Fermion masses from supersymmetric dynamics in proper time

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The proper-time superparticle formulation of the Dirac operator is extended by the introduction of supersymmetric dynamics for extra coordinates, representing multiplets of chiral quarks and leptons. The resulting internal Hamiltonian is interpreted as their mass-squared operator. Graded Kähler geometries are proposed for the additional coordinates, with the a numbers generating the spectrum of one fermion family, and the c numbers (in the holomorphic first-order formulation) generating bosonic excitations corresponding to family replication. A simplified SU(5) model is described which is compatible with the SU(5) minimal Higgs scenario [for one standard SU(5) quark and lepton generation plus singlet neutrino].

I. INTRODUCTION AND MAIN RESULTS

In the context of grand unified theories (GUT's), the structure of the fermion mass spectrum is usually a matter for fortuitous choices and/or fine-tuning of Yukawa couplings, leaving some "surviving" particles light, and others effectively to decouple by acquiring masses at some large unification scale. In the basic SU(5), SO(10), and E_6 models, family replication is handled simply by having (three) copies of a fundamental fermion multiplet[1]; supersymmetric GUT's fare little better.[2]

The situation is more promising for various higher-dimensional schemes such as Kaluza-Klein models, supergravity models, or superstrings regarded as compactifications. The Dirac operator acquires an internal piece and it is the topology of the internal manifold which determines the number of massless chiral fermions.[3] Unfortunately in these models there are usually severe technical difficulties which limit the extent to which definite predictions can be made.

On the other hand, the experimental observation of three generations of the basic quarks and leptons (provided the top quark is confirmed) cries out for some dynamical explanation. Technicolor and/or composite models have had some success,[4] although there is at present no evidence for quark or lepton substructure. At a different level there has been some considerable study of the numerics of the mass matrices and the Cabibbo-Kobayashi-Maskawa (CKM) matrix itself,[5] on the basis that this is the *only* trace of any hypothetical new interaction which might be responsible.

The scheme for fermion mass proposed in the present work departs from field theory by utilizing the first-quantized picture (technically the proper-time formalism for the Dirac operator; see below). In the spirit of the higher-dimensional schemes mentioned above, it is proposed to extend the formalism by the adjunction of extra internal coordinates, which carry a group action (that of the gauge group of the model), and whose quantization and dynamics determine the particle states (internal quantum numbers) and their mass spectrum, respectively. The extra coordinates can be either a numbers (fermionic) or c numbers (bosonic), and it is natural to conjecture that the excitations of the former correspond to particles grouped in one *family*, while excitations of the latter (below some continuum threshold) give rise to family *replications*.

The first-quantized or proper-time formalism derives from the "world-line" formulation of Feynman[6] and thus is as old as renormalizable quantum field theories, although it is often regarded merely as a technique for extracting Green's functions and in functional manipulations.[7] In the case of a particle moving in curved space,[8] it is a form of a nonlinear σ model,[9] and superparticle constructions have been related to superfields.[10] It is occasionally advocated as an alternative calculational scheme to perturbative quantum field theory.[11] For present purposes it is sufficient that any desired Green's function of a given second-quantized theory should be reproducible in the associated first-quantized picture (cf. string theories where the string field theory should reproduce the amplitudes of the first-quantized string).

The supersymmetric proper-time action for a Dirac particle moving in external gravitational and Yang-Mills fields, with extended dynamics, is

$$S=\int dt\left[\frac{1}{2}\dot{x}^{\mu}g_{\mu\nu}(x)\dot{x}^{\nu}+\frac{1}{2}i\psi^{\mu}\nabla_t\psi^{\nu}g_{\mu\nu}(x)+i\bar{\xi}\dot{\xi}+\bar{\xi}\dot{x}^{\mu}A_{\mu}\xi+\frac{1}{2}\bar{\xi}\psi^{\mu}F_{\mu\nu}\psi^{\nu}\xi\right], \tag{1}$$

and forms the starting point of the present work. Here $x^{\mu}(t),\psi^{\mu}(t),\mu=0,\ldots,3$ are vectorial c-number and a-

number coordinates, respectively, representing the particle's spatial and spin degrees of freedom; $A_\mu(x)$ and $F_{\mu\nu}(x)$ are the usual Yang-Mills potential and field strength, taking their values in some representation of the Lie algebra of the gauge group, the metric $g_{\mu\nu}(x)$ implies that the particle is moving in curved space; and ξ^a, $\bar{\xi}^{\bar{a}}$, a, $\bar{a} = 1, 2, \ldots, n$, are a set of $2n$ internal coordinates. Below, they acquire additional dynamics through the curved space which they parametrize [via a metric $k_{a\bar{b}}(\xi,\bar{\xi})$], and an interaction potential $W(\xi,\bar{\xi})$ [cf. (3) to (5) and (9) to (14) below].

The action (1) can be cast in superfield form and is given in Sec. II. It is supersymmetric for the following infinitesimal coordinate transformations (where ε is an a number):

$$\delta x^\mu = \varepsilon\psi^\mu, \quad \delta\psi^\mu = -\varepsilon\nabla_t x^\mu,$$

$$\delta\xi = -\varepsilon\psi^\mu A_\mu \xi, \quad \delta\bar{\xi} = -\bar{\xi}\psi^\mu A_\mu \varepsilon.$$

The main results of the present study, of extensions of (1) and their quantization, are as follows.

