives

$$N^k=0 . \tag{2.12}$$

Coordinate conditions can be imposed either before or after the variation. When imposed *after* the variation, the Einstein law of gravitation *in vacuo*, or its canonical equivalent (2.9) and (2.10), remains valid. The conditions prescribe only the spacetime coordinate system or, in the canonical formalism, the passage from one hypersurface to another. Thus the Gaussian time condition reduces the Hamiltonian (2.8) to the form $h+H_{\mathbf{N}}$, and the complete Gaussian conditions reduce it even further to

$$h=\int_{\Sigma}d^3x\,H^G(x)\ . \tag{2.13}$$

The constraints (2.9) still hold, and the reduced Hamiltonian (2.13) thus still weakly vanishes.

When the coordinate conditions are imposed *before* the variation, the rules of the game are entirely changed. The metric γ_{KL} can no longer be freely varied. There are two equivalent ways of handling this situation. The first one is to express γ_{KL} in terms of some freely variable quantities so that the coordinate conditions are identically satisfied. The second way is to adjoin the coordinate conditions to the action by Lagrange multipliers and vary these as well as the metric γ_{KL} freely.[28]

The ADM decomposition allows us to treat the Gaussian coordinate condition in the first way. We shall discuss the situation at the level of the canonical action principle (2.5). When the Gaussian time condition (2.11) is imposed, only N^k and g_{kl},p^{kl} are freely variable. The canonical data g_{kl},p^{kl} then do not need any longer to satisfy the super-Hamiltonian constraint $H^G=0$. When the complete Gaussian conditions (2.11) and (2.12) are imposed, only g_{kl} and p^{kl} are freely variable. Both the super-Hamiltonian and the supermomentum constraints are then suspended. The Hamiltonian (2.13) no longer vanishes; instead of being a mere combination of the constraints, it acquires the status of a true Hamiltonian. Let us analyze the details of the formalism obtained by imposing the complete Gaussian conditions.

In the following considerations, $t=T$ and $x^k=X^k$. Because the constraints are suspended, the evolution can start from arbitrary canonical data $g_{kl}(x),p^{kl}(x)$. These are evolved by the true Hamiltonian (2.13) according to the Hamilton equations

$$\dot g_{kl}(x)=\{g_{kl}(x),h\},\quad \dot p^{\,kl}(x)=\{p^{kl}(x),h\}\ . \tag{2.14}$$

The former constraints can now be considered as some ordinary nonvanishing dynamical variables:

$$\begin{aligned}M(x;\mathbf{g},\mathbf{p}]&:=-g^{-1/2}(x)H^G(x)\ ,\\ M_k(x;\mathbf{g},\mathbf{p}]&:=g^{-1/2}(x)H^G{}_k(x)\ .\end{aligned} \tag{2.15}$$

For that role, we chose their scalar form and called them by different names.

The dynamical variables (2.15) are evolved by the Hamiltonian (2.13). Because the super-Hamiltonian (2.6) and supermomentum (2.7) obey the Dirac "algebra,"[29]

$$\{H^G(x),H^G(x')\}=g^{kl}(x)H^G{}_k(x)\delta_{,l}(x,x')-(x\leftrightarrow x')\ , \tag{2.16}$$

$$\{H^G{}_k(x),H^G(x')\}=H^G(x)\delta_{,k}(x,x')\ , \tag{2.17}$$

$$\{H^G{}_k(x),H^G{}_l(x')\}=H^G{}_l(x)\delta_{,k}(x,x')-(kx\leftrightarrow lx')\ , \tag{2.18}$$

and the Hamiltonian (2.13) is an integral of $H^G(x)$, we see that

$$\begin{aligned}(g^{1/2}(x)M(x))^{\cdot}&=\{g^{1/2}(x)M(x),h\}\\ &=-(g^{1/2}(x)M^k(x))_{,k}\ ,\end{aligned} \tag{2.19}$$

and

$$(g^{1/2}(x)M_k(x))^{\cdot}=\{g^{1/2}(x)M_k(x),h\}=0\ . \tag{2.20}$$

By Eq. (2.20), the dynamical variables $g^{1/2}(x)M_k(x)$ are constants of motion. Equation (2.19) is a continuity equation for $M(x)$ whose current is given by $M^k(x)$.

Let us now treat the coordinate conditions in the second way: Instead of varying the Hilbert action under the auxiliary conditions (2.1) and (2.2), we adjoin them to the action by the multipliers $M(X)$ and $M_k(X)$ (we will justify the reuse of these symbols below):

$$S[\gamma_{KL},M,M_k]=S^G[\gamma_{KL}]+S^F[\gamma_{KL},M,M_k]\ , \tag{2.21}$$

with

$$\begin{aligned}S^F[\gamma_{KL},M,M_k]&\\ :=\int_{\mathcal{M}}d^4X(&-\tfrac{1}{2}M(X)|\gamma(X)|^{1/2}(\gamma^{00}(X)+1)\\ &+M_k(X)|\gamma(X)|^{1/2}\gamma^{0k}(X))\ ,\end{aligned} \tag{2.22}$$

and vary γ_{KL}, M, and M^k freely. The variation of S^F with respect to the metric γ_{KL} introduces a source term into the Einstein law of gravitation.[30] We shall provide its physical interpretation in the next section.

After the Dirac-ADM rearrangement, the action (2.21) and (2.22) assumes the form

$$\begin{aligned}S[g_{kl},p^{kl},N,N^k,M,M_k]&\\ =\int_{\mathbb{R}}dt\int_{\Sigma}d^3x(&p^{kl}\dot g_{kl}-NH^G-N^kH^G{}_k\\ &-\tfrac{1}{2}g^{1/2}M(N-N^{-1})+g^{1/2}M_kNN^k)\end{aligned} \tag{2.23}$$

on the Gaussian foliation $t=T$ and in the Gaussian frame coordinates $x^k=X^k$. Its variation with respect to M and M_k yields Eqs. (2.11) and (2.12). The variation with respect to N and N^k leads then to Eqs. (2.15) which fix the multipliers M and M_k as functions of the canonical variables g_{kl},p^{kl}. When we use Eqs. (2.11), (2.12), and (2.15) to eliminate the multipliers N,N^k and M,M_k from the action (2.23), we obtain the canonical action

$$S[g_{kl},p^{kl}]=\int_{\mathbb{R}}dt\left[\int_{\Sigma}d^3x\,p^{kl}\dot g_{kl}-h\right]\ , \tag{2.24}$$

which yields the Hamilton equation (2.14).

III. PARAMETRIZED ACTION AND THE STRUCTURE OF THE REFERENCE FLUID

The Hilbert action S^G is invariant under arbitrary transformations of the spacetime coordinates X^K. The action S^F whose variation enforces the coordinate conditions breaks this invariance. As a result, the field equations obtained by varying the total action (2.21) hold only in the Gaussian coordinate system. One knows, however, that one can always restore the diffeomorphism invariance of the action by its parametrization.[3] This process consists of expressing the privileged (Gaussian) coordinates $X^K=(T,X^k)$ as functions of arbitrary (label) coordinates x^α,

$$X^K=X^K(x^\alpha) , \tag{3.1}$$

and adjoining these functions to the original field variables.

The new action $S[\gamma_{\alpha\beta},M_K,X^K]$ is uniquely determined by the requirements that it be invariant under transformation of x^α and reduce to the old action $S[\gamma_{KL},M_K]$ when the Gaussian coordinates are used as the labels:

$$S[\gamma_{\alpha\beta},M_K,X^K=\delta^K_\alpha x^\alpha]=S[\gamma_{KL},M_K] . \tag{3.2}$$

The Hilbert action is invariant by itself, and one does not need to adjoin the functions (3.1) to the metric variables γ_{KL} to satisfy Eq. (3.2):

$$S^G[\gamma_{\alpha\beta}]=\int_{\mathcal{M}} d^4x\,|\gamma|^{1/2}R[\gamma_{\alpha\beta}] . \tag{3.3}$$

The condition (3.2) is, however, needed to determine the parametrized action S^F:

$$\begin{aligned} S^F(\gamma_{\alpha\beta},M_K,X^K] &= \int_{\mathcal{M}} d^4x(-\tfrac{1}{2}M|\gamma|^{1/2}(\gamma^{\alpha\beta}T_{,\alpha}T_{,\beta}+1) \\ &\quad + M_k|\gamma|^{1/2}\gamma^{\alpha\beta}T_{,\alpha}X^k{}_{,\beta}) . \end{aligned} \tag{3.4}$$

Under the transformations of x^α, both the Gaussian coordinates $X^K(x^\alpha)=(T(x^\alpha),X^k(x^\alpha))$ and the multipliers $M_K(x^\alpha)=(M(x^\alpha),M_k(x^\alpha))$ behave as scalars. The action (3.4) is thus manifestly invariant under arbitrary transformations of x^α. By comparing the new action (3.4) with the old action (2.22), one easily verifies Eq. (3.2).

The coordinate invariance of the parametrized action

$$S[\gamma_{\alpha\beta},M_K,X^K]=S^G[\gamma_{\alpha\beta}]+S^F[\gamma_{\alpha\beta},M_K,X^K] \tag{3.5}$$

implies that the equations of motion obtained by varying $X^K(x^\alpha)$ follow from the equations obtained by varying the remaining variables $\gamma_{\alpha\beta}$ and M_K. The action S^F describes a source of the gravitational field. We shall call this source the *Gaussian reference fluid.* We have chosen this term with care: We shall show that the source behaves very much as an actual fluid. The variables $X^K(x^\alpha)$ play the role of the velocity potentials, and their variation yields the Euler hydrodynamical equations.[31]

The variation of the action with respect to $\gamma_{\alpha\beta}$ leads to the Einstein law of gravitation

$$G^{\alpha\beta}=\tfrac{1}{2}T^{\alpha\beta} , \tag{3.6}$$

in which

$$T^{\alpha\beta}=2|\gamma|^{-1/2}\delta S^F/\delta\gamma_{\alpha\beta} \tag{3.7}$$

is the energy-momentum tensor of the reference fluid. The variation of the multipliers M_K yields the Gaussian coordinate conditions

$$\gamma^{\alpha\beta}T_{,\alpha}T_{,\beta}+1=0 , \tag{3.8}$$

and

$$\gamma^{\alpha\beta}T_{,\alpha}X^k{}_{,\beta}=0 . \tag{3.9}$$

Because of the invariance of the action, the Euler equations of the fluid follow from the Einstein law (3.6) and (3.7) and the coordinate conditions (3.8) and (3.9).

The Gaussian frame and Gaussian time have a number of important geometric features which stem from the coordinate conditions (3.8) and (3.9). Introduce the vector field

$$U^\alpha:=-\gamma^{\alpha\beta}T_{,\beta} , \tag{3.10}$$

which is the four-velocity of the Gaussian frame. By Eq. (3.8) U^α is unit and timelike; under the assumption that the Gaussian time T grows from the past to the future, it is also future pointing. Moreover, U^α is irrotational and geodesic:

$$U_{[\alpha,\beta]}=0 \quad\text{and}\quad U^\alpha{}_{;\beta}U^\beta=0 . \tag{3.11}$$

The symmetric tensor $U_{(\alpha;\beta)}$ is purely spatial:

$$U_{(\alpha;\beta)}U^\beta=0 ; \tag{3.12}$$

it describes the expansion and shear of the Gaussian frame.[32]

The projection

$$h^{kl}:=\gamma^{\alpha\beta}X^k{}_{,\alpha}X^l{}_{,\beta} \tag{3.13}$$

is the Gaussian three-metric on the leaves $T(x^\alpha)=\text{const}$. It behaves as a contravariant tensor under transformations (2.3) of the Gaussian coordinates X^k. It can be used for raising and lowering the Gaussian indices. The vectors $(U^\alpha,\ X^\alpha_k=h_{kl}\gamma^{\alpha\beta}X^l{}_{,\beta})$ form an orthogonal basis. These vectors are tangent to the coordinate lines of the Gaussian system. Like $U_{(\alpha;\beta)}$, the symmetric tensor $X_{k(\alpha;\beta)}$ is purely spatial:

$$X_{k(\alpha;\beta)}U^\beta=0 . \tag{3.14}$$

From the multipliers M_k and the cobasis elements $X^k{}_{,\alpha}$, we can form a spacetime covector

$$M_\alpha:=M_kX^k{}_{,\alpha},\quad M_\alpha U^\alpha=0 . \tag{3.15}$$

Under the transformation (2.3) of the Gaussian coordinates, M_k behaves as a covector. This ensures that the action (3.4) remains invariant and that M_α of Eq. (3.15) depends only on the Gaussian frame, not on the choice of the Gaussian coordinates. This brings us to the point at which we can write the energy-momentum tensor (3.7) of the reference fluid in a physically transparent way:

$$T^{\alpha\beta}=MU^\alpha U^\beta+M^{(\alpha}U^{\beta)} . \tag{3.16}$$

The tensor (3.16) has the form of the Eckart energy-momentum tensor of a heat-conducting fluid.[33] The vector U^α is the four-velocity of the fluid, the multiplier M is its mass density, and the vector M^α constructed from the multipliers M_k is the heat flow. (From now on, we shall call M_k the heat multipliers.) The energy-momentum tensor (3.16) does not have any stress part $\Theta^{\alpha\beta}$; the coordinate fluid thus behaves as a dust. When we impose only the time condition, but not the frame condition, $M_k=0$, the heat flow vanishes, and $T^{\alpha\beta}$ describes an incoherent dust:

$$T^{\alpha\beta}=MU^{\alpha}U^{\beta} \ . \tag{3.17}$$

By virtue of the Einstein law (3.6), $T^{\alpha\beta}$ is covariantly conserved:

$$T^{\alpha\beta}{}_{;\beta}=0 \ . \tag{3.18}$$

The conservation law (3.18) can be projected into the cobasis $U_\alpha, X_{k\alpha}$. Because of Eqs. (3.12) and (3.14), these projections yield two continuity equations:

$$(U_\alpha T^{\alpha\beta})_{;\beta}=0=(X_{k\alpha}T^{\alpha\beta})_{;\beta} \ . \tag{3.19}$$

From the structure (3.16) of the energy-momentum tensor, we get the concrete form of the conserved currents:

$$(MU^{\beta}+M^{\beta})_{;\beta}=0 \ , \tag{3.20}$$

and

$$(M_k U^{\beta})_{;\beta}=0 \ . \tag{3.21}$$

Equations (3.20) and (3.21) are the Euler hydrodynamical equations of the Gaussian reference fluid. They can also be obtained by varying the Gaussian coordinates $X^K(x^\alpha)$. Equation (3.20) tells us that the rest mass of a comoving element of dust increases when heat flows into the region. Similarly, Eq. (3.21) tells us that the quantities M_k in the comoving element are conserved. This is best brought out when we write Eqs. (3.20) and (3.21) in the Gaussian system of coordinates $t=T$, $x^k=X^k$, which is the comoving system of the fluid. Then $U^\alpha=\delta^\alpha_0$, $X^k_\alpha=\delta^k_\alpha$, and $h^{kl}=g^{kl}$. Equation (3.20) and (3.21) thereby assume the form (2.19) and (2.20).

