PHYSICAL REVIEW D VOLUME 43, NUMBER 12 15 JUNE 1991

Particles with small violations of Fermi or Bose statistics

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(Received 15 January 1991)

I discuss the statistics of "quons" (pronounced to rhyme with muons), particles whose annihilation and creation operators obey the q-deformed commutation relation (the quon algebra or q-mutator) which interpolates between fermions and bosons. Topics discussed include representations of the quon algebra, proof of the TCP theorem and clustering, violation of the usual locality properties, and experimental constraints on violations of the Pauli exclusion principle (i.e., Fermi statistics) and of Bose statistics.

I. INTRODUCTION

Within the last three years, two new approaches to particle statistics (in three or more space dimensions) have been studied in order to provide theories in which the Pauli exclusion principle (i.e., Fermi statistics) and/or Bose statistics can be violated by a small amount. One of these approaches uses deformations[1-10] of the trilinear commutation relations of Green[11] and Volkov.[12] (Deformations of algebras and groups, in particular, quantum groups, are a subject of great interest and activity at present. The extensive literature on this subject can be traced from Ref. 13.) The particles which obey this type of statistics, called "parons," have a quantum field theory which is local, but some states of such theories must have negative squared norms (i.e., there are negative probabilities in the theory). The negative squared norms first appear in many-particle states; it does not seem that they decouple from positive squared norm states (as, in contrast, the corresponding states do decouple in manifestly covariant quantum electrodynamics). Thus theories with parons seem to have a fatal flaw.

The other approach uses deformations of the bilinear Bose and Fermi commutation relations.[14-20] The particles which obey this type of statistics, called "quons," have positive-definite squared norms for a range of the deformation parameter, but the observables of such theories fail to have the usual locality properties. This failure raises questions about the validity of relativistic quon theories, but does not cause a problem with nonrelativistic quon theories. (As I prove below, the TCP theorem and clustering hold for free relativistic quon theories, so even relativistic quon theories may be consistent.)

Still other approaches to violations of statistics were given in Refs. 21–23. One should also recall the early work of Gentile[24] which proposed to interpolate between Fermi and Bose statistics by allowing up to k particles (instead of only one) to occupy a single state. (Gentile's model does not give a small violation of statistics, but it does interpolate.) An interpolation between Bose and Fermi statistics using parastatistics of increasing order was studied in Ref. 25; this also does not give a small violation.

The purpose of this article is to discuss quons, and to consider several issues related to statistics in the light of these recent developments. In Sec. II, I discuss the quon algebra, $a_k a_l^\dagger - q a_l^\dagger a_k = \delta_{kl}$. In Sec. III, I discuss the case of a single oscillator. Section IV considers Fock-like representations of quons. Section V describes operators which annihilate or create particles in a given place in a many-particle state. In Sec. VI, I discuss the use of the $q=0$ operators as standard building blocks for the operators with q in $-1 \leq q \leq 1$. In Sec. VII, I construct the free quon field and show that it obeys the TCP theorem despite the fact that the observables of the theory are not local. Section VIII deals with experiments relevant to small violations of the exclusion principle (i.e., of Fermi statistics) and of Bose statistics. Concluding remarks and directions for further research are given in Sec. IX.

II. THE QUON ALGEBRA

The quon algebra (or q-mutator)

$$a_k a_l^\dagger - q a_l^\dagger a_k = \delta_{kl} \quad (1)$$

is a deformation of the Bose and Fermi algebras which interpolates between these algebras as q goes from 1 to -1 on the real axis. The quon algebra, supplemented by the vacuum condition

$$a_k|0\rangle = 0, \quad (2)$$

determines a (Fock-like) representation in a linear vector space. For $-1 < q < 1$, the squared norms of all vectors made by limits of polynomials of the creation operators $a_k^\dagger$ are strictly positive. For $q = \pm 1$ the squared norms of all such vectors are never negative; vectors which are not totally symmetric (antisymmetric) under permutations have zero norms. Any relation of the form