Following early work on supersymmetric quantum mechanics,[12] it was shown that there was a natural supersymmetric structure to the Dirac operator, leading to quantum-mechanical proofs of the Atiyah-Singer index theorem.[13] Of primary concern are the spatial and spinor degrees of freedom [associated with the $X_\mu(t)$ and $\psi_\mu(t)$ coordinates, and typically a fixed representation of the gauge group is considered, e.g., the fundamental representation for SU(n)]. In the manifestly supersymmetric heat-kernel treatment[14] internal a-number coordinates were introduced merely as a convenient bookkeeping device in the non-Abelian case. For a treatment of the Abelian case in a different context, see Ref. 15. It was emphasized in Ref. 16 that the internal coordinates could be c numbers *or* a numbers. The last step is the projection onto the desired internal representation, and quantization of the spacetime sector leads to the Hamiltonian

$$H = \not{D}^2, \tag{2}$$

so that the appropriate supertrace in the Euclidean regime yields the Atiyah-Singer index.[13]

From the viewpoint of the present study, the above formulations represent simply the flat-space limit of the general case in which the internal coordinates have nontrivial dynamics [$k_{a\bar{b}} \equiv \delta_{a\bar{b}}$ and $W \equiv 0$; see (11) below]. The Hamiltonian acts on the entire space of the quantized operators $\hat{\xi}^a$, $\hat{\bar{\xi}}^{\bar{a}}$, a, $\bar{a} = 1, \ldots, n$ corresponding to ξ^a, $\bar{\xi}^{\bar{a}}$, a, $\bar{a} = 1, \ldots, n$ (which determines the internal quantum numbers), rather than being projected onto a fixed representation as is implicit in (2). It also acquires additional pieces, which give rise to the mass-squared operators of the set of particles (quarks and leptons) carrying the internal quantum numbers.

The following toy example is illustrative of the general situation and serves as a paradigm for the full analysis of Secs. II–IV. Consider the flat-space, zero-field case where the internal action is [cf. (11)]

$$S_I = \int dt(i\bar{\xi}\dot{\xi} - \Lambda\bar{\xi}\xi), \tag{3}$$

and the coordinates are taken to be a set of n complex a numbers. (See also Ref. 17.) Then

$$\pi = \delta L/\delta\dot{\xi} = -i\bar{\xi},$$

$$H_I = \dot{\xi}\pi - L = \Lambda\bar{\xi}\xi,$$

and dealing with the constraint leads to the conventional fermionic quantization

$$\{\hat{\xi}, \hat{\bar{\xi}}\} = 1. \tag{4}$$

Thus the total Hamiltonian is

$$\begin{aligned} H &= H_{\text{sp}} + H_I \\ &= \not{\partial}^2 \otimes 1 + 1 \otimes \Lambda N_I, \end{aligned} \tag{5}$$

acting on spinor wave functions carrying a 2^n-dimensional representation of the gauge group G, generated by the usual Fock realization of (4). It is natural to interpret the internal piece in (5) as the mass-squared operator for these particles, leading in this case to the mass spectrum $\sqrt{\Lambda N}$, $N = 0, 1, \ldots, n$.

In Sec. II the question of suitable generalizations of (3) to curved internal spaces is taken up. It is suggested that in the general case the coordinates ξ, $\bar{\xi}$ describe a homogeneous (graded) Kähler manifold, the group action being that of the gauge group of the model [the example of SU(n/1)/SU(n)⊗U(1) is given explicitly].

In Sec. III a simple model with gauge group SU(5) is considered. A generalized chiral projection is introduced which (for appropriate fermionic quantization in the internal sector) restricts the internal quantum numbers to the $\mathbf{1}_L$, $\mathbf{10}_L$, and $\overline{\mathbf{5}}_L$ of two standard generations of SU(5) plus a right-handed neutrino; a further projection to even or odd internal occupation numbers is also possible in the absence of additional mixing terms. A flat-space model, with a two-parameter potential, is shown to be compatible with the phenomenology of the standard SU(5) minimal Higgs-boson-mass generation for this case.

Finally, Sec. IV treats a further extension to graded coordinates ξ^a, z, $\bar{\xi}^{\bar{a}}$, $\bar{z}$, where the n complex a numbers ξ^a, $\bar{\xi}^{\bar{a}}$, $a = 1, \ldots, n$ are augmented by a single complex c-number coordinate z and its conjugate $\bar{z}$. Together the graded coordinates parametrize a graded Kähler manifold, with a typical metric of the form

$$\mathcal{G} = \begin{pmatrix} g_{a\bar{b}} & g_{a\bar{z}} \\ g_{z\bar{b}} & g_{z\bar{z}} \end{pmatrix}. \tag{6}$$

In this picture it is tentatively suggested how family replications may arise (with a finite number of families), and how fermion mixings may arise between families.

It should be pointed out that "internal Grassmann coordinates" have also been advocated via Kaluza-Klein-type models[18] and more generally[19] (see also Ref. 20).

However, the present framework tends towards first-order rather than second-order actions, and spinor degrees of freedom are treated via vector-valued a-number coordinates (rather than spinors). On the other hand, the "phenomenological superparticle"[21] also entails *spacetime* supersymmetry (rather than supersymmetry in proper time, as considered here).