IV. ENERGY CONDITIONS

The use of Gaussian conditions as auxiliary conditions in the Hilbert action principle is a purely formal device. It is quite surprising that the energy-momentum tensor (3.16) which emerges from this procedure seems to describe a simple physical system, namely, heat-conducting dust. If we had such a material at our disposal and scattered it throughout space, we could identify the Gaussian coordinates with its physical state variables. Figuratively speaking, we could anchor the Gaussian coordinates into the physical world lines of the dust. However, unlike an abstract Gaussian system, the dust can be real only if its energy-momentum tensor satisfies appropriate energy conditions.[34]

The energy conditions ensure that the energy density and energy current measured by an arbitrarily moving local observer have reasonable physical properties. The observer is completely characterized by its four-velocity V^α, $V^\alpha V_\alpha=-1$. At the very least, a physical system must satisfy the *weak energy condition*: The energy density cannot be negative, i.e.,

$$T_{\alpha\beta}V^\alpha V^\beta \geq 0 \quad \forall V^\alpha\colon\ V^\alpha V_\alpha=-1 \ . \tag{4.1}$$

Indeed, one can also justifiably argue that the energy flow S^α should never become spacelike:

$$S^\alpha S_\alpha \leq 0 \quad \forall V^\alpha\colon\ V^\alpha V_\alpha=-1 \ , \quad S^\alpha := -T^{\alpha\beta}V_\beta \ . \tag{4.2}$$

The *dominant energy condition* requires that both inequalities (4.1) and (4.2) be satisfied.

In singularity theorems, one often needs as an input the *strong energy condition* which, in addition to Eq. (4.1), requires that

$$T_{\alpha\beta}V^\alpha V^\beta \geq -\tfrac{1}{2}T^\beta_\beta \quad \forall V^\alpha\colon\ V^\alpha V_\alpha=-1 \ . \tag{4.3}$$

In principle, a physical system may violate the condition (4.3).

The energy conditions can be expressed in terms of eigenvectors and eigenvalues of the energy-momentum tensor. Classified by its eigenvectors, any $T^{\alpha\beta}$ necessarily belongs to one of the following types: *type* I: $T^{\alpha\beta}$ has three spacelike eigenvectors and one timelike eigenvector; *type* II: $T^{\alpha\beta}$ has two spacelike eigenvectors and a double lightlike eigenvector; *type* III: $T^{\alpha\beta}$ has one spacelike eigenvector and a triple lightlike eigenvector; and *type* IV: $T^{\alpha\beta}$ has two spacelike eigenvectors and no timelike or lightlike eigenvector.

To satisfy the weak energy condition, the energy-momentum tensor must be either of type I or II. Moreover, its eigenvalues must satisfy certain inequalities. To write them down, we shall denote those eigenvalues which belong to spacelike eigenvectors by $\pi_{(a)}$ (these are the principal stresses). We shall call the eigenvalue which belongs to a timelike or a lightlike eigenvector $-\mu$ (for a type-I tensor, μ is the proper energy density).

An energy-momentum tensor of type I satisfies the weak energy condition if

$$\mu \geq 0 \quad \text{and} \quad \pi_{(a)} \geq -\mu \quad \text{for } a=1,2,3 \ . \tag{4.4}$$

A tensor of type II satisfies the weak energy condition if

$$\mu \geq 0 \quad \text{and} \quad \pi_{(a)} \geq 0 \quad \text{for } a=1,2 \ , \tag{4.5}$$

and, moreover, if

$$T_{\alpha\beta}L^\alpha L^\beta > 0 \ , \tag{4.6}$$

where L^α is the lightlike vector orthogonal to the spacelike eigenvectors and linearly independent of the lightlike eigenvector.

The dominant and strong energy conditions subject the eigenvalues to additional inequalities. A tensor of type I satisfies the dominant energy condition if it satisfies Eq. (4.4) and

$$\pi_{(a)} \leq \mu \quad \text{for } a=1,2,3 \ . \tag{4.7}$$

It satisfies the strong energy condition if it satisfies Eq. (4.4) and

$$\sum_{a=1}^{3} \pi_{(a)} \geq -\mu \ . \tag{4.8}$$

A tensor of type II satisfies the dominant energy condition if it satisfies Eqs. (4.5) and (4.6) and

$$\pi_{(a)} \leq \mu \quad \text{for } a=1,2 \ . \tag{4.9}$$

For a tensor of type II, the strong and weak energy conditions are the same.

Let us now decide when the Gaussian reference fluid satisfies the energy conditions. First of all, it is obvious that any vector orthogonal to U^α and M^α is a spacelike eigenvector of the coordinate fluid (3.16) belonging to the eigenvalue 0. Therefore,

$$\pi_{(1)}=\pi_{(2)}=0 \tag{4.10}$$

is a doubly degenerate eigenvalue and its eigenvectors fill a two-dimensional spacelike plane. This means that the energy-momentum tensor (3.16) cannot be of type III. It also means that if $T^{\alpha\beta}$ is of type I, the weak energy condition implies the strong energy condition, and if it is of type II, the weak energy condition implies both the dominant and strong energy conditions.

Let us first discuss $T^{\alpha\beta}$ when $M^\alpha=0$. The energy-momentum tensor must have this form if we impose only the time condition, but not the frame condition, or it may happen to have it through a particular choice of the initial data. Every vector orthogonal to U^α is then a spacelike eigenvector with the eigenvalue 0. This becomes triply degenerate:

$$\pi_{(1)}=\pi_{(2)}=\pi_{(3)}=0 \ . \tag{4.11}$$

U^α itself is a timelike eigenvector of $T^{\alpha\beta}$ with the eigenvalue $-\mu=M$. The weak energy condition is satisfied for

$$M\geq 0 \ , \tag{4.12}$$

and the dominant and strong energy conditions are satisfied as well.

Next, let $M^\alpha\neq 0$. We know that the spacelike eigenvectors with the eigenvalue (4.10) fill a plane. The remaining eigenvectors of $T^{\alpha\beta}$, if any, must lie in the orthogonal plane which is spanned by M^α and U^α:

$$Y^\alpha=M^\alpha-\lambda U^\alpha \ . \tag{4.13}$$

The eigenvalue equation $T^\alpha_\beta Y^\beta \sim Y^\alpha$ tells us that the coefficient λ in Eq. (4.13) is actually the eigenvalue and that it must satisfy the characteristic equation

$$\begin{aligned}&\lambda^2+M\lambda+|\mathbf{M}|^2=0 \ ,\\ &|\mathbf{M}|^2:=\gamma^{\alpha\beta}M_\alpha M_\beta=h^{kl}M_kM_l \ .\end{aligned} \tag{4.14}$$

The solutions of Eq. (4.14) are

$$\lambda_\pm=\tfrac{1}{2}(-M\pm(M^2-4|\mathbf{M}|^2)^{1/2}) \ . \tag{4.15}$$

We see that for $|M|<2|\mathbf{M}|$ there are no real eigenvalues, and $T^{\alpha\beta}$ is thus of type IV.

For $|M|=2|\mathbf{M}|$, $T^{\alpha\beta}$ has a double eigenvalue $\lambda_+=\lambda_-=:-\mu=-\frac{1}{2}M$ with the lightlike eigenvector

$$Y^\alpha=M^\alpha+\tfrac{1}{2}MU^\alpha \ . \tag{4.16}$$

The coordinate fluid is then of type II. The second lightlike vector in the plane M^α, U^α is

$$L^\alpha=-M^\alpha+\tfrac{1}{2}MU^\alpha \ . \tag{4.17}$$

The weak energy condition (4.5) and (4.6) again amounts to Eq. (4.12). As we have already noted, the weak energy condition in this case implies the dominant and strong energy conditions.

For $|M|>2|\mathbf{M}|$, $T^{\alpha\beta}$ has two real eigenvalues (4.15). If $M<0$, Y^α_+ is timelike and Y^α_- is spacelike. Because $\mu=-\lambda_+<0$, the weak energy condition is then violated. If $M>0$, Y^α_+ is spacelike and Y^α_- is timelike. The eigenvalues

$$\begin{aligned}&\mu=-\lambda_-=\tfrac{1}{2}(M+(M^2-4|\mathbf{M}|^2)^{1/2})>0 \ ,\\ &\pi_{(1)}=\pi_{(2)}=0 \ ,\\ &\pi_{(3)}=\lambda_+=\tfrac{1}{2}(-M+(M^2-4|\mathbf{M}|^2)^{1/2})<0\end{aligned} \tag{4.18}$$

satisfy the weak energy condition which, as mentioned, also implies the strong energy condition. Moreover, the eigenvalues (4.18) satisfy also the dominant energy condition (4.7).

To summarize, the weak energy condition for the Gaussian reference fluid requires

$$M\geq 2|\mathbf{M}| \ . \tag{4.19}$$

If it is satisfied, the dominant and strong energy conditions are satisfied as well. When we impose only the Gaussian time condition, the Gaussian fluid is an incoherent dust (3.17). The weak energy condition then reduces to Eq. (4.12); it again implies the dominant and strong energy conditions. The reference fluid which obeys Eq. (4.19) can be proclaimed to be, if not really real, then at least realistic.

The fundamental physical fields (like the electromagnetic field) typically satisfy the energy conditions by the way in which their energy-momentum tensor is constructed from the field variables. Relativistic fluids satisfy the energy conditions by virtue of their equations of state. The Gaussian fluid does not have any equation of state; the energy condition (4.19) is simply an inequality involving the Lagrange multipliers M, M_k and the metric tensor. It is thus conceivable that it is satisfied by the initial data, but gets violated in the dynamical evolution.

A little reflection shows that this cannot happen for an incoherent dust (3.17). If, in the Gaussian system of coordinates $t=T$ and $x^k=X^k$, we start with $M(x)\geq 0$ at $t=0$, Eqs. (2.19) and (2.20) ensure that $g^{1/2}(t,x)M(t,x)=g^{1/2}(x)M(x)\geq 0$. However, for a heat-conducting fluid it is easy to give initial data $g_{kl}(x), p^{kl}(x)$ such that $M(t,x)$ and $M_k(t,x)\neq 0$, defined by Eqs. (2.15), satisfy the energy condition at $t=0$, but violate it for a later t. Let us present an example of such data that, shortly before $t=0$, the Gaussian fluid is of type I and satisfies the energy condition, at $t=0$ it becomes type II while still satisfying the energy condition, but slightly later than $t=0$ it becomes type IV and thus violates the energy condition.

Study the dynamical variable

$$F(x;\mathbf{g},\mathbf{p}]:=M(x)-2|\mathbf{M}(x)| \ . \tag{4.20}$$

The sign of F determines whether the reference fluid satisfies the energy conditions and it gives its type:

$$F\begin{cases}\left.\begin{matrix}>0\ , & \text{type I}\\ =0\ , & \text{type II}\end{matrix}\right\}\text{energy conditions satisfied}\ ,\\ <0, \ \text{type IV}\}\ \text{energy conditions violated}\ .\end{cases} \tag{4.21}$$

As we evolve $g_{kl}(x)$ and $p^{kl}(x)$ by Eq. (2.14) along a Gaussian foliation, Eqs. (2.19) and (2.20) yield

$$(g^{1/2}F)^{\cdot}=-(g^{1/2}M^k)_{,k}+2|\mathbf{M}|^{-1}(p^{kl}-\tfrac{1}{2}pg^{kl})M_kM_l\ . \tag{4.22}$$

To prove our point, we give $g_{kl}(x)$ and $p^{kl}(x)$ such that $F(x)=0$ and $(g^{1/2}(x)F(x))^{\cdot}<0$: We endow a three-torus $\mathbb{T}^3$ with a flat metric $g_{kl}=\delta_{kl}$ in the Cartesian angle coordinates $x^k=(x,y,z)$, $x^k\in\langle 0,2\pi\rangle$, and put

$$\begin{aligned}&p^{11}=p^{22}=p^{33}=\frac{2}{\sqrt{3}}(\alpha^2+2\alpha)^{1/2}\ ,\\ &p^{12}=\alpha\sin x,\quad p^{13}=\alpha\cos x,\quad p^{23}=0\ ,\end{aligned} \tag{4.23}$$

where $\alpha>0$ is a positive constant. We get

$$\begin{aligned}&M=4\alpha\ ,\\ &M_k=2\alpha(0,\ -\cos x,\ \sin x)\implies(g^{1/2}M^k)_{,k}=0\ ,\end{aligned} \tag{4.24}$$

and

$$|\mathbf{M}|=2\alpha\ ,$$

and hence

$$F=0,\quad \dot F=-\frac{4}{\sqrt{3}}\alpha(\alpha^2+2\alpha)^{1/2}<0\ , \tag{4.25}$$

as desired,

We can summarize our discussion by two statements: The Gaussian reference fluid is realistic only if the multipliers M,M_k satisfy the inequalities (4.19). Even if it is realistic at one time, it can become unrealistic later. This foreshadows the problems encountered in interpreting the Schrödinger equation obtained by quantizing geometry whose source is the reference fluid.

