Complete description of gauge-invariant fields in a pure gauge theory

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The basic set of gauge-invariant local as well as nonlocal fields is constructed for a non-Abelian pure gauge theory with the gauge group SU(N). It is shown that the basic set of local gauge-invariant fields are local generators of the nonlocal gauge-invariant fields. This provides a complete description of gauge-invariant fields in a pure gauge theory and thereby provides a complete solution to the problem of observables at the classical level.

All the known fundamental interactions among the elementary particles in nature seem to be mediated by gauge fields. Even without matter, these fields are quite complex except for the case of electromagnetic fields which is described by an Abelian theory. This has to do with the geometric richness that goes with a self-interacting non-Abelian gauge theory. In this Brief Report, we set aside the geometric structure and take an algebraic approach. One of the problems we hope to tackle here is that of observables. Though in quantum theory the question remains open, in classical theory we solve the problem by constructing a complete and irreducible set of local as well as nonlocal gauge-invariant fields and explicitly demonstrate the relationship between these two classes of invariants. This provides a complete description of gauge-invariant fields and completely solves the problem of observables in a classical theory. It should be mentioned here that these works have appeared before in parts,[1–4] but it is only now that the complete picture has emerged, and here we report it for the first time.

The mathematical framework for our construction is the invariants of the theory of matrices transforming adjointly under the action of the group SU(N). The two fundamental theorems of the invariants of the theory relevant for us are[5–7] the following.

First fundamental theorem. Let $X_1, X_2, \ldots, X_n, \ldots$ be a set of $N\times N$ matrices transforming adjointly:

$$x_i \to g^{-1} x_i g \ ,$$

under the action of the group SU(N). The basic set of invariants that can be constructed out of these matrices consist of monomial traces of bounded degree given by

$$T^{(k)}_{i_1 \cdots i_k} = \mathrm{tr}(x_{i_1} \cdots x_{i_k}) \ , \tag{1}$$

where the indices $(i_1, \ldots, i_k)$ are not necessarily different, and $k \le 2^N - 1$ for $N \neq 3$ and $k \le 6$ for $N=3$.

Second fundamental theorem. Every relation among the invariant polynomials follows from the Hamilton-Cayley identity applied to the basic monomials described above.

The term "basic" in the first fundamental theorem means "complete and irreducible" when the group is SU(2). For SU(N), $N \ge 3$ the exact formula is not known. Therefore, the term "basic" in this case means that monomial traces of the theorem contain a basis for the algebra of invariant polynomials. All other invariants are polynomials in this small "basic" set of monomials. As far as the second fundamental theorem is concerned, an irreducible set of the relation is known for the case of SU(2),[2] For higher groups, it is impracticable to list these relations. However, as in the case of SU(2), the constraints form a finite set of polynomial relations.[6,7] Below, we use only the first fundamental theorem for our construction. For details[2] and the relevance of the second fundamental theorem, in large-N quantum theory, see Ref. 4.

Before we apply the first theorem, we note the following two elementary lemmas.[2]

Lemma 1. There are no nontrivial local gauge-invariant fields that can be constructed out of affinely transforming variables, such as the gauge potential A_μ, alone.

Lemma 2. There are no nontrivial nonlocal gauge-invariant variables that can be constructed out of variables.

$$W(C_{xy}) = P \exp\left[\int_x^y (-ig A_\mu dx^\mu)\right] \ , \tag{2}$$

the path-ordered exponential for the open loop C_{xy}.

The two lemmas provide us with the relevant variables for our construction. For the local case, it consists of the field strength $F_{\mu\nu}$ and its covariant derivatives of arbitrary order:

$$\{F_\nu\} = \{F_{\mu\nu}, D_{\lambda_1} F_{\mu\nu}, \ldots, D_{\lambda_1 \cdots \lambda_n} F_{\mu\nu}, \ldots\} \ .$$

For the nonlocal case, the relevant variables are the path-ordered exponentials, defined as

$$W(c,x) = P \oint_{c,x} \exp(-ig A_\mu dx^\mu) \tag{3}$$

for the closed loop C, passing through the point x. They are called the Wilson loops for the loop C.

Under gauge transformations, the local and the nonlocal variables described above transform as

$$\begin{aligned} F_\gamma(x) &\to g^{-1}(x) F_\gamma g(x) \ , \\ W(c,x) &\to g^{-1}(x) W(c,x) g(x) \ . \end{aligned} \tag{4}$$

Now, we can use the first fundamental theorem to write down the basic set of local and nonlocal gauge-invariant fields.