II. FERMION MULTIPLETS FROM ANTICOMMUTING KÄHLER GEOMETRY

Supersymmetric actions are most readily understood in a superfield notation, and it has been established[22] that the combination

$$X^\mu = x^\mu + i\theta\psi^\mu ,$$

together with the covariant derivative

$$D = \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t},$$

are the correct ingredients for the spatial and spin degree of freedom parts of the supersymmetric action (1) describing the Dirac operator, where the supersymmetry generator is

$$Q = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}.$$

Here we consider the extension to a set of complex internal superfield coordinates $\Phi^a, \bar{\Phi}^{\bar{a}}$, $a = 1, 2, \ldots, n$, where

$$\begin{aligned} \Phi^a &= \xi^a + \theta\eta^{\bar{a}}, \\ \bar{\Phi}^{\bar{a}} &= \bar{\xi}^{\bar{a}} + \theta\bar{\eta}^{\bar{a}}. \end{aligned} \tag{7}$$

(Here the Φ^a, ξ^a, $\bar{\Phi}^{\bar{a}}$, $\bar{\xi}^{\bar{a}}$ are considered a numbers, and η^a, $\bar{\eta}^{\bar{a}}$ c numbers. In the next section an additional complex c-number superfield

$$\begin{aligned} Z &= z + i\theta\zeta, \\ \bar{Z} &= \bar{z} + i\theta\bar{\zeta}, \end{aligned} \tag{8}$$

is considered). The action (1), in superfield form, is then[14]

$$S = \int dt[g_{\mu\nu}(X)DX^\mu\dot{X}^\nu - \bar{\Phi}^a D_A\Phi_a],$$

where $D_A = D + DX^\mu A_\mu(X)$ is the covariant derivative including the gauge potential $A_\mu(X)$.

Since we wish to consider the Φ^a on a par with X^μ, and the latter are related to curved spacetime coordinates, it is natural to consider Φ^a, $\Phi^{\bar{a}}$ as coordinates of a complex manifold, and to construct an action using appropriate geometrical objects. The $\Phi^a \sim \xi^a$, η^a differ, however, from the $X^\mu \sim x^\mu$, θ^μ since, as the flat space example (3) shows, a first-order action in the ξ^a, $\bar{\xi}^{\bar{a}}$ (and z, $\bar{z}$) is anticipated, while the η^a,$\bar{\eta}^{\bar{a}}$ (and ζ, $\bar{\zeta}$) become auxiliary (whereas the x^μ have a second-order action and the ψ^μ remain dynamical with a first-order action. In the case of extra bosonic coordinates, the action is in the first-order holomorphic formalism.). Thus a Riemannian structure alone is not sufficient; for example, terms $\bar{\xi}^{\bar{a}}k_{\bar{a}b}\nabla_t\xi^b$, even when ∇_t is the covariant time derivative, are not appropriate if ξ^a, $\bar{\xi}^{\bar{b}}$ are coordinates (rather than tangent vectors).

A refinement of a complex Riemannian geometry (which has, of course, arisen in conventional spacetime supersymmetry[23]) is a Kähler manifold. Here the existence of a certain closed two-form, plus compatibility with the complex structure leads to a scalar Kähler potential $K(\Phi, \bar{\Phi})$ for which in appropriate coordinates the metric becomes

$$\begin{aligned} k_{a\bar{b}} &= \frac{\partial}{\partial\Phi^a}\frac{\partial}{\partial\bar{\Phi}^{\bar{b}}}K, \quad k_{a\bar{b}} = -k_{\bar{b}a}, \\ k_{ab} &= 0, \quad k_{\bar{a}\bar{b}} = 0.. \end{aligned} \tag{9}$$

For the present case it is natural to consider homogeneous Kähler manifolds whose symmetry group is identified with the gauge group of the model. A concrete example of such a Kähler manifold parametrized by a-number coordinates is SU($n/1$)/SU(n)$\otimes$U(1), for which

$$K(\Phi, \bar{\Phi}) = \ln(1 + \bar{\Phi}\Phi) ,$$

$$g_{a\bar{b}} = \delta_{a\bar{b}}/(1 + \bar{\Phi}\Phi) - \Phi_a\bar{\Phi}_{\bar{b}}/(1 + \bar{\Phi}\Phi)^2 .$$

The existence of the scalar K leads to the possibility of defining natural covariants and invariants by appropriate differentiation: thus for example[23] the only Christoffel symbols are of the form

$$[\bar{a}bc] \equiv \frac{\partial}{\partial\bar{\Phi}^{\bar{a}}}\frac{\partial}{\partial\Phi^b}\frac{\partial}{\partial^c}K(\Phi, \bar{\Phi}) .$$