V. CANONICAL DESCRIPTION OF THE GAUSSIAN REFERENCE FLUID

The Lagrangian density

$$\begin{aligned}\mathcal{L}^F=&-\tfrac{1}{2}M|\gamma|^{1/2}(\gamma^{\alpha\beta}T_{,\alpha}T_{,\beta}+1)\\ &+M_k|\gamma|^{1/2}\gamma^{\alpha\beta}T_{,\alpha}X^k{}_{,\beta}\end{aligned} \tag{5.1}$$

of the Gaussian reference fluid (3.4) does not contain any derivatives of $\gamma_{\alpha\beta}$. The canonical form of the parametrized action (3.5) thus follows the pattern of all theories with nonderivative gravitational coupling:[35]

$$\begin{aligned}&S[g_{ab},M_K,X^K]\\ &\quad=\int_{\mathbb{R}}dt\int_\Sigma d^3x(p^{ab}\dot g_{ab}+P_K\dot X^K-NH-N^aH_a)\ ,\end{aligned} \tag{5.2}$$

with

$$H=H^G+H^F,\quad H_a=H^G{}_a+H^F{}_a\ . \tag{5.3}$$

The super-Hamiltonian of the coupled system is obtained by adding to the gravitational super-Hamiltonian (2.6) the energy density H^F of the fluid. Similarly, the supermomentum is the sum of the gravitational supermomentum (2.7) and the momentum density $H^F{}_a$ of the fluid. Because the fluid potentials X^K are spatial scalars, the momentum density $H^F{}_a$ must have the form[36]

$$H^F{}_a=P_KX^K{}_{,a}=PT_{,a}+P_kX^k{}_{,a}\ , \tag{5.4}$$

where $P_K=(P,P_k)$ are the momenta canonically conjugate to $X^K=(T,X^k)$. The only quantity which we need to calculate is H^F. For this purpose, it is justified to simplify the procedure by putting $N^a=0$.

After the ADM decomposition with $N^a=0$, the Lagrangian density of the fluid assumes the form

$$\begin{aligned}\mathcal{L}^F=&\tfrac{1}{2}g^{1/2}M(N^{-1}\dot T^2-NW^{-2})\\ &-g^{1/2}M_k(N^{-1}\dot T\dot X^k-NW^k)\ .\end{aligned} \tag{5.5}$$

We have introduced the abbreviations

$$W:=(1+g^{ab}T_{,a}T_{,b})^{-1/2}\ , \tag{5.6}$$

and

$$W^k:=g^{ab}T_{,a}X^k{}_{,b}\ , \tag{5.7}$$

for two kinds of "potential energies" of the fluid which run as coefficients through all our calculations.

We can now introduce the momenta

$$P:=\frac{\partial\mathcal{L}^F}{\partial\dot T}=g^{1/2}N^{-1}(M\dot T-M_k\dot X^k)\ , \tag{5.8}$$

and

$$P_k:=\frac{\partial\mathcal{L}^F}{\partial\dot X^k}=-g^{1/2}N^{-1}M_k\dot T\ . \tag{5.9}$$

From Eq. (5.9), we see that M_k and P_k are not independent; they satisfy the constraints

$$M_{[k}P_{l]}=0\ . \tag{5.10}$$

As a consequence, not all the velocities $\dot X^k$, but only their combination $M_k\dot X^k$, can be expressed back as functions of the momenta. One can either adjoin the constraints (5.10) to the action by a new set of multipliers, or solve them immediately and not worry about them any more. The second option is simpler. The constraints (5.10) mean that M_k is parallel to P_k; one can thus express the heat multipliers M_k in terms of a single multiplier J and the momenta P_k:

$$M_k=-g^{-1/2}JP_k\ . \tag{5.11}$$

Equation (5.9) then reads

$$\dot T=NJ^{-1}\ , \tag{5.12}$$

and Eq. (5.8) can be inverted as

$$M_k\dot X^k=N(-g^{-1/2}P+MJ^{-1})\ . \tag{5.13}$$

The Hamiltonian density

$$\mathcal{H}^F := P\dot{T} + P_k \dot{X}^k - \mathcal{L}^F \tag{5.14}$$

depends only on the combinations (5.12) and (5.13) of the velocities. When expressed in terms of the momenta, it takes the form $\mathcal{H}^F = NH^F$, with

$$H^F = J^{-1}P + JW^k P_k + \tfrac{1}{2} g^{1/2} M (W^{-2} - J^{-2}) \ . \tag{5.15}$$

The action (5.2)–(5.4) and (5.15) still contains the multipliers M and J. The value of the multipliers is fixed by the condition that their variation leave the action stationary, which leads to

$$\frac{\partial H^F}{\partial M} = 0 = \frac{\partial H^F}{\partial J} \ . \tag{5.16}$$

This determines the multipliers in terms of the fluid variables:

$$J = +W \ , \tag{5.17}$$

and

$$M = g^{-1/2} W (P - W^2 W^k P_k) \ . \tag{5.18}$$

The positive sign in Eq. (5.17) is forced upon us by the requirement that T grow from the past to the future, i.e., that $N > 0 \Longrightarrow \dot{T} > 0$ in Eq. (5.12). The heat multipliers M_k are then given by Eqs. (5.11) and (5.17):

$$M_k = -g^{-1/2} W P_k \ . \tag{5.19}$$

We can now eliminate the multipliers from the action by substituting expressions (5.17) and (5.18) back into Eq. (5.2). The energy density (5.15) of the reference fluid becomes thereby a linear homogeneous function of the momenta:

$$H^F = n^K P_K = W^{-1} P + W W^k P_k \ . \tag{5.20}$$

This is a remarkable feature, and we shall devote the rest of this paper to its exploration.

The coefficients

$$n^K = (W^{-1}, W W^k) \tag{5.21}$$

can be identified with the components of the unit normal to the hypersurface in the Gaussian system of coordinates:

$$\gamma_{KL} n^K n^L = -1, \quad \gamma_{KL} n^K X^L{}_{,a} = 0 \ . \tag{5.22}$$

Equations (5.22) follow from the coordinate conditions (2.1) and (2.2) and from the connection

$$g_{ab} = \gamma_{KL} X^K{}_{,a} X^L{}_{,b} = -T_{,a} T_{,b} + h_{kl} X^k{}_{,a} X^l{}_{,b} \tag{5.23}$$

between the Gaussian metric h_{kl} on $T=$const and the induced metric g_{ab} on Σ. Note that Eq. (5.23) can be inverted for the Gaussian metric:

$$h^{kl} = X^k{}_{,a} X^l{}_{,b} (g^{ab} - W^2 T^a T^b) \ , \tag{5.24}$$

with $T^a := g^{ab} T_{,b}$. The energy condition (4.19) on the multipliers (5.18) and (5.19) can then be expressed as limitations on the range of the canonical momenta P_K.

The spacetime action (3.4) of the reference fluid remains invariant under arbitrary transformations (2.3) of the Gaussian coordinates X^k. This invariance must be reflected in the canonical formalism as an internal symmetry of the Hamiltonian.[37] Let us find the generators of this symmetry.

The transformations (2.3) form a group, namely, DiffΣ. Its Lie algebra, LDiffΣ, consists of all (complete) vector fields $\boldsymbol{\Theta} = \Theta^k(X)\partial_k$ on Σ, closed under the Lie bracket operation $-[\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2]$. A transformation (2.3) induces a point transformation of the canonical variables $X^k(x), P_k(x)$:

$$X^{k'}(x) = X^{k'}(X^l(x)) \ , \tag{5.25}$$

Each vector field $\boldsymbol{\Theta}(X)$ generates a one-parameter subgroup of DiffΣ; the corresponding one-parameter subgroup of canonical transformations (5.25) is generated by the dynamical variable

$$P(\boldsymbol{\Theta}) := \int_\Sigma d^3x \ \Theta^k(X(x)) P_k(x) \tag{5.26}$$

according to the equations

$$\frac{dX^k(x)}{d\sigma} = \{X^k(x), P(\boldsymbol{\Theta})\} \ , \qquad \frac{dP_k(x)}{d\sigma} = \{P_k(x), P(\boldsymbol{\Theta})\} \ . \tag{5.27}$$

The generators $P(\boldsymbol{\Theta})$ represent LDiffΣ because

$$\{P(\boldsymbol{\Theta}_1), P(\boldsymbol{\Theta}_2)\} = P(-[\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2]) \ . \tag{5.28}$$

One can verify that the canonical transformation (5.27) generated by $P(\boldsymbol{\Theta})$ does not affect the dynamics of the reference fluid coupled to gravity, i.e., that it leaves invariant the Hamiltonian $H_N + H_{\mathbf{N}}$ of the system:

$$\{H_N + H_{\mathbf{N}}, P(\boldsymbol{\Theta})\} \approx 0 \quad \forall N, \mathbf{N} \ \text{and} \ \forall \boldsymbol{\Theta} \ . \tag{5.29}$$

Because the gravitational super-Hamiltonian and supermomentum do not depend on the fluid variables, Eq. (5.29) reduces to the requirement

$$\{H^F(x), P(\boldsymbol{\Theta})\} = 0 = \{H^F{}_a(x), P(\boldsymbol{\Theta})\} \tag{5.30}$$

on the energy and momentum densities of the fluid. It is easy to check that Eqs. (5.30) are satisfied by expressions (5.20) and (5.4).

The generators $P(\boldsymbol{\Theta})$ which leave the Hamiltonian invariant [Eq. (5.29)] are by the same token constants of motion. For the general state of the fluid, these constants do not vanish. Because

$$P(\boldsymbol{\Theta}) = 0 \ \forall \boldsymbol{\Theta} \ \Longrightarrow \ P_k = 0 \ \Longrightarrow \ M_k = 0 \ , \tag{5.31}$$

the vanishing of all $P(\boldsymbol{\Theta})$'s implies that the reference fluid does not conduct heat and reduces thereby to an incoherent dust.

Let us relate the canonical formalism for the reference fluid to the spacetime approach. In theories with nonderivative gravitational coupling, the energy density H^F and the momentum density $H^F{}_a$ are the $\perp\perp$ and $\perp\parallel$ projections of the energy-momentum tensor on Σ:[38]

$$T_{\alpha\beta} n^\alpha n^\beta = g^{-1/2} H^F, \quad T_{\alpha\beta} n^\alpha X^\beta{}_{,b} = g^{-1/2} H^F{}_b \ . \tag{5.32}$$

Moreover, H^F alone determines the remaining stress components of $T_{\alpha\beta}$:[39]

$$T_{ab} := T_{\alpha\beta} X^{\alpha}{}_{,a} X^{\beta}{}_{,b} = 2g^{-1/2} \frac{\partial H^F}{\partial g^{ab}} . \tag{5.33}$$

The right-hand sides of Eqs. (5.32) and (5.33) belong to the canonical formalism. The energy-momentum tensor on the left-hand sides follows from the spacetime approach. Equations (5.32) and (5.33) connect these two standpoints.

For the Gaussian reference fluid, H^F and $H^F{}_a$ are given by Eqs. (5.20) and (5.4). Expressions (5.18) and (5.19) determine the mass density M and heat flow M_k as dynamical variables on the phase space. By differentiating H^F with respect to g^{ab}, one obtains

$$T_{ab} = M T_{,a} T_{,b} - M_k T_{,(a} X^k{}_{,b)} . \tag{5.34}$$

On the other hand, the spacetime energy-momentum tensor is given by Eq. (3.16). One can easily check Eqs. (5.32) and (5.33) by evaluating the projections of $T_{\alpha\beta}$ in the Gaussian coordinates, referring to Eq. (5.21) for the normal n^K.

The canonical formalism vastly simplifies when one chooses not to impose the Gaussian frame conditions, but only the Gaussian time condition. The derivations become straightforward, and we leave them to the reader. The final results can be obtained from those we have presented by putting $P_k = 0$ in Eqs. (5.4), (5.20), and (5.18):

$$H^F = W^{-1} P, \quad H^F{}_a = T_{,a} P , \tag{5.35}$$

and

$$M = g^{-1/2} W P . \tag{5.36}$$

The Gaussian clock T tells us how to draw the hypersurfaces of constant time, but the phase space (T, P) lacks the markers X^k which would identify the spatial points. Such a shift of the conceptual framework has far-reaching consequences for the quantum theory.

VI. CANONICAL REPRESENTATION OF SPACETIME DIFFEOMORPHISMS

The energy density (5.20) and momentum density (5.4) of the reference fluid are linear combinations of the momenta P_K. Inversely, the momenta can be expressed as linear combinations of these densities:

$$P_K = -n_K H^F + X^a_K H^F{}_a . \tag{6.1}$$

Here

$$-n_K = -\gamma_{KJ} n^J, \quad X^a_K = \gamma_{KJ} X^J{}_{,b} g^{ba} \tag{6.2}$$

is the cobasis dual to the normal basis $(n^K, X^K{}_{,a})$. Let X^a_k denote the inverse matrix to $X^k{}_{,a}$:

$$X^k{}_{,a} X^a_j = \delta^k_j, \quad X^k{}_{,a} X^b_k = \delta^b_a . \tag{6.3}$$

Equations (6.2) and (5.21), (5.23) then give

$$\begin{aligned} -n_K &= (W^{-1}, -W^{-1} T_{,k}) , \\ X^a_K &= (-T^a, X^a_k + T^a T_{,k}) , \quad T^a := g^{ab} T_{,b} . \end{aligned} \tag{6.4}$$

Equations (6.1) attain thereby the form

$$P = W^{-1} H^F - T^a H^F{}_a , \tag{6.5}$$

and

$$P_k = -W^{-1} T_{,k} H^F + (X^a_k + T^a T_{,k}) H^F{}_a . \tag{6.6}$$

This arrangement leads to an alternative form of the constraints:[40]

$$\Pi_K := P_K + h_K = 0, \quad h_K := -n_K H^G + X^a_K H^G{}_a , \tag{6.7}$$

or

$$\Pi := P + W^{-1} H^G - T^a H^G{}_a = 0 , \tag{6.8}$$

and

$$\Pi_k := P_k - W^{-1} T_{,k} H^G + (X^a_k + T^a T_{,k}) H^G{}_a = 0 . \tag{6.9}$$

The old constraints (5.3), $H = 0 = H_a$, and the new constraints (6.7) are completely equivalent. However, the momenta P_K in Eq. (6.7) are clearly separated from the rest of the canonical variables. This has an important consequence for the constraint algebra. The old constraints (5.3) are the super-Hamiltonian and supermomentum of a system nonderivatively coupled to gravity; as such, they close according to the same Dirac relations (2.16)–(2.18) as their purely gravitational counterparts. On the other hand, the Poisson brackets of the new constraints (6.7) *strongly* vanish:

$$\{ \Pi_J(x), \Pi_K(x') \} = 0 . \tag{6.10}$$

This remarkable fact follows by a neat argument which circumvents the tedious algebra: Because the new constraints are equivalent to the old constraints and the Poisson brackets of the old constraints weakly vanish, the Poisson brackets of the new constraints must also weakly vanish. However, the form (6.7) of the new constraints ensures that these brackets do not depend on the momenta P_K. The constraints (6.7) thus cannot help in any way to turn those brackets into zeros; this means that the brackets must vanish strongly.