$$c_1 A_k A_l^\dagger - c_2 A_l^\dagger A_k = c_3 \delta_{kl} \quad (3)$$

can be brought into the form of (1) by the change of normalization

$$a_k = (c_1/c_3)^{1/2} A_k,\ a_k^\dagger = (c_1/c_3)^{1/2} A_k^\dagger,$$

with $q = c_2/c_1$, provided $c_1/c_3 > 0$ and $c_3 \neq 0$. The choice made in (1) is particularly convenient for the interpolation between Bose and Fermi statistics. The choice $a_k a_l^\dagger - q a_l^\dagger a_k = q^{-N}\delta_{kl}$, where N is the number operator, studied by some authors does not have a convenient interpolation. The Bose ($q = 1$) and Fermi ($q = -1$) cases are, of course, of special interest; another special case is $q = 0$ ("infinite statistics") which I discussed earlier.[15] The range $-1 < q < 1$ shares with the case of $q = 0$ the property of having positive-norm vectors belonging to all representations of the symmetric (i.e., the permutation) group. Also, as in the case of $q = 0$, Eqs. (1) and (2) allow the calculation of the vacuum-to-vacuum matrix element of any polynomial in the a's and $a^\dagger$'s.

Note that no commutation rule can be imposed on aa or $a^\dagger a^\dagger$. Furthermore, no such rule is needed to calculate vacuum matrix elements of polynomials in the a's and $a^\dagger$'s. All such matrix elements can be calculated by moving annihilation operators to the right using (1) until they annihilate the vacuum according to (2) or by moving creation operators to the left again using (1) until they annihilate the vacuum on the left according to the adjoint of (2). The relation

$$a_k a_l - q a_l a_k = 0 \tag{4}$$

between two a's which one might guess in analogy with the Bose and Fermi commutation rules holds only when $q^2 = 1$; and requires that $q = \pm 1$ in (1); i.e., (4) can hold only in the Bose and Fermi cases. To see this, interchange k and l in (4) and put the result back in the initial relation. It is worth remarking that the commutation or anticommutation relations between two a's or between two $a^\dagger$'s are also not needed to calculate matrix elements in the Fock representation [in which (2) holds] of Bose or Fermi statistics. [The fact that the a's (or the $a^\dagger$'s) commute or anticommute when $q \to \pm 1$ can be seen using the expansion of the general-q operators in the $q = 0$ operators (see Sec. VI).] Neither are these needed for normal ordering, i.e., to expand a product of a's and $a^\dagger$'s as a sum of terms in which creation operators always stand to the left of annihilation operators. Wick's theorem for quon operators is similar to the usual Wick's theorem; the only difference is that the terms acquire powers of q. I give the precise algorithm in Sec. VII.

Although the qualitative results are often the same, $q = 0$ (Refs. 15, 27, and 28) is much simpler than general q. For $q = 0$, all states formed by monomials in $a^\dagger$'s acting on the vacuum have norm one, regardless of whether the $a^\dagger$'s have the same or distinct labels. Monomial states are orthogonal unless the same set of $a^\dagger$'s occur in the same order. The space of states is obviously a Hilbert space; i.e., it has a positive-definite metric. To construct the operators for the energy, momentum, angular momentum, etc., in terms of the annihilation and creation operators, it suffices to construct a set of number operators, n_i, which obey the usual commutation relations

$$[n_i, a_j]_- = -\delta_{ij} a_j . \tag{5}$$

Then the energy operator, for example, is

$$E = \sum_i \epsilon_i n_i , \tag{6}$$

where ϵ_i is the single-particle energy . For our $q = 0$ example of infinite statistics, both the number operator and the transition number operator have an infinite series expansion in terms of the a's and $a^\dagger$'s:

$$n_i = a_i^\dagger a_i + \sum_k a_k^\dagger a_i^\dagger a_i a_k + \sum_{k_1,k_2} a_{k_1}^\dagger a_{k_2}^\dagger a_i^\dagger a_i a_{k_2} a_{k_1} + \cdots + \sum_{k_1,k_2,\cdots k_s} a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_s}^\dagger a_i^\dagger a_i a_{k_s} \cdots a_{k_2} a_{k_1} + \cdots . \tag{7}$$