The supersymmetric action whose Hamiltonian provides the mass-squared operator of a set of fermions carrying the appropriate internal quantum numbers is

$$S_I = \int dt\, d\theta \left[\frac{1}{2}\left(D_t\bar{\Phi}^{\bar{a}}\frac{\partial}{\partial\bar{\Phi}^{\bar{a}}}K - D_t\Phi^a\frac{\partial}{\partial\Phi^a}K \right) - V\left(\bar{\Phi}, \Phi\right) \right] , \tag{10}$$

where V is a potential (self-interaction term) and K is the Kähler potential. After integrating over θ and eliminating η, $\bar{\eta}$ one finds [cf. (3)]

$$S_I = \int dt\, d\theta \left[\frac{1}{2}i\left(\dot{\bar{\xi}}^{\bar{a}}\frac{\partial}{\partial\bar{\xi}^{\bar{a}}}K - \dot{\xi}^a\frac{\partial}{\partial\xi^a}K \right) - W\left(\bar{\xi}, \xi\right) \right] , \tag{11}$$

where

$$W\left(\bar{\xi}, \xi\right) = \frac{1}{2}\frac{\partial}{\partial\Phi^a}Vk^{a\bar{b}}\frac{\partial}{\partial\bar{\Phi}^{\bar{b}}}V$$

and

$$\begin{aligned} k^{a\bar{b}}k_{\bar{b}c} &= \delta^a{}_c, \\ k_{\bar{a}b}k^{b\bar{c}} &= \delta_a{}^{\bar{c}} \end{aligned} \tag{12}$$

Finally, the Hamiltonian corresponding to (11) is

$$H_I = W(\xi,\bar{\xi}). \tag{13}$$

Quantization of first-order actions of this sort can be carried out by Dirac or Becchi-Rouet-Stora-Tyutin (BRST) methods.[24] The result is that the ξ, $\bar{\xi}$ become fermionic operators $\hat{\xi}$, $\hat{\bar{\xi}}$ satisfying the (in general nonlinear) anticommutation relations

$$\{\hat{\xi}^a, \hat{\bar{\xi}}^{\bar{b}}\} = k^{a\bar{b}}(\hat{\xi},\hat{\bar{\xi}}). \tag{14}$$

In Sec. III particular choices of K and V are made in a toy model. There the internal space turns out to have its full complement of 2^n states, but from the point of view of phenomenology, it should be emphasized that appropriate nonlinear realizations of (14) may entail fewer states.

III. A SIMPLIFIED SU(5) MODEL

In this section the general scheme outlined above is specialized to the $n = 5$ case in an attempt to describe the SU(5) model. The quark-lepton mass matrix in the latter, generated in the usual scenario from Higgs-Yukawa couplings after spontaneous symmetry breaking, must therefore be reproduced by the dynamics of the additional anticommuting coordinates. A natural choice for the latter would be a nontrivial Kähler geometry [using, for example, the manifold SU(5/1)/SU(5)⊗U(1) mentioned above], and interactions such that some type of dynamical symmetry breaking would reproduce the fermion mass matrix. Such a proposal depends on further analysis of the quantum mechanics of this class of model (cf. Ref. 24), and work along these lines is in progress. In particular, the explicit treatment in this proper-time picture of the gauge degrees of freedom is required (cf. Ref. 15) before the complete correspondence with the spontaneous symmetry breaking and Higgs mechanism of the second-quantized formalism is apparent. In the meantime, however, a simpler strategy, equivalent as far as fermion mass generation is concerned to the introduction of Higgs vacuum expectation values in Yukawa couplings in the field-theory version, is considered: namely, the restriction to flat space, but with explicit symmetry-breaking terms.

For illustrative purposes we therefore take in (10) the (flat) Kähler potential

$$K(\Phi,\bar{\Phi}) = \frac{1}{2}(\bar{\Phi}\Phi) \equiv \frac{1}{2}\sum_{a=1}^{5}\bar{\Phi}^{\bar{a}}\Phi^a, \qquad g_{a\bar{a}} = \delta_{a\bar{a}}, a,\ \bar{a} = 1,\ldots,5, \tag{15}$$

together with the superpotential

$$V = (\bar{\Lambda}_a\Phi^a + \Lambda_{\bar{a}}\bar{\Phi}^{\bar{a}})[1 + \mu_1(\bar{\Phi}\Phi) + \tfrac{1}{2}\mu_2(\bar{\Phi}\Phi)^2], \tag{16}$$

in which $\bar{\Lambda}_a$, $\Lambda_{\bar{a}}$ are (complex-conjugate) constant c numbers, $a = 1,\ldots,5$, and μ_1 and μ_2 are real parameters (playing the role of arbitrary couplings).

Clearly field redefinitions of the form

$$\Phi' = \Phi[1 + \mu_1(\bar{\Phi}\Phi) + \tfrac{1}{2}\mu_2(\bar{\Phi}\Phi)^2] \tag{17}$$

will modify the appearance of (10), (11) by simplifying the potential at the expense of the appearance of interactions derived from the metric; indeed (17) simply means that the superfields Φ,$\bar{\Phi}$ experience an interaction with constant external fields $\bar{\Lambda}$, Λ, whereas in (16), although the manifold is manifestly flat, the external potential interactions involve some rescaling of Φ,$\bar{\Phi}$. As we shall see, the parameters μ_1 and μ_2 are essential if the standard SU(5) scenario is to be reproduced.