Equation (6.10) underlies the method by which the Lie algebra LDiff$\mathcal{M}$ of the spacetime diffeomorphism group Diff$\mathcal{M}$ is homomorphically mapped into the Poisson algebra of the dynamical variables on the phase space of the Gaussian coordinate fluid coupled to gravity. It was originally proved by direct but lengthy calculations relying heavily on special properties of the Gaussian coordinate conditions, in particular, on their ultralocality.[15] Here Eq. (6.10) follows by a general argument easily applicable to other coordinate conditions. We have thus devised an algorithm leading to the canonical representation of LDiff $\mathcal{M}$ for a large class of coordinate conditions. We shall show how it works for an important case of the DeDonder harmonic coordinate conditions in a separate paper.[20]

Once the momenta P_K conjugate to the privileged coordinates X^K are separated [Eq. (6.7)], and the new constraints are shown to satisfy Eq. (6.10), the canonical representation of LDiff$\mathcal{M}$ becomes straightforward. The elements of LDiff$\mathcal{M}$ are vector fields $\mathbf{u},\mathbf{v},\ldots$, on $\mathcal{M}$, and the Lie bracket is their commutator $-[\mathbf{u},\mathbf{v}]$. In the privileged Gaussian coordinates, the vector fields are characterized by their components $u^K(X)$ and $v^K(X)$. When restricted to an embedding $X^K=X^K(x)$, the vector fields become dynamical variables over the configuration space of the coordinate fluid. When we use them to smear the new constraints on Σ,

$$\mathbf{u}\rightarrow\Pi(\mathbf{u}):=\int_\Sigma d^3x\, u^K(X(x))\Pi_K(x)\ , \tag{6.11}$$

we map each vector field $\mathbf{u}$ into a dynamical variable over the phase space of the coordinate fluid coupled to gravity. Equations (6.7) and (6.10) guarantee that

$$\{\Pi(\mathbf{u}),\Pi(\mathbf{v})\}=\Pi(-[\mathbf{u},\mathbf{v}])\ , \tag{6.12}$$

i.e., that the mapping (6.11) is a homomorphism from LDiff$\mathcal{M}$ into the Poisson bracket over the phase space. The geometric meaning of the dynamical variables (6.11) is extensively discussed in our previous papers.[15]

VII. REDUCED CANONICAL FORMALISMS

The Gaussian reference fluid introduces a privileged frame of reference and a privileged foliation into spacetime, but the canonical formalism we have developed is capable of describing the change of the gravitational field in an arbitrary frame of reference and along an arbitrary foliation: We can follow the change of the field along any one-parameter family of embeddings $X^K=X^K(t,x)$. The timelike lines of a constant x^a do not need to coincide with those of the Gaussian frame and the spacelike hypersurface of a constant t with those of the Gaussian foliation.

The general canonical formalism simplifies when one requires either the frame to be Gaussian or the foliation to be Gaussian, or both of these at once. Appropriate canonical variables are eliminated from the canonical formalism, and one works with a reduced phase space. The questions one can naturally ask are thereby limited, but the reduced canonical description provides a suitable framework for their discussion. We shall introduce three versions of the reduced canonical formalism and use them later in the quantum theory.

In the first version, we label the points of spacelike hypersurfaces by the Gaussian coordinates,

$$X^k=x^k\ , \tag{7.1}$$

but leave the hypersurfaces themselves arbitrary. The condition (7.1) stipulates that the x-frame be Gaussian. We specify a hypersurface by giving T as a function of X^k:

$$T=T(X^k)\ . \tag{7.2}$$

It follows that

$$g_{kl}=h_{kl}-T_{,k}T_{,l}\quad\text{and}\quad W^k=g^{kl}T_{,l}\ . \tag{7.3}$$

The expressions for H^F, M, and M_k do not simplify. However,

$$H^F{}_k=P_k+T_{,k}P\ . \tag{7.4}$$

We can solve the supermomentum constraint with respect to P_k and eliminate P_k by substituting this solution back into the action. In this process, it is convenient to rescale simultaneously the lapse function and the super-Hamiltonian:

$$\bar{N}=WN,\quad \bar{H}=W^{-1}H\ . \tag{7.5}$$

The reduced action assumes the form

$$S[T(X),g_{kl}(X);P(X),p^{kl}(X);\bar{N}(X)] =\int_{\mathbb{R}}dt\int_\Sigma d^3X(P\dot{T}+p^{kl}\dot{g}_{kl}-\bar{N}\bar{H})\ , \tag{7.6}$$

with

$$\bar{H}=P-g^{kl}T_{,k}H^G{}_{,l}+W^{-1}H^G\ . \tag{7.7}$$

The remaining variables T, g_{kl}, P, p^{kl}, and $\bar{N}$ are now all considered as functions of the Gaussian coordinates X^k.

The meaning of the rescaled quantities $\bar{N}$ and $\bar{H}$ is revealed by evaluating the Poisson bracket

$$\dot{T}(X)=\{T(X),\bar{H}_{\bar{N}}\}=\bar{N}(X)\ . \tag{7.8}$$

The rescaled lapse function tells us what is the Gaussian time separation between the neighboring spacelike hypersurfaces $T=T(X,t)$ and $T=T(X,t+dt)$ along the world line $X^k=$const of the Gaussian frame. [In comparison, the lapse function $N(x)$ determines the *proper time* separation between the neighboring hypersurfaces measured in the *normal* direction.] The rescaled super-Hamiltonian $\bar{H}$ thus generates the change along the world lines of the Gaussian frame per unit Gaussian time. The argument evoked in connection with Eq. (6.10) guarantees that the rescaled super-Hamiltonians have strongly vanishing Poisson brackets:

$$\{\bar{H}(X),\bar{H}(X')\}=0\ . \tag{7.9}$$

In the second version of the reduced canonical formalism, we confine ourselves to the leaves of the Gaussian foliation,

$$T=t\ , \tag{7.10}$$

but leave the x-frame arbitrary,

$$X^k=X^k(T,x)\ . \tag{7.11}$$

As a result, many dynamical variables simplify:

$$W=1\quad\text{and}\quad W^k=0\ , \tag{7.12}$$

$$M=g^{-1/2}P,\quad M_k=-g^{-1/2}P_k\ , \tag{7.13}$$

$$H^F=P,\quad H^F{}_a=X^k{}_{,a}P_k\ , \tag{7.14}$$

and

$$g_{ab}=h_{kl}X^k{}_{,a}X^k{}_{,b}\ . \tag{7.15}$$

We can solve the super-Hamiltonian constraint with respect to P and eliminate P by substituting this solution back into the action,

$$S[X^k(x),g_{ab}(x);P_k(x),p^{ab}(x);N^a(x)]$$
$$=\int_{\mathbb{R}} dT \int_\Sigma d^3x\,(P_k(x)\dot{X}^k(x)+p^{ab}(x)\dot{g}_{ab}(x) -H^G(x)-N^a(x)H_a(x)) \ . \qquad (7.16)$$

The remaining variables are given on the leaves (7.10) of the Gaussian foliation as functions of arbitrary spatial coordinates x^a. The variables $X^k(x)$ tell us how the labels x^a are connected with the Gaussian frame. The true Hamiltonian

$$h=\int_\Sigma d^3x\, H^G(x) \qquad (7.17)$$

evolves the variables along the world lines of the Gaussian frame from one Gaussian hypersurface to another. The supermomentum

$$H_a=X^k{}_{,a}P_k+H^G{}_a \ , \qquad (7.18)$$

smeared by the shift vector N^a, generates the change which leaves the Gaussian hypersurface fixed, but displaces its points into new positions. The Poisson brackets of the supermomenta (7.18) represent the algebra of spatial diffeomorphisms LDiffΣ [Eq. (2.18)], and the supermomentum constraints commute with the Hamiltonian (7.17):

$$\{H_a(x),h\}=0 \ , \qquad (7.19)$$

The maximal reduction of the canonical formalism is accomplished by staying on the Gaussian foliation and parametrizing its points by Gaussian coordinates. The reduced action is obtained either by putting $T=t$ in the action (7.6) and solving the constraint $\overline{H}=0$ with respect to P, or by putting $X^k=x^k$ in the action (7.16) and solving the constraint $H_a=0$ with respect to P_k. The reduced action

$$S[g_{kl}(X),p^{kl}(X)]=\int_{\mathbb{R}} dT\left[\int_\Sigma d^3X\, p^{kl}(X)\dot{g}_{kl}(X)-h\right] \qquad (7.20)$$

depends only on the Gaussian metric

$$g_{kl}(X)=h_{kl}(X) \ , \qquad (7.21)$$

and its conjugate momentum. These variables are propagated by the true Hamiltonian (7.17), with $X^k=x^k$. There are no constraints left in the theory.

VIII. ENERGY CONDITIONS IN HAMILTONIAN FORMALISM

We have seen that the strong and dominant energy conditions for the Gaussian reference fluid are equivalent to the weak energy condition. The weak energy condition requires that the energy density of the fluid measured by an arbitrary local observer be non-negative. This requirement has a simple transcription in terms of the canonical variables. The energy density H^F measured by an observer who is moving with the four-velocity n^α normal to the hypersurface is given by Eq. (5.20). The weak energy condition thus amounts to the statement that

$$H^F(x;X^K,P_K,g_{ab}]\geq 0 \qquad (8.1)$$

on an arbitrary hypersurface.[41] The super-Hamiltonian constraint allows us to reexpress this condition entirely in terms of the geometric variables:

$$H^G(x;g_{ab},p^{ab}]\leq 0 \ . \qquad (8.2)$$

The inequality (8.2) selects the physically permissible region of the phase space: The geometric data $g_{ab}(x),p^{ab}(x)$ are subject to the restriction (8.2), the (spacelike) hypersurface $X^K(x)$ is arbitrary, and the embedding momenta $P_K(x)$ are determined from the constraints (6.7).

By introducing the step function

$$\Theta(z)=\begin{cases}0 & \text{for } z\leq 0\\ 1 & \text{for } z>0 \ ,\end{cases} \qquad (8.3)$$

we can replace the inequality (8.2) by an equality

$$\Theta(H^G(x))=0 \quad \forall x\in\Sigma \ , \qquad (8.4)$$

i.e., by a system of constraints. This is handy for discussing the role of energy conditions in quantum theory. The step function $\Theta(H^G)$ is the characteristic function of the forbidden region of the phase space: Its value is 1 if the phase-space point lies in the forbidden region and 0 if it lies outside of it, i.e., if it lies in the permissible region.

To guarantee that the reference fluid be realistic, let us adjoin the energy conditions (8.4) to the super-Hamiltonian and supermomentum constraints. Unfortunately, the extended system of constraints is no longer first class. The supermomentum constraints remain first class because the energy conditions are invariant under spatial diffeomorphisms. Formally, this follows from the closure relation (2.17) and the well-known property $z\delta(z)=0$ of the δ function:

$$\{\Theta(H^G(x)),H_a(x')\}=\delta(H^G(x))\{H^G(x),H^G{}_a(x')\}$$
$$=H^G(x)\delta(H^G(x))\delta_{,a}(x,x')$$
$$=0 \ . \qquad (8.5)$$

However, the super-Hamiltonian constraints are not first class because their Poisson brackets with the energy conditions turn out to be nontrivial. By virtue of the closure relations (2.16) and Eq. (5.33),

$$\{\Theta(H^G(x)),H(x')\}$$
$$=\delta(H^G(x))(H^{Ga}(x)\delta_{,a}(x,x')-(x\leftrightarrow x') +g^{1/2}G_{abcd}T^{ab}p^{cd}\delta(x,x')) \ . \qquad (8.6)$$

Even if the constraints (6.7) and the energy conditions (8.4) are satisfied, the Poisson brackets (8.6) do not vanish. This reflects the fact noted in Sec. IV that the energy conditions for a heat-conducting Gaussian fluid are not preserved in time. The Poisson brackets of the energy conditions themselves are also nontrivial:

$$\{\Theta(H^G(x)),\Theta(H^G(x'))\} =\delta(H^G(x))\delta(H^G(x'))(H^{Ga}(x)\delta_{,a}(x,x')-(x\leftrightarrow x')) \ . \tag{8.7}$$

For a heat-conducting Gaussian reference fluid, neither the constraints (6.7) nor the energy conditions (8.4) can help us to turn the expression (8.7) into zero. Because the Poisson brackets (8.6) and (8.7) do not weakly vanish, the extended system of constraints is not first class.

The standard procedure for completing a system of constraints into a first-class system is to keep adding the constraints obtained from the Poisson brackets to the original system until the enlarged system closes (or becomes inconsistent). By following this procedure, we can prove that the system (6.7) and (8.4) can be closed if we add to it the additional constraints

$$P_k(x)=0 \ . \tag{8.8}$$

These constraints switch off the heat conduction, $M_k=0$, and turn the Gaussian fluid effectively into an incoherent dust.