(As far as I know, this is the first case in which the number operator, Hamiltonian, etc., for a free field are of infinite degree.) In verifying that (5) is valid, the contributions with $\cdots aa$ coming from a given term in n_i cancel against a contribution from the next term in n_i so that the commutator telescopes to give the stated result. There is an analogous formula for the transition operator n_{ij} which annihilates a particle in state j and creates a particle in state i. Just replace $a_i^\dagger a_i$ in (7) by $a_i^\dagger a_j$. The transition operator obeys

$$[n_{ij}, a_k]_- = -\delta_{ik} a_j . \tag{8}$$

(The number operator $n_k = n_{kk}$.)

Operators which obey (5) or (8) are composite Bose operators. Their eigenvalues are additive for product states. Thus the construction (7) gives a composite Bose operator, which can be an observable, in terms of quon operators in analogy with the construction of composite Bose operators from Fermi operators [for example, the current $j_\mu(x) = \bar\psi(x)\gamma_\mu\psi(x)$, where ψ is a Fermi operator] or from para-Bose or para-Fermi operators (for example, the current $j_\mu(x) = (i/2)[\phi^\dagger(x), \partial_\mu\phi(x)]_+$, where ϕ is a para-Bose operator). The condition that observables must be composite Bose operators leads to conservation of statistics which states that all interactions must involve an even number of fermions or parafermions and an even number of paraparticles (except for cases in which p parafields can occur when the order of the parastatistics is p).[26] I expect that conservation of statistics must also hold for quons and, in particular, that a single quon cannot couple to normal fields.[29,30] I plan to discuss the conservation of statistics for quons in detail elsewhere. [Conservation of statistics, which holds for both nonrelativistic and for relativistic theories, is a more primitive property than the spin-statistics theorem. The rule that a composite state with an even (odd) number of fermions is a boson (fermion) is a consequence of conservation of statistics.]

III. SINGLE OSCILLATOR

Just as results are simple for $q = 0$, they are also simple for a single mode; in particular,[31–33] give simple proofs that the single oscillators act in a positive metric (Hilbert) space. Freund and Nambu[31] do this by giving a representation of a single q-mutator oscillator a in terms of a Bose oscillator b :

$$a = f(N_0)b, \quad f^2(N_0) = (1 - q^{N_0+1})/(N_0 + 1)(1 - q), \quad N_0 = b^\dagger b. \tag{9}$$

[Substitute the *Ansätz* for a in the q-mutator relation, (1), and solve the resulting recursion relation. N_0 is the number operator for b.] An analogous representation of a holds using a single infinite statistics operator c obeying Eq. (1) with $q = 0$,

$$a = g(N_0)c, \quad g^2(N_0) = (1 - q^{N_0})/(1 - q); \tag{10}$$

now N_0 is the number operator for c which has an infinite degree expansion in terms of c and $c^\dagger$. Unfortunately, these *Ansätze* do not seem to generalize to more than one oscillator. The number operator for a single mode can be written[32] $n = \sum_{j=1}^{\infty}(1-q^j)^{-1}(1-q)^j a^{\dagger j} a^j$.

IV. FOCK-LIKE REPRESENTATION FOR GENERAL Q

To construct the Fock-like Hilbert space for quons, we must calculate the scalar products between vectors made by polynomials in the creation operators acting on the vacuum state. To calculate these scalar products, it suffices to give the scalar products between vectors which are made from monomials in the $a^\dagger$'s acting on the vacuum. When the degrees of the monomials differ, the scalar product vanishes. The scalar products are non-vanishing only when the labels of the $a^\dagger$'s in one vector are a permutation of the labels of the $a^\dagger$'s in the other vector. [I use permutations of places, rather than permutations of labels. Place permutations remain defined when the same label occurs more than once. So, for example, the permutation (132) means that the label that was in the first place goes to the third place, the label that was in the third place goes to the second place, and the label that was in the second place goes to the first place. Permutations are multiplied from right to left.] When the labels are distinct, the scalar product is

$$\begin{aligned} M_{P,Q}(q) &= (Pa^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle, Qa^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle) \\ &= (a^\dagger(k_{P^{-1}1})\cdots a^\dagger(k_{P^{-1}n})|0\rangle, a^\dagger(k_{Q^{-1}1})\cdots a^\dagger(k_{Q^{-1}n})|0\rangle) = q^{i(PQ^{-1})}, \end{aligned} \tag{11}$$