With (15), (16) in place the internal Hamiltonian $W(\bar{\xi},\xi)$ [cf. (11) and (12)] becomes

$$\begin{aligned} W(\bar{\xi},\xi) = {} & \Lambda\bar{\Lambda}(1 + 2\mu_1\bar{\xi}\xi + \mu_1^2\bar{\xi}\xi^2 + \mu_1\mu_2\bar{\xi}\xi^3 + \tfrac{1}{4}\bar{\xi}\xi^4) \\ & -\Lambda\bar{\xi}\xi\bar{\Lambda}[(2\mu_1 + 3\mu_1^2 + 2\mu_2)\bar{\xi}\xi + 5\mu_1\mu_2\bar{\xi}\xi^2 \\ & +2\mu_2^2\bar{\xi}\xi^3], \end{aligned} \tag{18}$$

where $\Lambda\bar{\Lambda} \equiv \sum_{a=1}^{5}\Lambda_{\bar{a}}\bar{\Lambda}_a$, $\Lambda\bar{\xi}\xi\bar{\Lambda} \equiv \sum_{a,b=1}^{5}\Lambda_{\bar{a}}\bar{\xi}^{\bar{a}}\xi^b\bar{\Lambda}_b$, and in view of (14), after quantization,

$$\{\hat{\xi}^a, \hat{\bar{\xi}}^{\bar{b}}\} = \delta^{a\bar{b}}, \tag{19}$$

[which is another reason why (15), (16) give a more straightforward starting point than (25)].

Before examining the spectrum of (18) and working out the mass-squared operator, the fermion content is restricted by introducing a generalized chiral projection on the states. In the field theory description this is just the restriction to chiral spinors (with, for example, the eigenvalue of the Hermitian γ_5 matrix being -1 in the left-handed case). In the present (proper time) case, it is necessary to interpret this in terms of the quantized operators corresponding to the ψ^μ coordinate of (1), where [cf. (14)]

$$\{\psi^\mu, \psi^\nu\} = 2\eta^{\mu\nu}$$

in the flat-space case. Generically one must take $d/2$ pairs of fermionic creation and annihilation operators α, $\alpha^\dagger$, in space-time dimension d (cf. Ref. 25), and the ψ's are combinations such as $\alpha \pm \alpha^\dagger$ with appropriate numerical factors. Clearly ψ^μ (like γ^μ) changes the chirality of states, and, as signaled by the selection rule $\delta N_{\rm sp} = \pm 1$ (where $N_{\rm sp}$ is the total spinor occupation number in this representation), γ_5 can be identified with $(-1)^{N_{\rm sp}}$ using, for example, $(\alpha + \alpha^\dagger)(\alpha - \alpha^\dagger) = (-1)^{N_\alpha}$. The Weyl projection for left-handed states would therefore be $(-1)^{N_{\rm sp}} = -1$.

In the present case where both spinor and internal states are given in an occupation number basis we introduce

$$\mathcal{N} = N^{\rm sp} + N^I$$

and demand the generalized chiral projection

$$(-1)^{\mathcal{N}} = -1. \tag{20}$$

From (19) and (20) if we regard the operators $\bar{\xi}^{\bar{a}}$, $a = 1, 2, \ldots, 5$ as creating SU(5) representations from a vacuum state, and include the required correlation with spinor chirality, we have the following states:

N_I	0	1	2	3	4	5
SU(5)	$\mathbf{1}_L$	$\mathbf{5}_R$	$\mathbf{10}_L$	$\overline{\mathbf{10}}_R$	$\bar{\mathbf{5}}_L$	$\mathbf{1}_R$

At this stage, a further restriction may be introduced which is valid for this model but may not necessarily be so in general [even within this class of model, we can entertain generalizations such as the SU(5)-symmetric interaction $V' = \Delta(\varepsilon_{abcde}\Phi^a\Phi^b\Phi^c\Phi^d\Phi^e + \text{H.c.})$]: namely, that in the model (18) the internal occupational number is conserved, so attention can be restricted to just the even or odd sectors. We choose for example

$$(-1)^{N_I} = +1 \,. \tag{21}$$

An alternative would be to implement charge-conjugation symmetry so that the R fields can be regarded as the charge conjugate of their L counterparts, $\psi_R \simeq (\psi_L)^c$. In fact if $\mathcal{M}$ is an internal operator which satisfies $\Delta N_I = \pm 1$ then the $\Delta\mathcal{N} = \pm 1$ projections of $\mathcal{Q} \equiv i\,\partial\!\!\!/ - \mathcal{M}$ satisfy $\{\mathcal{Q}_+, \mathcal{Q}_-\} = \partial^2 + \mathcal{M}^2$, and there is a supersymmetry of the *massive* Dirac operator.

Thus the net result of the generalized chiral condition (20) is to introduce the correct correlation between spinor chirality and SU(5) quantum number such that the fermions (21) describe *two* standard SU(5) generations (plus singlet right-handed neutrino); in cases where internal fermion number is conserved, the additional restriction (22) allows for just *one* generation (plus singlet neutrino), namely, $\mathbf{1}_L$ ($N_I = 0$), $\mathbf{10}_L$ ($N_I = 2$), and $\bar{\mathbf{5}}_L$ ($N_I = 4$). In this connection it should be noted that in the curved case SU(5/1)/SU(5)⊗U(1) a suitable (Dyson?) type quantization may admit the *atypical*, 31-dimensional representation of SU(5/1), giving one less singlet neutrino (for comments on superalgebra representations and anomalies see Sec. V). In order to evaluate the mass-squared operator (18) in the occupation number basis, the question of operator ordering must first be addressed. As shown for example in Ref. 26, the path-integral quantization of a given operator $V(\hat{\zeta}, \hat{\bar{\zeta}})$ leads to an action of the form (3) but with the potential $W(\xi, \bar{\xi})$ being the normal kernel of $V(\hat{\xi}, \hat{\bar{\xi}})$. Denoting N_I by N hereafter we have