We have adopted an alternative description of the dust by choosing not to impose the Gaussian frame conditions, but only the Gaussian time condition. This removes the canonical variables X^k, P_k from the phase space. In that case, we can prove directly that the constraints (6.7) and the energy conditions (8.4) form a first-class system. By virtue of Eqs. (5.34)–(5.36) and the constraints, Eq. (8.6) assumes the form

$$\{\Theta(H^G(x)),H(x')\} \approx\delta(H^G(x))(H^G(x)W(x)T^a(x)\delta_{,a}(x,x')-(x\leftrightarrow x') -H^GW^2G_{abcd}T^aT^bp^{cd}\delta(x,x')) \ . \tag{8.9}$$

The expression (8.9) vanishes because $H^G(x)\delta(H^G(x))=0$, and the differentiated energy conditions (8.4) yield

$$H^G_{\ ,a}(x)\delta(H^G(x))=0 \ . \tag{8.10}$$

This reaffirms that the energy conditions for the Gaussian incoherent dust are preserved in time, a result which we have previously reached by the spacetime approach. Similarly, Eq. (8.7) reduces to

$$\{\Theta(H^G(x)),\Theta(H^G(x'))\} \approx\delta(H^G(x))\delta(H^G(x')) \times(H^G(x)W(x)T^a(x)\delta_{,a}(x,x')-(x\leftrightarrow x'))=0 \ . \tag{8.11}$$

This foreshadows our conclusion that energy conditions in quantum theory can be consistently imposed only for the Gaussian incoherent dust.

The final question we would like to clarify is the relationship of the energy conditions (8.1) and (8.2) to those we have imposed in Sec. IV on the mass and heat multipliers M,M_k:

$$F(x):=M(x)-2|\mathbf{M}(x)|\geq 0 \ . \tag{8.12}$$

In the canonical formalism, the multipliers become dynamical variables (5.18) and (5.19) on the extended phase space $(g_{ab}(x),p^{ab}(x),X^K(x),p_K(x))$ of the system. The energy conditions (8.12) thus restrict the range of the canonical data. One can impose them on an arbitrary embedding $X^K(x)$, but this is redundant. Unlike the energy density $H^F(x)$, the multipliers M and M_k are spacetime scalars. In canonical formalism, a spacetime scalar is a spatial scalar which is unaffected by any tilt or bending of a hypersurface about a fixed spacetime point.[42] In other words, a spatial scalar $\phi(x)$ is a spacetime scalar if

$$\{\phi(x'),H_N\}=0 \tag{8.13}$$

for all $N(x)$ which vanish at x'. This condition means that the Poisson bracket of a spacetime scalar with the super-Hamiltonian must be proportional to an undifferentiated δ function:

$$\{\phi(x'),H(x)\}=\psi(x')\delta(x',x) \ . \tag{8.14}$$

It is straightforward to check that the multipliers (5.18) and (5.19) and the super-Hamiltonian (2.6) and (5.20) satisfy Eq. (8.14). Similarly, by the same criterion, the Gaussian metric (5.24) is a collection of spacetime scalars. It follows that the dynamical variable $F(x;\mathbf{g},\mathbf{p},X^K,P_K]$ whose sign determines whether the reference fluid satisfies the energy conditions is also a spacetime scalar.

These considerations show that if $F(x)\geq 0$ on a given foliation, say, on the Gaussian foliation $T(x)=t$, $X^k(x)=x^k$, then $F(x)\geq 0$ on an arbitrary embedding. This highlights the difference between the two forms, (8.1) and (8.2), and (8.12), of the energy conditions. The conditions (8.1) and (8.2) must be imposed on an arbitrary embedding; conditions (8.12) need to be imposed only on a given foliation. The energy conditions (8.12) are thus suitable for application in the reduced canonical formalism (7.16) or (7.20).

Unfortunately, the new form of the energy conditions still does not help us with the evolution and compatibility problems. As with the old form of the energy conditions,

$$\{\Theta(-F(x)),H(x')\}\approx 0 \tag{8.15}$$

and

$$\{\Theta(-F(x)),\Theta(-F(x'))\}\approx 0 \tag{8.16}$$

only for an incoherent Gaussian dust.

IX. REFERENCE FLUID IN QUANTUM GEOMETRODYNAMICS

To quantize geometry which interacts with the reference fluid, we turn the canonical variables into operators

$$\hat{g}_{ab}(x)=g_{ab}(x)\times, \quad \hat{X}^K(x)=X^K(x)\times \ , \qquad \hat{p}^{ab}(x)=-i\frac{\delta}{\delta g_{ab}(x)}, \quad \hat{P}_K(x)=-i\frac{\delta}{\delta X^K(x)} \ , \tag{9.1}$$

substitute them into the constraints, and require that the physical states $\Psi[X^K,g_{ab}]$ be annihilated by the constraint operators. For the metric field *in vacuo*, such a procedure leads to the Wheeler-DeWitt equation,[43] which is analogous to a Klein-Gordon equation. However, here the momenta P_K of the coordinate fluid enter into the constraints linearly, and the resulting equation thus resembles a Schrödinger equation. This is best exhibited when the constraints are taken in their new form (6.7):

$$\hat{\Pi}_K(x)\Psi[X^J,g_{ab}]=0 . \quad (9.2)$$

Because the momenta P_K are separated from the rest of the canonical variables, Eq. (9.2) assumes the form

$$i\frac{\delta\Psi}{\delta X^K(x)}=h_K(x;X^J,\hat{g}_{ab},\hat{p}^{ab}]\Psi . \quad (9.3)$$

This is a first-order functional differential equation in the embedding variables $X^K(x)$. Therefore, if $\Psi[X^K,g_{ab}]$ is known on an initial embedding $X_0^K(x)$, Eq. (9.3) determines it on an arbitrary embedding $X^K(x)$.

The evolution of the state by Eq. (9.3) is consistent only if the classical Poisson bracket relation (6.10) holds also for the commutator:

$$\frac{1}{i}[\hat{\Pi}_J(x),\hat{\Pi}_K(x')]=0 . \quad (9.4)$$

Equation (9.4) is an integrability condition for the functional differential equation (9.3); it ensures that the evolution of state from $X_0^K(x)$ to $X^K(x)$ does not depend on the foliation which connects X_0^K with $X^K(x)$. The validity of Eq. (9.4) depends on the factor ordering of the operators $\hat{g}_{ab}(x)$ and $\hat{p}^{ab}(x)$ in $h_K(x)$. It is not at all clear that there exists a factor ordering which makes the operators $\hat{\Pi}_K(x)$ well defined and still commuting. If there is no such factor ordering, the functional differential equation (9.3) becomes inconsistent.

Even for much simpler systems than canonical gravity, one expects that Schwinger terms[44] in the commutators bring an anomaly into Eq. (9.4). This happens, e.g., under the standard factor ordering for a parametrized linear field theory on a flat background[45] or for a bosonic string.[46] We have shown that in these cases the factor ordering can be modified so that the anomaly disappears from the commutator (9.4).[22,45] Our further discussion makes sense only under the assumption that a similar thing can be achieved in quantum gravity.

The tangential projection of the quantum constraint (9.2) is the supermomentum constraint

$$\hat{H}_a(x)\Psi[X^K,g_{bc}]=0 . \quad (9.5)$$

Its geometric content is the same as in vacuum geometrodynamics: It ensures that the state Ψ does not depend on the system of coordinates x^c on Σ. Formally,

$$\Psi[X^K(x'),g_{a'b'}(x')]=\Psi[X^K(x),g_{ab}(x)] \quad (9.6)$$

for

$$X^K(x)=X^K(x'(x)) ,$$
$$g_{ab}(x)=g_{c'd'}(x'(x))X_a^{c'}(x)X_b^{d'}(x) . \quad (9.7)$$

This means that we can parametrize Σ by any coordinates we want; in particular, we can parametrize it by the spatial Gaussian coordinates X^k. Such a decision corresponds to the first version of the reduced formalism of Sec. VII. The state becomes then a functional of $T(X^k)$ and of the induced metric $g_{kl}(X^m)$ expressed in the spatial Gaussian coordinates:

$$\Psi[T(X),g_{kl}(X)]:=\Psi[T(x),\ X^k(x)=\delta_a^k x^a,\ g_{ab}(x)] . \quad (9.8)$$

If we change the Gaussian coordinates by Eq. (2.3), the functional (9.8) changes as well:

$$\Psi'[T'(X'),g_{k'l'}(X')]$$
$$:=\Psi[T(x),\ X^k(x)=X^k(X^{l'}=\delta_a^{l'}x^a),\ g_{ab}(x)] . \quad (9.9)$$

By virtue of Eqs. (9.6) and (9.7), it still holds that

$$\Psi'[T'(X'),g_{k'l'}(X')]=\Psi[T(X),g_{mn}(X)] \quad (9.10)$$

for

$$T(X)=T'(X'(X)) ,$$
$$g_{mn}(X)=g_{k'l'}(X'(X))X_m^{k'}(X)X_n^{l'}(X) . \quad (9.11)$$

However, the state $\Psi[T(X),g_{mn}(X)]$ is in general not form invariant: $\Psi'[T'(X'),g_{k'l'}(X')]$ is a different functional of $T'(X'),g_{k'l'}(X')$ than $\Psi[T(X),g_{mn}(X)]$ is of $T(X)$ and $g_{mn}(X)$. The reduced state functional (9.8) thus does not need to satisfy the constraint

$$(T_{,k}(X)\hat{P}(X)-2\hat{p}^l_{k|l}(X))\Psi[T(X),g_{mn}(X)]=0 . \quad (9.12)$$

As a result, we can prescribe Ψ on an initial embedding $T(X)$ as an arbitrary functional of six independent function variables $g_{mn}(X)$. The form-invariant states (9.12) are an exception rather than the rule; Eq. (9.12) holds only if the parametrized state functional $\Psi[T(x),X^k(x),g_{ab}(x)]$ is unchanged by the transformation (2.3) of the Gaussian coordinates, i.e., if in correspondence with the classical equation (5.31),

$$\hat{P}_k(x)\Psi=0 . \quad (9.13)$$

Such states do not depend on the fluid variables $X^k(x)$. In any case, the spatial deparametrization

$$T(x),X^k(x)\mapsto T(X)$$

naturally leads to the elimination of the supermomentum constraint for the quantum states.

The reduced state functional (9.8) is evolved in time by the Schrödinger equation

$$i\frac{\delta\Psi}{\delta T(X)}=(-g^{kl}(x)T_{,k}(X)H^G{}_l(X)$$
$$+W^{-1}(X)H^G(X))\hat{\Psi} , \quad (9.14)$$

corresponding to the classical super-Hamiltonian (7.7). As in the fully parametrized theory, the functional differential equation (9.14) is consistent only if Eq. (7.9) holds for the commutator of the reduced constraint operators.

In the functional Schrödinger equation (9.14), the frame is fixed, but the hypersurfaces on which the state is registered are arbitrary. We shall now reduce the formalism in the second way discussed in Sec. VII, by leaving the frame arbitrary, but restricting the states $\Psi[T(x),X^k(x),g_{ab}(x)]$ to the hypersurfaces $T(x,t)=t$ of the Gaussian foliation:

$$\Psi(t;X^k(x),g_{ab}(x)]:=\Psi[T(x,t)=t,X^k(x),g_{ab}(x)] \ . \tag{9.15}$$

The reduced states (9.15) are ordinary functions of the Gaussian time $t=T$, but they are still functionals of the remaining variables $X^k(x),g_{ab}(x)$. By the chain rule,

$$\begin{aligned}\partial_t\Psi(t;X^k(x),g_{ab}(x)]&=\int_\Sigma d^3x\frac{\delta\Psi[T(x;t),X^k(x),g_{ab}(x)]}{\delta T(x;t)}\partial_t T(x;t)\bigg|_{T(x;t)=t}\\&=\int_\Sigma d^3x\frac{\delta\Psi(T(x),X^k(x),g_{ab}(x)]}{\delta T(x)}\bigg|_{T(x)=t}\end{aligned} \tag{9.16}$$

The reduced states thus satisfy the Schrödinger equation

$$i\,\partial_t\Psi(t;X^k(x),g_{ab}(x)]=\hat{h}\Psi(t;X^k(x),g_{ab}(x)] \ , \tag{9.17}$$

in which $\hat{h}$ is the operator version

$$\hat{h}=\int_\Sigma d^3x\, H^G(x;\hat{g}_{ab}(x),\hat{p}^{\,ab}(x)] \tag{9.18}$$

of the classical Hamiltonian (7.17). They also satisfy the reduced form (7.18) of the supermomentum constraint (9.5):

$$-iX^k{}_{,a}(x)\frac{\delta\Psi}{\delta X^k(x)}+\hat{H}^{\,G}{}_a(x)\Psi=0 \ . \tag{9.19}$$

The ultimate reduction of the states is achieved by both fixing the frame and restricting the states (9.8) to the leaves of the Gaussian foliation:

$$\Psi(t;g_{kl}(X)]:=\Psi[T(X,t)=t,\, g_{kl}(X)] \ . \tag{9.20}$$

The fixation of the frame eliminates the supermomentum constraint, and the restriction to the Gaussian foliation converts the functional Schrödinger equation (9.14) into an ordinary Schrödinger equation:

$$i\,\partial_t\Psi(t;g_{kl}(X)]=\hat{h}\Psi(t;g_{kl}(X)] \ , \tag{9.21}$$

whose Hamiltonian

$$\hat{h}=\int_\Sigma d^3X\, H^G(X;\hat{g}_{ab}(X),\hat{p}^{\,kl}(X)] \tag{9.22}$$

is constructed from the Gaussian metric (7.21) and its conjugate momentum.