The first fundamental theorem for local gauge-invariant fields. The basic set of local gauge-invariant fields in a non-Abelian pure gauge theory with the gauge group SU(N) consists of trace monomials of bounded degree given by

$$\mathrm{tr}(F_{\gamma_1}\cdots F_{\gamma_k})\ , \qquad (5)$$

where the indices $(\gamma_1,\ldots,\gamma_k)$ are not necessarily different and $k\leq 2^N-1$ for $N\neq 3$ and $k\leq 6$ for $N=3$.[2]

The first fundamental theorem for nonlocal gauge-invariant fields. Single-loop case. The complete and irreducible set of gauge-invariant fields that can be constructed out of the Wilson loop $W(c,x)$ for a loop c passing through the point x in a pure gauge theory with gauge group SU(N) consists of

$$\{\mathrm{tr}W,\mathrm{tr}W^2,\ldots,\mathrm{tr}W^N\}\ , \qquad (6)$$

where $W=W(c,x), W^2=W(c^2,x)=W(c\cdot c,x)=$Wilson loop for the loop traversing c twice, etc.[3]

Multiloop case. Let $c_1,c_2,\ldots,c_n,\ldots$ be a set of loops passing through the point x and let

$$W_i=W(c_i,x)=P\oint\exp(-igA_\mu dx^\mu)\ ,$$
$$W_i^2=W(c_i\cdot c_i,x),\ \ \text{etc}\ .$$

The basic set of nonlocal gauge-invariant variables that one can construct out of Wilson loops for loops $(c_1,c_2,\ldots,C_n,\ldots)$ passing through the point x in a pure gauge theory with gauge group SU(N) consists of the trace monomials of bounded degree given by

$$\mathrm{tr}(W_{i_1}^{\gamma_1}W_{i_2}^{\gamma_2}\cdots W_{i_j}^{\gamma_k})\ , \qquad (7)$$

where $(i_1,i_2,\ldots,i_j)$ are not necessarily different and $(\gamma_1+\gamma_2+\cdots+\gamma_k)\leq 2^N-1$ for $N\neq 3$ and ≤ 6 for $N=3$.[3]

To show that the basic set of local gauge-invariant fields are local generators of the nonlocal gauge-invariant fields, we use the coordinate gauge[8–10]

$$x^\mu A_\mu(x)=0\ . \qquad (8)$$

In this gauge, the potential A_μ can be written as

$$A_\mu(x)=\int_0^1 tx^\nu F_{\mu\nu}(tx)dt\ , \qquad (9)$$

where t is some parameter, and $A_\mu(x)$ and $F_{\mu\nu}(x)$ are analytic in their arguments. Moreover, in this gauge the conventional expansion of the field strength $F_{\mu\nu}(x)$ in Taylor series,

$$F_{\mu\nu}(x)=F_{\mu\nu}(0)+x^\beta\partial_\beta F_{\mu\nu}(0)+\cdots\ , \qquad (10)$$

can be replaced by[11]

$$F_{\mu\nu}(x)=F_{\mu\nu}(0)+x^\beta D_\beta F_{\mu\nu}(0)+\cdots\ , \qquad (11)$$

where ∂_β and D_β are ordinary and covariant derivatives, respectively.

Assume now that the contour considered for the Wilson loop is centered at the origin, and the field strength and the potential are analytical functions in their arguments. We use Eq. (9) to write the Wilson loop as

$$W(c)=\sum_{n=0}^{\infty}\frac{(-ig)^n}{n!}\mathrm{tr}P\int_0^1 t_1dt_1\oint_c F_{\mu_1\nu_1}(t_1x_1)x_{1\mu_1}dx_{1\nu_1}\cdots\int_0^1 t_ndt_n\oint_c F_{\mu_n\nu_n}(t_nx_n)x_{n\mu_n}dx_{n\nu_n}\ . \qquad (12)$$

Each field strength tensor in Eq. (12) can be expanded in a Taylor series. Combining this fact with Eq. (11), we obtain a new infinite series for the Wilson loop. Each term in the series contains, apart from the geometric factors, the dynamical variables in the form $\mathrm{tr}(F_{\nu_1}\cdots F_{\nu_n})$ where as before each F_ν is either the field strength or its covariant derivative of some order. Now we use the first fundamental theorem for the local gauge-invariant fields which proves the following theorem.

The theorem on local generators for Wilson loops. The basic set of local gauge-invariant fields given by the first fundamental theorem on local gauge-invariant fields are local generators for a gauge-invariant Wilson loop provided the potential and the field strength are analytic.

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