$$W = (\bar{\xi}\xi) = \sum_{a=1}^{5} \bar{\xi}^a \xi^a$$

corresponds to $V = (\hat{\bar{\xi}}\hat{\xi}) = N$

and to $W = (\bar{\xi}\xi)^2 \equiv \sum_{a,b=1}^{5} \bar{\xi}^{\bar{a}}\bar{\xi}^{\bar{b}}\xi^b\xi^a$

corresponds $V = \sum_{a,b=1}^{5} \hat{\bar{\xi}}^{\bar{a}}\hat{\bar{\xi}}^{\bar{b}}\hat{\xi}^b\hat{\xi}^a = N(N-1)$

and so on. The complete correspondence is readily given if for the constant "external" fields Λ, $\bar{\Lambda}$ the values $(\Lambda) = \Lambda(1,0,0,0,0)$, $(\bar{\Lambda}) = \bar{\Lambda}(1,0,0,0,0)$ are taken, and the operator $\hat{\bar{\xi}}_1\hat{\xi}_1$ denoted N_ν. Defining $\Xi = \sum_a \bar{\xi}^a\xi^a$ and $\Gamma = \sum_{a,b} \bar{\Lambda}_{\bar{a}}\bar{\xi}^{\bar{a}}\xi^b\Lambda_b$ we find simply

$$\Xi^n \to N(N-1)(N-2)(\cdots)(N-n+1)\,,$$

$$\Gamma^n \to N_\nu N(N-1)(\cdots)(N-n+2).$$

The mass-squared operator can finally be constructed with the help of these results. Identification of physical states is by means of the SU(5) and SU(4) basis chain if it is noted that N and N_ν uniquely identify representations: for example, for $N = 2$, $N_\nu = 0, 1$ correspond respectively to the 6- or 4-dimensional representations of SU(4) contained in the **10** of SU(5). The complete results are given in Table I.

Table I gives the mass-squared operator for one SU(5) generation (plus singlet neutrino) and can be interpreted by comparison with standard SU(5) Higgs-induced mass generation. Suppose, for example, that the Lagrangian for a set of chiral fermions $\{\psi_{A\alpha}\}$ is

$$\mathcal{L} = \bar{\psi}^{A\alpha}(i\partial\!\!\!/)_\alpha^\beta\psi_{A\beta} - \tfrac{1}{2}M^\dagger_{AB}C_{\alpha\beta}\bar{\psi}^{A\alpha}\bar{\psi}^{B\beta} + \tfrac{1}{2}M^{AB}(C^{-1})^{\alpha\beta}\psi_{A\alpha}\psi_{B\beta}\,, \tag{22}$$

where $C_{\alpha\beta}$, $(C^{-1})^{\alpha\beta}$ are the charge-conjugation matrix and its inverse, and the mass matrices $M^\dagger_{AB}$,M^{AB} are assumed to be derived from Yukawa couplings. For example, if the set of (left-handed) chiral fermions is $\psi_A \simeq \psi$, ψ_{ab}, ψ^a corresponding to the **1**, **10**, $\bar{\mathbf{5}}$ representations of SU(5), then with appropriate index conventions

$$(M_{AC}) = \begin{pmatrix} M & M^{cd} & M_c \\ M^{ab} & M^{abcd} & M_c^{ab} \\ M_a & M_a^{cd} & N_{ac} \end{pmatrix} = \begin{pmatrix} M & 0 & Y_1H_c \\ 0 & Y_{10}\varepsilon^{abcd}H_e & Y_M(\delta_c^aH^b - \delta_c^bH^a) \\ Y_1H_a & Y_M(\delta_c^aH^b - \delta_c^bH^a) & N_{ac} \end{pmatrix},$$

TABLE I. Comparison of mass-squared eigenvalues of a simplified SU(5) model [see Eqs. (18) and (19)] with the conventional Yukawa-Higgs mechanism (23), in the occupation number basis, together with the conventional particle labels.

SU(5)	N_I	N_ν	SU(4) (name)	(Mass)^2 (proper time)	(Mass)^2 (Higgs)
1	0	0	**1** (ν')	1	Y_1^2
10	2	0	**6** (u, u^c)	$1+4\mu_1+2\mu_1^2$	$4Y_{10}^2$
		1	**4** (d, e^+)	$1+2\mu_1-\mu_1^2-\mu_2$	$\frac{1}{2}Y_M^2$
$\bar{\mathbf{5}}$	4	0	$\bar{\mathbf{4}}$ (d^c, e)	$1+8\mu_1+12\mu_1^2$ $+24\mu_1\mu_2+6\mu_2^2$	$\frac{1}{2}Y_M^2$
		1	**1** (ν)	$1+6\mu_1-6\mu_2+3\mu_1^2$ $-6\mu_1\mu_2-6\mu_2^2$	Y_1^2

where Y_1, Y_M, Y_{10} are Yukawa couplings, H_a, $H^a=(H_a)^\dagger$ is the vacuum expectation value of the Higgs field (taking the minimal choice of a single complex **5**), and M is an additional (Majorana) mass which is allowed for the singlet neutrino. Then for this choice,

$$(M^\dagger M)^A{}_E = \begin{pmatrix} \bar{M}M+\bar{M}^eM_e & 0 & \bar{M}M_e \\ 0 & \bar{M}^c_{ab}M^{ef}_c & 0 \\ M\bar{M}^a & 0 & \bar{M}^aM_e+\bar{M}^a_{cd}M^{cd}_e \end{pmatrix}. \tag{23}$$