The comparison of the full formalism with its reduced versions highlights the role of the Gaussian fluid in quantum geometrodynamics. In vacuum geometrodynamics, there is no natural way of identifying spacetime events or, what is the same thing, of identifying world lines of a frame or leaves of a foliation. By virtue of the supermomentum constraint, the states Ψ do not depend on the metric $g_{ab}(x)$, but only on the geometry $\mathbf{g}(x)$. By virtue of the super-Hamiltonian constraint, not even the geometry $\mathbf{g}(x)$, but at best only its conformally invariant part is measurable. The introduction of the Gaussian fluid coupled to the geometry fundamentally changes this state of affairs. Spacetime events X can be identified by observing the fluid variables X^K. The embeddings $X^K=X^K(x)$ are thus physically fixed with respect to the material background provided by the fluid. The functionals $\Psi[X^K,g_{ab}]$ describe the state of the metric field on such embeddings. By virtue of the supermomentum constraints, they are invariant under arbitrary transformations of the spatial labels x^a. However, that does not imply that they depend only on the geometry $\mathbf{g}(x)$, because their construction involves also the fluid variables $X^K(x)$. One can eliminate the frame variables X^k by using them as the labels x^a. The reduced functionals $\Psi[T(X);g_{kl}(X)]$ describe the state of the metric field $g_{kl}(X)$ given in the Gaussian frame on an arbitrary hypersurface $T=T(X)$. The fluid variables X^k identify the points of the Gaussian frame, and one can thus meaningfully talk about measuring the metric in this frame rather than about measuring the mere geometry $\mathbf{g}(x)$. Moreover, the fluid potential $T(X)$ identifies the hypersurface on which the metric $g_{kl}(X)$ is being observed. The super-Hamiltonian constraint does not restrict the measurability of the full metric, but rather determines the evolution of the state from one hypersurface to another. The reduced states are no longer invariant under transformations of X^k. They can be made invariant under transformations of arbitrary labels x^a only at the price of adjoining the fluid variables $X^k(x)$ back to the potential $T(X)$ and the metric $g_{kl}(X)$, i.e., by returning from the state $\Psi[T(X),g_{kl}(X)]$ to the original state $\Psi[X^K(x),g_{ab}(x)]$ by the parametrization process. The original states thus characterize the state of the metric field $g_{ab}(x)$ given on the embedding $X^K(x)$ in the system of coordinates x^a related to the Gaussian frame coordinates X^k by the transformation $X^k=X^k(x^a)$.

Instead of asking about the state of the metric field on an arbitrary hypersurface Σ, one can less ambitiously ask only about the state of the Gaussian metric on the leaves of the Gaussian foliation. The answer to this restricted question is provided by the state functional $\Psi(t;g_{kl}(X)]$. By allowing only a one-parameter family of hypersurfaces, one needs only an ordinary Schrödinger equation to describe the change of such a restricted state. By adjoining the fluid variables $X^k(x)$ back to the metric variables $g_{kl}(X)$, one again arrives at a parametrized functional $\Psi(t;X^k(x),g_{ab}(x)]$. This functional describes the state of the metric field $g_{ab}(x)$ on the leaf $T=t$ of the

Gaussian foliation in the system of coordinates x^a related to the Gaussian frame coordinates X^k by the transformation $X^k = X^k(x^a)$.

This conceptual scheme entirely changes when we replace the heat-conducting Gaussian fluid by an incoherent dust, i.e., when we impose the Gaussian time condition, but not the Gaussian frame conditions. The velocity potential $T(x)$ of the Gaussian dust is still at our disposal to identify the hypersurfaces, but we lack the markers $X^k(x)$ to fix the world lines of the Gaussian frame of reference. The classical constraints have the form (5.3) and (5.35). After quantization, they become the restrictions

$$i\frac{\delta\Psi}{\delta T}(x) = (W(x)H^G(x))\hat{\Psi} , \tag{9.23}$$

and

$$(T_{,a}(x)\hat{P}(x) + \hat{H}^G{}_a(x))\Psi = 0 , \tag{9.24}$$

on the state functional $\Psi[T(x), g_{ab}(x)]$.

Again, the supermomentum constraint ensures that $\Psi[T(x), g_{ab}(x)]$ does not depend on the particular labeling x^a of Σ. However, there are no fluid variables X^k which could be used as the special labels. Therefore, on a given hypersurface $T = T(x)$, we cannot measure the metric components $g_{ab}(x)$, but only the geometry $\mathbf{g}(x)$.

The state is evolved from one hypersurface to another by the Schrödinger equation (9.23). This equation is different from the Schrödinger equation (9.14) obtained by the spatial deparametrization based on the Gaussian frame. The state functional $\Psi[T(X), g_{kl}(X)]$ in Eq. (9.14) does not satisfy the supermomentum constraint (9.12) and hence depends on the metric $g_{kl}(X)$ viewed as a collection of six scalar functions rather than on the three-geometry $\mathbf{g}(x)$. The solution space of Eq. (9.14) is thus larger than that of Eq. (9.23) and (9.24). It contains many states which do not satisfy the constraint (9.12). However, for those states which satisfy Eq. (9.12), i.e., for the parametrized states with the symmetry (9.13), Eq. (9.14) reduces back, modulo familiar factor-ordering problems, to Eq. (9.23).

Like Eq. (9.14), Eq. (9.23) can also be restricted to the Gaussian foliation $T(x,t) = t$. The state function

$$\Psi(t;\mathbf{g}) = \Psi[T(x,t) = t, \mathbf{g}(x)] \tag{9.25}$$

satisfies again the Schrödinger equation

$$i\,\partial_t \Psi(t;\mathbf{g}) = \hat{h}\Psi(t;\mathbf{g}) . \tag{9.26}$$

However, while the state function (9.15) depends on all components of the metric $g_{kl}(X)$, the state function $\Psi(t;g]$ depends, for each t, only on the three-geometry $\mathbf{g}(x)$.

We shall now proceed with interpreting the Schrödinger equations based on the introduction of the Gaussian fluid variables. We shall see that the conceptual framework relying on the heat-conducting Gaussian fluid encounters serious difficulties, while the modified schema relying on the Gaussian incoherent dust is very nearly viable.

X. ENERGY CONDITIONS AND INTERPRETATION OF QUANTUM GEOMETRODYNAMICS

When one tries to give a probabilistic interpretation to the solutions $\Psi[\mathbf{g}]$ of the Wheeler-DeWitt equation, one encounters familiar difficulties. From $\Psi[\mathbf{g}]$, one can (at least formally) construct a conserved current,[6] but like the Klein-Gordon current, this does not yield a positive-definite inner product. For the Klein-Gordon equation in a stationary spacetime, one can restrict the solutions to those which have a positive energy as measured by the stationary observer. The Klein-Gordon inner product on the space of positive-energy solutions becomes positive definite.[47] In a dynamical spacetime, one cannot consistently speak about positive-energy solutions, and the probabilistic interpretation of the Klein-Gordon equation fails. The superspace which plays the role of the background for the Wheeler-DeWitt equation is not stationary[48] and the interpretation of the state functional $\Psi[\mathbf{g}]$ thus remains problematic.[49]

One can try to resolve such problems by turning the Wheeler-DeWitt equation into a Schrödinger equation. An introduction of the reference fluid is one way of accomplishing this goal.[50] We have seen that the Gaussian time $T(x)$ provides a clock with respect to which the state evolves according to a functional Schrödinger equation (9.3), (9.14), or (9.23). As in a parametrized field theory, such an equation possesses (at least formally) a conserved current which leads to the standard positive-definite inner product.[51] For the three forms of the functional Schrödinger equation we have mentioned, the norm of the state is given by the expressions

$$\langle\Psi|\Psi\rangle = \int Dg_{ab}\Psi^*[T, X^k, g_{ab}]\Psi[T, X^k, g_{ab}] , \tag{10.1}$$

$$\langle\Psi|\Psi\rangle = \int Dg_{kl}\Psi^*[T, g_{kl}]\Psi[T, g_{kl}] , \tag{10.2}$$

and

$$\langle\Psi|\Psi\rangle = \int Dg_{ab}\Psi^*[T, g_{ab}]\Psi[T, g_{ab}] . \tag{10.3}$$

The functional integral in Eq. (10.2) is taken over all Riemannian metrics $g_{kl}(X)$, while that in Eqs. (10.1) and (10.3) over all Riemannian geometries $\mathbf{g}(x)$, i.e., over the equivalence classes of Riemannian metrics modulo the spatial diffeomorphisms DiffΣ. By virtue of the Schrödinger equation, the normal (10.1) does not depend on the embedding $T(x), X^k(x)$ on which it is evaluated. Similarly, the norm (10.2) does not depend on $T(X)$ and the norm (10.3) on $T(x)$.

The norms (10.1)–(10.3) enable us to turn the space of physical states into a Hilbert space. The operators representing physical observables must be self-adjoint with respect to these norms. In particular, the metric $g_{kl}(x)$ and the conjugate momentum $p^{kl}(x)$ must be represented by the operators $\hat{g}_{kl}(x)$ and $\hat{p}^{kl}(x)$, which are self-adjoint with respect to the norm (10.2). More generally, a classical observable

$$O[X^K(x), g_{ab}(x), p^{ab}(x)] \tag{10.4}$$

is any functional of the canonical variables $g_{ab}(x), p^{ab}(x)$ and of the embedding $X^K(x)$, which is invariant under DiffΣ:

$$\{O,H_a(x)\}=0 \tag{10.5}$$

(Ref. 52). Such an observable must be represented by an operator

$$\hat{O}=O[X^K(x),\hat{g}_{ab}(x),\hat{p}^{ab}(x)] \tag{10.6}$$

which commutes with the supermomentum,

$$\frac{1}{i}[\hat{O},\hat{H}_a(x)]=0 \ , \tag{10.7}$$

and which is self-adjoint with respect to the norm (10.1). It follows that the reduced observable

$$O[T(X),\hat{g}_{kl}(X),\hat{p}^{kl}(X)] := O[T(x),\ X^k(x)=\delta^k_a x^a,\ \hat{g}_{ab}(x),\hat{p}^{ab}(x)] \tag{10.8}$$

is self-adjoint with respect to the norm (10.2). In particular, this applies to the Hamiltonian (7.7) of the reduced canonical formalism. Indeed, this Hamiltonian must be self-adjoint for the norm (10.2) to be independent of the hypersurface $T(X)$ and the norm (10.1) to be independent of the embedding. This implies that the gravitational super-Hamiltonian $H^G(X)$ and supermomentum $H^G{}_a(X)$, which enter as the building blocks into the Hamiltonian (7.7), may be subject to quite a different factor ordering than the corresponding expressions in vacuum geometrodynamics.

A similar caution is needed when defining observables based on the Gaussian incoherent dust. A classical observable $O[T(x),g_{ab}(x),p^{ab}(x)]$ is an arbitrary functional of the canonical variables $g_{ab}(x),p^{ab}(x)$ and of the hypersurface $T(x)$, which is invariant under DiffΣ [Eq. (10.5)]. Again, such an observable must be represented by an operator

$$\hat{O}=O[T(x),\hat{g}_{ab}(x),\hat{p}^{ab}(x)] \tag{10.9}$$

which commutes with the supermomentum [Eq. (10.7)], and which is self-adjoint with respect to the norm (10.3).

It is tempting to believe that the integrands of the norm integrals (10.1)–(10.3) represent the probability densities for the metric (or the geometry) to have a definite value on a hypersurface specified by the Gaussian clock. Let us spell such a proposal in the necessary detail.

The expression

$$\Psi^*[T(X),g_{kl}(X)]\Psi[T(X),g_{kl}(X)] \tag{10.10}$$

constructed from a solution $\Psi[T(X),g_{kl}(X)]$ of Eq. (9.14), is to be interpreted as the probability density that, on the hypersurface of the Gaussian time $T(X)$, the metric $g_{kl}(X)$ measured in the Gaussian frame X^k be found in the cell Dg_{kl} centered about $g_{kl}(X)$.

The expression

$$\Psi^*[T(x),X^k(x),g_{ab}(x)]\Psi[T(x),X^k(x),g_{ab}(x)] \tag{10.11}$$

constructed from a solution $\Psi[T(x),X^k(x),g_{ab}(x)]$ of Eq. (9.3), is to be interpreted as the probability density that, on the embedding $T(x),X^k(x)$, the metric $g_{ab}(x)$, which is measured in the system of coordinates x^a connected to the Gaussian system of coordinates X^k by the transformation $X^k=X^k(x^a)$, be found in the cell Dg_{ab} centered about $g_{ab}(x)$.

The expression

$$\Psi^*[T(x),g_{ab}(x)]\Psi[T(x),g_{ab}(x)] \tag{10.12}$$

constructed from a solution $\Psi[T(x),g_{ab}(x)]$ of Eqs. (9.23) and (9.24), is to be interpreted as the probability density for the geometry $\mathbf{g}(x)$, which is represented on the hypersurface $T=T(x)$ by the metric $g_{ab}(x)$, to be found in the cell $D\mathbf{g}$ centered about $\mathbf{g}(x)$.

We shall now argue that such an interpretation of the state functional, in spite of appearances, is not tenable. The crux of the matter is that the clock variable $T(x)$ and the frame variables $X^k(x)$ are physically realizable only when the reference fluid satisfies the appropriate energy conditions. When the energy conditions are violated, the variables $T(x),X^k(x)$ cannot be identified by observing the physical properties of a real system, and the interpretation of expressions (10.10)–(10.12) thus loses a sound epistemological foundation. On the other hand, if the energy conditions are imposed as additional restrictions on the state functionals, such restrictions prevent us from interpreting the expressions (10.10)–(10.12) as probability densities. This is a dilemma out of which we see no easy escape.

The energy conditions in the Hamiltonian formalism can be considered as an additional system of constraints (8.4). In quantum theory, such constraints should be turned into operators which annihilate the physical states:

$$\hat{P}^G_+(x)\Psi=0 \ . \tag{10.13}$$

The operator version of the step function,

$$\hat{P}^G_+(x):=\Theta(\hat{H}^G(x)) \ , \tag{10.14}$$

is simply the projector into the space spanned by the eigenfunctions of $\hat{H}^G(x)$ corresponding to the positive eigenvalues $H^G(x)$:

$$\hat{H}^G(x)\Psi_{H^G(x')}=H^G(x)\Psi_{H^G(x')},\quad H^G(x)>0 \ . \tag{10.15}$$

In other words, the physical Hilbert space should be spanned only by the eigenfunctions of $\hat{H}^G(x)$ with nonpositive eigenvalues.