$i(P)$ = number of inversions in going from the natural order to the order given by P = minimum number of transpositions of successive integers necessary to bring $1,\ldots,n$ into $P1,\ldots Pn$. Here and in the discussion of the inversion table below the permutations are regarded as acting on symbols. (An inversion occurs when a number j occurs to the left of a number $i < j$.) The inversion table of a permutation $P : (1,2,\ldots,n) \to (P1,P2,\ldots,Pn) = (Pj)$ is a set of n integers; the leftmost integer is the number of times numbers larger than 1 appear to the left of 1 in (Pj) (this first integer can have values from 1 to $n-1$), the jth integer is the number of times numbers larger than j appear to the left of j (this jth integer can have values from 0 to $n-j$) and the last integer is the number of times numbers larger than n appear to the left of n (this last integer must be 0). The inversion number $i(P)$ is the sum of the integers in the inversion table of P. Each P in S_n has one and only one inversion table; i.e., the inversion table labels permutations uniquely.[34] One can calculate $i(P)$ by representing a permutation $P : (1,\ldots,n) \to (P1,\ldots,Pn)$ by writing the integers in natural order on one line and writing the integers in permuted order on a line underneath. Draw a set of arrows from the original position of each integer to its permuted position. Then $i(P)$ is the number of times these directed lines cross. Reversing the lines gives the permutation P^{-1}; clearly the number of crossings remains the same, so $i(P) = i(P^{-1})$. The inversion number of an even (odd) permutation is even (odd). Note that $i(P)$ is not a class function on S_n. For example, $i((12)) = 1$, but $i((13)) = 3$, so i splits a class, and $i((12)(34)) = 2$, and $i((123)) = 2$, so i has the same value on permutations in different classes. The matrix $M_{P,Q}(q)$ is real, symmetric for real q, provided, as was shown above, $i(P) = i(P^{-1})$. This symmetry, together with the dependence of M_n on PQ^{-1} only, shows that the rows and columns of M_n are permutations of each other. Note that $M_{PR,QR}(q) = M_{P,Q}(q)$, but $M_{RP,RQ}(q) \neq M_{P,Q}(q)$ in general.

Explicit calculations which I give below show that in the open set $-1 < q < 1$ all representations of the symmetric group occur for two and three particles, with the symmetric (antisymmetric) representations more heavily weighted as q approaches one (negative one). In this range of q all norms are positive[17–19] so all representations occur for all n, and very likely the statements about the weighting of the representations also hold for all n. Thus quon theories allow small violations of Bose or Fermi statistics. The problem of negative squared norms which occurs for paron theories (based on a deformation of parastatistics) does not arise for quons.