Since the mass-squared matrix derived in Table I is diagonal in the SU(5) basis, for comparison the Majorana mass M must be dropped here [a model with $\Delta N=2$, e.g., one containing $V'=\Delta(\varepsilon_{abcde}\Phi^a\Phi^b\Phi^c\Phi^d\Phi^e+\text{H.c.})$, would be required to reproduce such mixing terms], leaving the diagonal elements (for **1**,**10** and $\bar{\mathbf{5}}$, respectively):

$$(M^\dagger M)_1 = {Y_1}^2|H|^2,$$

$$\begin{aligned}(M^\dagger M)^{ef}_{ab} &= ({Y_M}^2-\tfrac{1}{2}{Y_{10}}^2)\\ &\quad\times(\delta^e_aH_bH^f-\delta^f_aH_bH^e-\delta^e_bH_aH^f\\ &\quad\quad+\delta^f_bH_aH^e)\\ &\quad+{Y_{10}}^2\tfrac{1}{2}(\delta^e_a\delta^f_b-\delta^e_a\delta^f_b)\\ (M^\dagger M)^a_e &= 2{Y_M}^2\delta^a_e|H|^2+({Y_1}^2-2{Y_M}^2)H^aH_e.\end{aligned}$$

It is now straightforward to interpret (24) in the SU(5) $\supset$ SU(4) basis assuming $(H_a)\simeq\Lambda(1,0,0,0,0)$. The results are given in the last column in Table I, together with the conventional particle labels for the various states.

Crucial to the Higgs-induced mass scenario (and hence the mass-squared matrix) is the charge conjugation symmetry which allows the particle states of the $\bar{\mathbf{4}}$ of SU(4) [in the SU(5) $\bar{\mathbf{5}}$ multiplet] to be identified with the charge conjugates of the states of **4** of SU(4) [in the SU(5) **10**], so that the spectrum comprises four Dirac fermions d, e, with the familiar SU(5) result $m_d=m_e$ at the tree level. Unfortunately, in the model based on (22) the charge conjugation symmetry is not manifest for the generic parameters μ_1, μ_2 (see Table I). The symmetry can be imposed *a posteriori* by demanding that the entries in Table I follow the same pattern as those derived from Yukawa couplings, namely,

$$1=1+6\mu_1-6\mu_2+3\mu_1^2-6\mu_1\mu_2-6\mu_2^2,$$

$$1+2\mu_1-\mu_1^2-\mu_2=1+8\mu_1+12\mu_1^2+24\mu_1\mu_2+6\mu_2^2.$$

In fact, there are three nontrivial solutions to these equations (aside from $\mu_1=\mu_2=0$), each corresponding to fixed ratios of $m_e=m_d$: m_u: (ν_{mixing}). However, these "predictions" should not be taken seriously in this simplified model. In a more sophisticated version, the requisite charge conjugation symmetry should appear automatically (cf. Ref. 7), and allowance would also be made for a (large) Majorana mass parameter. The main point of the simplified model is that it is compatible with standard field theory, and moreover that generalizations are conceivable within the present framework which go beyond the field theory description.

IV. FAMILY REPLICATION FROM GRADED KÄHLER GEOMETRY

As pointed out in Sec. I, the supersymmetric formalism goes through equally well with the additional coordinates being *either* a numbers *or* c numbers, and it is natural to associate the (bosonic) excitations of the latter with family replications.

In the Kähler geometry consider for example a potential

$$\mathcal{K}(\bar{\xi},\xi,z,\bar{z})=K(\bar{\xi}\xi)h^{-1}(\bar{z},z),$$

where as before $\xi^a,\bar{\xi}^{\bar{a}}$, $a=1,2,\ldots,n$ are a numbers, but z is an additional c number with its complex conjugate $\bar{z}$, and K, h^{-1} are invertible functions of $(\bar{\xi}\xi)$

($\equiv \sum_{a=1}^{n} \bar{\xi}^{\bar{a}} \xi^a$), and $z, \bar{z}$, respectively. Then the Kähler metric is

$$(\mathcal{G}) = \begin{pmatrix} g_{a\bar{b}} & g_{a\bar{z}} \\ g_{z\bar{b}} & g_{z\bar{z}} \end{pmatrix} = \begin{pmatrix} k_{a\bar{b}} h^{-1} & -\bar{\xi}^{\bar{a}} K' h_{\bar{z}}^{-1} \\ \xi^b K' h_z^{-1} & h_{z\bar{z}}^{-1} \end{pmatrix} , \quad (24)$$

where

$$k_{a\bar{b}} = \frac{\partial}{\partial \xi^a} \frac{\partial}{\partial \bar{\xi}^{\bar{b}}} K$$