Call $\mathcal{F}$ the Hilbert space of physical states (9.2), $\mathcal{F}_0$ the space of states on a fixed hypersurface $X^K(x)$ which are invariant under spatial diffeomorphisms [Eq. (9.5)], and $\mathcal{F}^+\subset\mathcal{F},\mathcal{F}_0^+\subset\mathcal{F}_0$ the corresponding subspaces on which the energy conditions (10.13) are satisfied.

The proposal (10.13) on implementing the energy conditions fails on two counts. First, the energy conditions get in general violated in the dynamical evolution. As in the classical theory, this means that

$$\frac{1}{i}[\hat{P}^G_+(x),\hat{\Pi}_K(x')]\Psi\neq 0 \quad \text{for } \Psi\in\mathcal{F}^+ \ . \tag{10.16}$$

Second, like the corresponding Poisson brackets (8.7), the projectors (10.14) do not commute on $\mathcal{F}^+$,

$$\frac{1}{i}[\hat{P}^G_+(x),\hat{P}^G_+(x')]\Psi\neq 0 \quad \text{for } \Psi\in\mathcal{F}^+ \ , \tag{10.17}$$

and hence do not possess a common set of eigenfunctions. This implies that the whole idea of spanning $\mathcal{F}^+$ by such eigenfunctions is inconsistent.

A standard way of handling this situation is to keep adding to the original system the constraints obtained by commutators until the extended system of constraints closes. This parallels the procedure leading to Eq. (8.8) of the classical theory. The closure is achieved by adding to the constraints (9.2) and (9.13) the secondary constraints

$$\hat{P}_k(x)\Psi=0 . \qquad (10.18)$$

This amounts to reducing the heat-conducting fluid to an incoherent dust. Equation (10.18) means that the state functional Ψ cannot depend on the fluid variables $X^k(x)$. Moreover, for Eq. (10.18) not to get violated in a measurement process, any admissible observable $\hat{O}$ must commute with $\hat{P}_k(x)$ and cannot thus depend on $X^k(x)$. The fluid variables $X^k(x)$ are thus not observable, and the canonical pair $X^k(x),P_k(x)$ can be dropped from the phase space. The resulting formalism is equivalent to that obtained by imposing the energy conditions in quantum geometrodynamics (9.23) and (9.24) of the Gaussian incoherent dust. For such a system, the Poisson bracket relations (8.7) and (8.10) carry over, modulo factor ordering difficulties, into the commutator relations

$$\frac{1}{i}[\hat{P}^{\,G}_{+}(x),\hat{H}(x')]\Psi=0=\frac{1}{i}[\hat{P}^{\,G}_{+}(x),\hat{P}^{\,G}_{+}(x')]\Psi \qquad (10.19)$$

on $\mathcal{F}^+$, and the energy conditions (10.13) are thereby consistently implemented.

We must now decide whether it is still appropriate to interpret the expression (10.12) as the probability density.

Surely, the functional integral (10.3) gives the conserved norm of the state $\Psi\in\mathcal{F}^+$; however, by the general rules of quantum theory, its integrand can be interpreted as the probability density only if the metric operator $\hat{g}_{ab}(x)$ [or rather the geometry operator $\hat{\mathbf{g}}(x)$] is a multiplication operator on $\mathcal{F}_0^+$.

The geometry operator $\hat{\mathbf{g}}(x)$ is defined as a multiplication operator indirectly, by turning an arbitrary invariant functional $G[g_{ab}]$ of the metric into a multiplication operator. Because $G[g_{ab}]$ depends only on the equivalence classes of the metrics modulo DiffΣ,

$$\hat{H}^{G}{}_{a}(x)G[g_{ab}]=0 , \qquad (10.20)$$

it can be interpreted as a functional $G[\mathbf{g}]$ of the geometry. An implementation of $\hat{G}$ as a multiplication operator

$$\hat{G}:=G[\mathbf{g}]\times \qquad (10.21)$$

then amounts to defining $\hat{\mathbf{g}}(x)$ as a multiplication operator.

The multiplication operator $\hat{G}$ commutes with the supermomentum operator $\hat{H}_a(x)$,

$$\frac{1}{i}[\hat{G},\hat{H}_a(x)]=0 , \qquad (10.22)$$

and hence it keeps the state functional Ψ in the space $\mathcal{F}_0$ of the constraint (9.24). However, it does not commute with the projectors (10.14), because even classically

$$\{G,\Theta(H^G(x))\}=\delta(H^G(x))\frac{\delta G}{\delta g_{ab}(x)}G_{abcd}(x)p^{cd}(x) \neq 0 . \qquad (10.23)$$

The action of $\hat{G}$ on a functional $\Psi[T(x),g_{ab}(x)]$ which satisfies the energy conditions (10.13) thus usually leads to a state functional which does not satisfy the energy conditions. For this reason, the multiplication operator $\hat{G}[\mathbf{g}]$ throws the state functional out of the Hilbert space $\mathcal{F}_0^+$, and the integrand (10.12) of the norm integral (10.3) cannot be interpreted as the probability density for the three-geometry $\mathbf{g}(x)$.

This does not mean that such a probability density does not exist, but only that its identification requires a much more complicated procedure. To find the correct probability amplitude, one should construct the operators $\hat{\mathbf{g}}(x)$ and $\hat{\mathbf{p}}(x)$ [defined again indirectly through DiffΣ-invariant functionals of $g_{ab}(x)$ and $p^{ab}(x)$] which would, on the one hand, satisfy the appropriate fundamental commutation relations

$$\frac{1}{i}[\hat{\mathbf{g}}(x),\hat{\mathbf{p}}(x')]=\delta(x,x') . \qquad (10.24)$$

and, on the other hand, commute with the supermomentum $\hat{H}_a(x)$ and with the projectors $\hat{P}^{\,G}_{+}(x)$. One should then solve the eigenvalue equation

$$\hat{\mathbf{g}}(x)\Psi_{\mathbf{h}(x')}=\mathbf{h}(x)\Psi_{\mathbf{h}(x')} , \qquad (10.25)$$

and, for a given state $\Psi[T(x),g_{ab}(x)]$ on a fixed hypersurface $T=T(x)$, find the true probability amplitude

$$\langle \mathbf{h}(x)|\Psi\rangle=\int D\mathbf{g}\,\Psi^*{}_{\mathbf{h}(x)}[T,g_{ab}]\Psi(T,g_{ab}] , \qquad (10.26)$$

which, of course, does not need to coincide with $\Psi[T(x),g_{ab}(x)]$. The expression

$$|\langle \mathbf{h}(x)|\Psi\rangle|^2 \qquad (10.27)$$

could then be interpreted as the true probability density for finding the three-geometry $\mathbf{h}(x)$. Such a procedure is closely analogous to the problem of finding the Newton-Wigner position operator of a relativistic particle, determining its eigenfunctions, and through the inner product in the space of positive-energy solutions of the Klein-Gordon equation, defining the probability amplitude for the localization of the particle.[53] Our discussion of the steps which are involved in finding the true probability amplitude clearly indicates that while the introduction of the Gaussian coordinate dust may solve the problem of obtaining a conserved positive inner product, the construction of meaningful observables, such as the spatial geometry $\hat{\mathbf{g}}(x)$, is still a difficult task. Any suggestion that the introduction of the Gaussian time, or of a similar realistic matter time variable, automatically solves the problem of how to interpret quantized geometry is thus to be taken with a pinch of salt.

A final word of caution is needed about our procedure of taking care of the energy conditions in quantum theory. One can object that the energy conditions (10.13) are unnecessarily strong and physically untenable. After all, one knows that even in an ordinary free-field theory on a given background, the expectation value of the energy *density* can become negative for suitable states in the Fock space.[54] In other words, if $H^F(x)$ denotes the energy density of the field, the projection operator

$$\hat{P}^G_+(x) := \Theta(-:\hat{H}^F(x):) , \quad (10.28)$$

constructed from the normal-ordered energy-density operator $:\hat{H}^F(x):$, does not annihilate all states Ψ in the Fock space. This phenomenon is closely associated with the appearance of the Schwinger terms in the algebra of the constraints of the parametrized field theory. One can thus justifiably argue that the requirement (10.13) as an expression of the energy conditions in quantum geometrodynamics should be substantially softened. An important point to keep in mind, however, is that it cannot be entirely dropped. Geometrodynamics is fundamentally different from an ordinary bosonic field theory: The classical energy density of a bosonic field is positive by its construction from the basic field variables, while the super-Hamiltonian $H^G(x)$ of the metric field is indefinite even classically. As a consequence, the (weak) energy conditions $\Theta(-H^F(x))=0$ for a classical bosonic field are identically satisfied, but the energy condition $\Theta(H^G(x))=0$ must be imposed to ensure the energy conditions $\Theta(-H^F(x))=0$ for the classical Gaussian fluid. After quantization, an ordinary bosonic field keeps the total energy positive, although, because of factor-ordering problems, it does not succeed to keep the energy density positive. The same thing can be expected in gravitation, but some energy conditions are needed to ensure that the breakdown of positivity is a necessary consequence of quantum effects rather than an artifact brought in by an indefinite nature of the classical expression. The formulation of such energy conditions lies outside the framework of our discussion which systematically sidestepped the issues associated with factor ordering and regularization.

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[1]The notion of a reference fluid was introduced under different names in a number of early publications aiming at a physical interpretation of the general theory of relativity. Einstein himself coined the charming term "mollusc" (Molluske); A. Einstein, *Uber die spezielle und die allgemeine Relativitätstheorie* (Vieweg, Braunschweig, 1920), Sec. 28, p. 67; *Relativity: The Special and the General Theory,* translated by R. W. Lawson, 17th ed. (Crown, New York, 1961). This form was expanded into "a mollusc of reference" by M. Born, *Die relativitätstheorie Einsteins und ihre physikalischen Grundlagen gemeinverständlich dargestellt* (Springer, Berlin, 1920); *Einstein's Theory of Relativity,* translated by H. L. Brose (Methuen, London, 1924), Sec. 7.7, p. 269. Hilbert, in the second of his famous communications on foundations of physics, formalized the idea that the coordinate system should be realized by a physical fluid carrying clocks which keep a causal time: He imposed a set of inequalities ensuring that the world lines of the reference frame be timelike and the leaves of the time foliation be spacelike. See D. Hilbert, 2 Mitt., Nachr. Ges. Wiss. Göttingen **53**, (1917). Hilbert's communications were republished in Math. Ann. **92**, 1 (1924), and in this form they appear in *David Hilbert Gessammelte Abhandlungen,* 2nd ed. (Springer, Berlin, 1970), Vol. 3. Hilbert used the term "proper space-time coordinate system" (eigentliches Raum-Zeit-Koordinatensystem). Reichenbach's conceptual and axiomatic analysis of the theory of relativity introduced the same concept under the name of "a real system" (reales System): H. Reichenbach, *Axiomatic der relativistischen Raum-Zeit-Lehre* (Vieweg, Braunschweig, 1924); *Axiomatization of the Theory of Relativity,* translated by M. Reichenbach (University of California Press, Berkeley, 1969); *Philosophie der Raum-Zeit-Lehre* (deGruyter, Berlin, 1928); *The Philosophy of Space and Time,* translated by M. Reichenbach and J. Freund (Dover, New York, 1957). The concept of the reference fluid found its way into standard textbooks on the general theory of relativity, e.g., into L. D. Landau and E. M. Lifshitz, *Teorija polja* (Gosundarstvennoe izdatel'stvo fiziko-matematiceskoj literatury, Moskva, 1962); *The Classical Theory of Fields,* translated by M. Hamermesh, 4th revised ed. (Pergamon, Oxford, 1975), Sec. 82; and, in a very fundamental and systematic way, into C. Møller, *The Theory of Relativity* (Clarendon, Oxford, 1952), Chap. 8, especially Sec. 8.8. Landau and Lifshitz emphasize that the reference system can be visualized as a "medium" (sreda) and Møller states that "The corresponding system of reference can be pictured by a real fluid." Amidst this terminological embarrassment of riches, we have decided to use a neutral term "reference fluid."

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[18]In his fundamental paper on general theory of relativity [A. Einstein, Ann. Phys. (Leipzig) **49**, 769 (1916)], Einstein noticed that the law of gravitation can be simplified by a special choice of coordinates (pp. 801 and 812). For this purpose, he used the unimodular $[|\det(\gamma_{\alpha\beta})|=1]$ coordinate condition. The view that coordinate conditions are necessary for extracting physical conclusions from a covariant theory was forcefully put forward by E. Kretschmann, Ann. Phys. (Leipzig) **53**, 575 (1917). Kretchmann proposed to use four independent curvature scalars as the privileged coordinates. This idea was later revived in connection with constructing observables in geometrodynamics by A. B. Komar, Phys. Rev. **111**, 1182 (1958). Hilbert chose the normal Gaussian coordinates as a prime example of his "proper space-time coordinate system" and used them in a discussion of the Cauchy problem for the coupled gravitational and electromagnetic field (see Ref. 1). In his 1854 inaugural lecture, Riemann introduced another important coordinate system, the Riemann normal coordinates: B. Riemann, Nachr. Ges. Wiss. Göttingen **13**, 13 (1968); in *B. Riemann: Gesammelte Mathematische Werke,* edited by H. Weber, 2nd ed. (Dover, New York, 1953); Nature **8**, 14 (1873), translated by W. K. Clifford. The spacetime Riemann normal coordinates are based on a spacetime event rather than on a spacelike hypersurface, and they are thus not of much use in canonical geometrodynamics. The spatial Riemann normal coordinates in canonical gravity were discussed by J. Nelson and T. Regge, Gen. Relativ. Gravit. **21**, 645 (1989). Natural coordinates of an observer moving along a timelike world line are the Fermi-Walker coordinates, introduced by E. Fermi, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. Rend. **31**, 184 (1922); **31**, 306 (1922), and A. G. Walker, Proc. R. Soc. Edinburgh **52**, 345 (1932). In quantum geometrodynamics, they are potentially important for discussing measurements performed by such an observer, but their role in the canonical formalism remains to be studied. One of the most important coordinate conditions in general relativity turned out to be the harmonic coordinate conditions introduced by T. DeDonder, *La gravifique einsteinienne* (Gauthier-Villars, Paris, 1921), and C. Lanczos, Phys. Z. **23**, 537 (1923). The foremost advocate of harmonic coordinates as *the* privileged coordinate system in general relativity was Fock. For a summary of his views, see V. Fock, *Teorija prostranstva, vremeni i tjagotenija* (Gosundarstvennoe izdatel'stvo techniko teoreticeskoj literatury, Moskva, 1956); *The Theory of Space, Time and Gravitation,* translated by N. Kemmer (Pergamon, New York, 1959). Harmonic coordinates were extensively used in linearized gravity, in the derivation of equations of motion for the gravitational sources (see V. Fock, this reference), in the analysis of the Cauchy problem [see Y. Choquet-Bruhat, in *Gravitation: an Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962), and S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973), Chap. 7], and in the "covariant approach" to quantum gravity [see B. S. DeWitt, Phys. Rev. **162**, 1195 (1967)]. In canonical geometrodynamics, the most widely used coordinate condition is the restriction of the leaves of the time foliation to maximal hypersurfaces (in asymptotically flat spacetimes) or to hypersurfaces of a constant mean extrinsic curvature (in compact spaces). The maximal slicing was proposed by A. Lichnerowicz, J. Math. Pures Appl. **23**, 37 (1944) and introduced into canonical geometrodynamics by P.A.M. Dirac, Phys. Rev. **114**, 924 (1959). The constant mean curvature condition was proposed by J. W. York, Phys. Rev. Lett. **28**, 1082 (1972). York used these slicing conditions for solving the initial-value problem. A summary of his techniques and further citations are given in Y. Choquet-Bruhat and J. W. York, in *General Relativity and Gravitation the Hundred Years After the Birth of Albert Einstein,* edited by P. Bergmann, J. Goldberg, and A. Held (Plenum, New York, 1980), and in J. W. York, in *Sources of Gravitational Radiation,* edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979). These two reviews are also to be consulted on the discussion of coordinate conditions limiting the spatial frame: The spatial harmonic coordinate condition or the "minimal distortion" coordinate condition. The literature on coordinate conditions is overwhelming. Our sample is merely intended to give a feeling for its relevance to quantum geometrodynamics.