is the Kähler metric for K alone. In the symmetry-breaking scheme of Sec. III [cf. (16) and (17)], where there is an interaction between the additional coordinates and a constant external potential (representing the vacuum expectation value of the Higgs field), the Hamiltonian becomes [cf. (11) to (13) in Sec. II]

$$H = \bar{\Lambda}_a g^{a\bar{b}} \Lambda_{\bar{b}} ,$$

where $g^{a\bar{b}}$ is now the inverse of the graded Kähler metric (24). Clearly $g^{a\bar{b}}$ is of the form

$$g^{a\bar{b}} = k^{a\bar{b}} h + l^{a\bar{b}} ,$$

where $l^{a\bar{b}}$ will typically be bilinear in K' and h_z^{-1}, $h_{\bar{z}}^{-1}$. Thus the structure of H is

$$H = h\bar{\Lambda}_a k^{a\bar{b}} \Lambda_{\bar{b}} + \bar{\Lambda}_a l^{a\bar{b}} \Lambda_{\bar{b}} \equiv h \otimes H_I + \Delta H, \quad (25)$$

where H_I is the mass-squared matrix due to the fermionic interactions alone, h after quantization plays the role of a multiplicative bosonic contribution to the Hamiltonian, and ΔH is a further mixing term which in general will not commute with $h \otimes H_I$.

Clearly if the bosonic states $|i\rangle$, $i = 1, \ldots, F$ are eigenstates of h, with eigenvalues λ_i, and the fermionic states $|\alpha\rangle$ are eigenstates of H_I with eigenvalues m_α^2, then

$$h \otimes H_I \, |i\rangle|\alpha\rangle = \lambda_i m_\alpha^2 |i\rangle \otimes |\alpha\rangle,$$

indicating that a basis for H is given by F copies of the fermionic eigenstates, and that diagonalization of H including the mixing term ΔH implies taking certain linear combinations of the $|i\rangle \otimes |\alpha\rangle$.

This structure, while not worked out in detail here, is therefore an attractive scenario for the origin of family replications, and for the generalized mixing matrices. In a proper time formulation including the gauge degrees of freedom, the dynamics should be such that the $|i\rangle \otimes |\alpha\rangle$ states emerge as the true gauge eigenstates. At the same time one might hope to interpret the states corresponding to the continuum of h in terms of some singular sector of the complete dynamics. Further work along these lines is in progress.

V. CONCLUSION

As explained at length in the introductory discussion (Sec. I), the strategy of analyzing fermion mass and mixing here proposed is to put aside the field-theory description in favor of a proper-time formulation with additional extended dynamics (Sec. II). This strategy is worked out in a simplified example (Sec. III), but the general formalism is robust and admits of further generalizations such as a mechanism for family replication (Sec. IV). The following general comments list some of the points in favor of the scheme as well as some problems to be clarified in future work:

(i) The introduction of extended dynamics in curved space suggests a geometrical origin of the fermion spectrum (cf. Ref. 27). The model has some resemblance to the noncommutative geometry of Connes,[27] while remaining in the tradition of higher-dimensional theories.

(ii) The generalized chiral projection (20) in the SU(5) case provides another way (among many) to explain the correlations between chirality and equivalence class of representation of the observed fermions; namely, that the **10** and $\bar{\mathbf{5}}$ go together, rather than, say, the **10** and the **5**. In this context a "grading by chirality" (cf. Ref. 28) is automatic given the way the states are constructed [see, e.g., the table following (20)].

(iii) The introduction of a graded Kähler manifold naturally leads to mixings between repeated families because of noncommuting parts of the mass-squared Hamiltonian [see (25)]. Presumably inclusion of the gauge degrees of freedom (cf. Ref. 15) would clarify why the $|i\rangle \otimes |\alpha\rangle$ states associated with $h \otimes H_I$ are by themselves gauge eigenstates [see (1)].

(iv) The simplified SU(5) model in Sec. III sets aside the geometrically derived dynamics [cf. (i) above] in favor of a flat space model with an explicit symmetry-breaking potential derived from an external field, equivalent to (but at this level rather simpler than) the Higgs mechanism for mass generation. In three nontrivial cases the simplified model was compatible with the SU(5) minimal Higgs-boson-mass scenario, with fixed ratios of $m_e \equiv m_d : m_u : (\nu_{\text{mixing}})$.

(v) The present study is at the first quantized, superparticle level; as mentioned above, the gauge degrees of freedom could be introduced explicitly also (cf. Ref. 15), and their interactions may generate the symmetry-breaking direction dynamically. The corresponding field theory description would then be interesting. In fact it could be argued that the complex c numbers z,$\bar{z}$ actually amount (in the field theory version) to the addition of one scalar field (rather than various multiplets) as in the usual Higgs mechanism.

(vi) A possible continuum sector in the bosonic part of the mass-squared Hamiltonian (as well as the discrete bound-state spectrum, which corresponds to family replication) defies interpretation in the present framework. However, it may turn out to signal some important regime of the corresponding field theory version and should not be rejected at this stage.

(vii) Finally, it should be pointed out that in the proper-time formulation there seems no need to have conventional canonical quantization of the fermionic degrees of freedom. Parastatistics or modular statistics[29] or perhaps even quantization based on a given Lie algebra ("compact quantum mechanics[30]") may be equally admissible, and may provide another route to family replication.

Extensions of the model, with these and related questions in mind, are currently under study.

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