[19]J. D. Brown, K. V. Kuchař, and C. G. Torre (unpublished).

[20]K. V. Kuchař and C. G. Torre (unpublished) handled the harmonic conditions through Lagrange multipliers. Unlike the Gaussian conditions, the harmonic conditions lead to dynamical multipliers and hence to a double extension of the phase space. The formalism can be reduced to an ordinarily extended phase space; the harmonic reference fluid attains thereby the structure of a collection of independent massless scalar fields. An alternative way of introducing the harmonic conditions is through a parametrized Fermi-type term [K. V. Kuchař and C. Stone (unpublished)].

[21]K. V. Kuchař (unpublished).

[22]K. V. Kuchař and C. G. Torre, J. Math. Phys. **30**, 1769 (1989); in *Einstein Studies II*, edited by J. Stachel (Birkhauser, Boston, in press).

[23]C. G. Torre, Phys. Rev. D **40**, 2588 (1989).

[24]Gauss introduced this system of coordinates in his study of two-dimensional surfaces. His 1827 paper "Disquisitiones generales circa superficies curvas" can be found in *Karl Friedrich Gauss Werke* (Kaestner, Göttingen, 1870), Vol. 4, p. 217; *General Investigations of Curved Surfaces of 1827 and 1825,* translated by J. C. Morehead and A. M. Hiltebeitel (Raven, New York, 1965). From the beginning, the Gaussian coordinates were frequently used in the general theory of relativity: cf. Hilbert (Ref. 1). For a geometric construction of Gaussian coordinates and simplifications which they bring, see, e.g., J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960), especially Sec. I.8.

[25]D. Hilbert, "Grundlagen der Physik," 1 Mitt., Nachr. Ges. Wiss. Göttingen, 395 (1915). For the reprinted versions, see Ref. 1.

[26]P.A.M. Dirac, Proc. R. Soc. London **A246**, 333 (1958); R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research* (Ref. 18).

[27]We are denoting spatial covariant derivatives by a vertical bar (|) and spacetime covariant derivatives by a semicolon (;).

[28]The same options arise when one wants to take into account holonomic constraints in analytical mechanics: One can express the configuration of the system in terms of the generalized coordinates so that the constraints are identically satisfied; the Hamilton variational principle then yields the Lagrange equations of the second kind. Alternatively, one can adjoin the constraints to the action by means of Lagrange multipliers; the variational principle then yields the Lagrange equations of the first kind, with multipliers introducing the forces from constraints (see, e.g., C. Lanczos, Ref. 3, Sec. I.5).

[29]P. A. M. Dirac, Can. J. Math. **3**, 1 (1951).

[30]The source term is analogous to the forces from constraints in the Lagrange equations of the first kind.

[31]The Euler hydrodynamical equations can be obtained from a variational principle either by varying the streamlines of the fluid (the Lagrange picture) or the fluid potentials (the Euler picture). Our action for the Gaussian reference fluid is given in the Euler picture. The Euler-type variational principle for perfect relativistic fluids was devised by B. F. Schutz, Phys. Rev. D **4**, 3559 (1971), who developed from it the Hamiltonian theory of the fluid interacting with gravity. Further references on this subject are B. F. Schutz and R. D. Sorkin, Ann. Phys. (N.Y.) **107**, 1 (1977); V. Moncrief, Phys. Rev. D **16**, 1702 (1977); D. Bao, J. E. Marsden, and R. Walton, Commun. Math. Phys. **99**, 319 (1985); L. Bombelli and R. J. Torrence, University of Calgary report, 1989 (unpublished). A Lagrange-type variational principle for relativistic perfect fluids was given by A. H. Taub, Phys. Rev. **94**, 1468 (1954). The Gaussian reference fluid is not a perfect fluid because it allows for the flow of heat.

[32]The round brackets symmetrize and the square brackets antisymmetrize a tensor in a pair of indices: $T_{\cdots(\alpha\cdots\beta)\cdots} := T_{\cdots\alpha\cdots\beta\cdots} + T_{\cdots\beta\cdots\alpha\cdots}$ and $T_{\cdots[\alpha\cdots\beta]\cdots} := T_{\cdots\alpha\cdots\beta\cdots} - T_{\cdots\beta\cdots\alpha\cdots}$.

[33]C. Eckart, Phys. Rev. **58**, 919 (1940); A. H. Taub, Phys. Rev. **74**, 328 (1948); G. A. Kluitenberg and S. R. deGroot, Physica **21**, 148 (1955), and earlier references cited therein. For a recent review of relativistic thermodynamics, see W. Israel and J. M. Stewart, in *General Relativity and Gravitation the Hundred Years After the Birth of Albert Einstein* (Ref. 18), Vol. 2.

[34]A thorough discussion of energy conditions and of algebraic types of the energy-momentum tensor is given by S. W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Ref. 18), Sec. 4.3.

[35]Theories with derivative gravitational coupling follow quite a different pattern. For comparisons, see K. V. Kuchař, J. Math. Phys. **17**, 801 (1976); **18**, 1589 (1977).

[36]Dirac (Ref. 3), pp. 59–60; K. V. Kuchař, J. Math. Phys. **17**, 792 (1978), Sec. 5.

[37]The concept of symmetry in the Dirac constraint formalism is discussed by K. V. Kuchař, J. Math. Phys. **25**, 1647 (1982).

[38]K. V. Kuchař, J. Math. Phys. **17**, 801 (1976), Eq. (8.22).

[39]K. V. Kuchař, J. Math. Phys. **17**, 801 (1976), Eq. (8.23).

[40]Cf. C. J. Isham and K. V. Kuchař, Ann. Phys. (N.Y.) **164**, 316 (1985), Eqs. (2.14) and (2.15). In this paper, the system of coordinates X^α on M is not necessarily Gaussian. To compare the equations given there with corresponding equations of the present paper, one must put the mapping Y leading to Eq. (2.3) equal to identity.

[41]The parenthesis in Eq. (8.2) indicates that H^F is a function of x; the square bracket means that H^F is a functional of g_{ab} and p^{ab}. This notation is systematically used in the rest of this paper. Cf. also Eqs. (2.15) and (4.20).

[42]In general, one can recognize that a collection of spatial tensors was generated by projecting a spacetime tensor ∥ and ⊥ to a hypersurface by testing their behavior under bendings and tilts: K. V. Kuchař, J. Math. Phys. **17**, 777 (1976), Sec. 10; **17**, 792 (1976), Sec. 3; **25**, 1647 (1982), Sec. 2D.

[43]J. A. Wheeler, in *Relativity, Groups and Topology,* edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1964); in *Battelle Rencontres: 1967 Lectures in Mathematics and Physics,* edited by C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968); DeWitt (Ref. 6).

[44]For the Schwinger terms in the commutator of the components of the energy-momentum tensor, see D. G. Boulware and S. Deser, J. Math. Phys. **8**, 1468 (1967).

[45]K. V. Kuchař, Phys. Rev. D **39**, 1579 (1989); **39**, 2263 (1989).

[46]See, e.g., M. Green, J. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987).

[47]The construction of the positive inner product in stationary spacetimes is discussed in A. Ashtekar and A. Magnon, Proc. R. Soc. (London) **A346**, 375 (1975); C. R. Acad. Sci. Paris **286**, 531 (1978); A. Magnon-Ashtekar, thesis Université de Clermont, 1976 (unpublished); B. S. Kay, Commun. Math. Phys. **62**, 55 (1978). It is summarized by K. V. Kuchař, in *Einstein Studies II* (Ref. 22).

[48]K. V. Kuchař, J. Math. Phys. **22**, 2640 (1981).

[49]K. V. Kuchař, in *Quantum Gravity II: A Second Oxford Symposium,* edited by C. J. Isham, R. Penrose, and D. W. Sciama (Clarendon, Oxford, 1981); Kuchař (Ref. 47).

[50]In principle, a Schrödinger equation can also be obtained by constructing a time variable from the extrinsic curvature (the extrinsic time). Such a time was first used in the linearized theory by Arnowitt, Deser, and Misner (see Ref. 26 and citations therein). Its advantages over the intrinsic time were analyzed by K. V. Kuchař, J. Math. Phys. **11**, 3322 (1970). The extrinsic time representation enabled Kuchař to quantize cylindrical gravitational waves by a midisuperspace functional Schrödinger equation: K. V. Kuchař, Phys. Rev. D **4**, 955 (1971). A general strategy for casting the Klein-Gordon equation into a Schrödinger equation by constructing an internal time and its conjugate momentum from the geometrodynamical variables is discussed in K. V. Kuchař, J.

Math. Phys. **11**, 3322 (1970). Examples of model systems in which a super-Hamiltonian quadratic in the momenta can be brought into a super-Hamiltonian linear in the momenta by a canonical transformation is given in K. V. Kuchař, J. Math. Phys. **19**, 390 (1978). The cylindrical gravitational waves mentioned in this footnote and the bosonic string (see Ref. 22) are specific examples of this procedure. York and his collaborators designed a powerful algorithm for solving the initial-value constraints in geometrodynamics on hypersurfaces of a constant extrinsic time. A summary of those techniques with citations of original papers can be found in Choquet-Bruhat and York, and York (Ref. 18). A disadvantage of the resulting Schrödinger equation is that the Hamiltonian is defined only implicitly, through a nonlocal prescription, and it is thus difficult to define the corresponding operator.

[51]The inner product for the functional Schrödinger equation is discussed in Ref. 48. In a free-field theory, a heuristic expression for the inner product can be given an exact meaning; see K. V. Kuchař, Phys. Rev. D **39**, 2263 (1989), Sec. IIA.

[52]People often define "observables" as dynamical variables which commute with *all* the constraints, including the super-Hamiltonian. Because the super-Hamiltonian generates the dynamical evolution of a covariant system, such a definition implies that an observable necessarily is a constant of motion. This certainly does not capture the ordinary meaning of the term "observable" in unconstrained quantum theory. We believe it is more appropriate to define an observable as a dynamical variable which commutes only with the supermomenta, i.e., which is gauge invariant under spatial diffeomorphisms.

[53]T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949). The description of the procedure and further references can be found, e.g., in S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961), or in S. R. deGroot and L. G. Suttorp, *Foundations of Electrodynamics* (North-Holland, Amsterdam, 1972).

[54]A simple example of such a state may be found in R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984), Chap. 14, problem 6. Negative-energy densities can also arise in a Minkowsian spacetime which is limited by boundaries or which has a nontrivial topology. The oldest example is the Casimir effect for the electromagnetic field in the region between two parallel reflecting planes: H. B. G. Casimir, Proc. K. Ned. Acad. Wet. **51**, 793 (1948). The simplest problem involving topology was discussed by S. A. Fulling, Phys. Rev. D **7**, 2850 (1973); B. S. Kay, *ibid.* **20**, 3052 (1979); a massless scalar field on a two-dimensional cylindrical Minkowskian spacetime gives rise to negative vacuum expectation values of the energy density and pressure. For the definition of these quantities on curved slices of the cylinder, see K. V. Kuchař, Phys. Rev. D **39**, 2263 (1989). A survey of further examples can be found in B. S. DeWitt, in *General Relativity: An Einstein Centenary Survey* (Ref. 8), or in N. D. Birrell and P. C. Davies (this reference). *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982). Curved backgrounds bring in additional effects. S. W. Hawking [Commun. Math. Phys. **18**, 301 (1970)] showed that the expectation value of the energy-momentum tensor cannot be conserved and at the same time satisfy the dominant energy condition. Ya. B. Zel'dovich and L. P. Pitaevsky [Commun. Math. Phys. **23**, 185 (1971)] discussed the mechanism by which spacetime curvature can induce an energy-momentum tensor which violates the dominant energy condition. L. Parker and S. A. Fulling [Phys. Rev. D **7**, 2357 (1973)] exhibited states of a quantized massive scalar field on the Robertson-Walker background which give rise to negative pressure terms sufficiently large to violate the energy conditions. The extensive literature on the vacuum expectation values of the renormalized energy-momentum tensor on a curved background is reviewed in N. D. Birrell and P. C. Davies (this reference).