For each eigenvalue $\lambda(q)$ and eigenvector v_+ there is an eigenvalue $\lambda(-q)$ and eigenvector v_-. Using the fact mentioned above that the inversion number $i(P)$ is even or odd depending on whether the permutation P is even or odd, one can choose a basis so that the matrix M_n is in two-by-two block form with the diagonal blocks having even powers of q and the off-diagonal blocks having odd powers of q. Each block is an $n!/2 \times n!/2$ matrix. Write the eigenvectors in the corresponding block form as $v_+ = (v_+^{(1)}, v_+^{(2)})$ and $v_- = (v_-^{(1)}, v_-^{(2)})$, where these are column vectors, although I wrote them here as row vectors to save space. Let J be the diagonal matrix with 1's in the first diagonal block and -1's in the second diagonal block. Then using the eigenvector equations $M_\pm v_\pm = \lambda_\pm(q)v_\pm$ and $JM_n(q)J = M_n(-q)$, we find that if there is an eigenvector $v_+(q) = (v^{(1)}(q), v^{(2)}(q))$ with eigenvalue $\lambda(q)$ then there is also an eigenvector $v_-(q) = (v^{(1)}(-q), -v^{(2)}(-q))$ with eigenvalue $\lambda(-q)$. For $n = 2, 3$ the eigenvectors are independent of q. This argument can be given in the context of the group algebra by defining J to be an element (not in the group algebra) whose square is the identity and which commutes with even permutations and anticommutes with odd permutations, i.e., $J^2 = e$, $JPJ = \delta_P P$, where δ_P is the signature of P and e is the identity of S_n. Then, again, $JM_n(q)J = M_n(-q)$ and the eigenvalue equation $M_n(q)v_\pm(q) = \lambda_\pm(q)v_\pm(q)$, where now $v_+(q) = \sum_P c(P,q)P$, leads to the pair of eigenvalues and eigenvectors $(\lambda_+(q), v_+(q))$ and $(\lambda_-(q), v_-(q))$, where $v_-(q) = \sum_P \delta_P c(P,-q)P$ and $\lambda_-(q) = \lambda_+(-q)$. One can easily find the eigenvectors and eigenvalues for the symmetric and antisymmetric representations for any n. The eigenvectors are $s = \sum_P P$ and $a = \sum_P \delta_P P$ and the eigenvalue for the antisymmetric representation follows from

$$M_n(q)a = \sum_P q^{i(P)}P\sum_Q \delta_Q Q = \sum_{P,Q} q^{i(P)}\delta_Q PQ = \sum_{P,R} q^{i(P)}\delta_{P^{-1}R}R = \left(\sum_P \delta_P q^{i(P)}\right)\sum_R \delta_R R,$$

where I used the facts that the character of a permutation equals the character of its inverse, which is true in general, and the character of a product of permutations is the product of the characters, which is true only for the symmetric and antisymmetric representations. The argument for the eigenvalue of the symmetric representation is similar. Thus the eigenvalues are $\sum_P q^{i(P)}$ and $\sum_P \delta_P q^{i(P)}$ for s and a, respectively. The discussion of the inversion table given above shows that these eigenvalues can be factored into $\lambda_s = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$ for s in S_n and $\lambda_a = (1-q)(1-q+q^2)\cdots(1-q+\cdots\pm q^{n-1})$ for a in S_n. These calculations can also be done using the matrices.

Since I showed that for $q = 0$, $M_n(q) = \mathbf{1}$, which is positive definite, a demonstration that $\det M_n(q)$ has no zeroes for q in $-1 < q < 1$ suffices to prove that M_n is positive definite in this interval. It also suffices to see that the eigenvalues of M are positive. For $n = 2, 3$, the eigenvalues which I give below are indeed positive in this range.

It takes less space to write the matrices as elements of the group algebra of the relevant S_n rather than explicitly as matrices. Thus $M_2 = e + q(12)$, $M_3 = e + q((12)+(23)) + q^2((123)+(132)) + q^3(13)$, and the general case is $M_n = \sum q^{i(P)}P$. Here e is the identity and the permutations act on places, rather than on specific numbers. I label the representations by their Young tableaux $(l_1, \ldots, l_j)$, where the l's are the number of boxes in each row. For $n = 2$, only the symmetric and antisymmetric representations occur; the representations, eigenvalues and eigenvectors are

$$(2):\quad 1+q;\quad e+q(12), \tag{12}$$

$$(1,1):\quad 1-q;\quad e-q(12). \tag{13}$$

For $n = 3$, the symmetric, mixed, and antisymmetric representations occur. The eigenvectors and eigenvalues are

$$(3):\quad (1+q)(1+q+q^2);\quad \sum P, \tag{14}$$

$$\begin{aligned}&(2,1):\quad (1+q)^2(1-q);\\ &e-(132)-(13)+(23),\\ &e+(132)-2(123)-(13)-(23)+2(12);\end{aligned} \tag{15}$$

$$\begin{aligned}&(2,1):\quad (1+q)(1-q)^2;\\ &e-(132)+(13)-(23),\\ &e+(132)-2(123)+(13)+(23)-2(12);\end{aligned} \tag{16}$$

$$(1,1,1):\quad (1-q)(1-q+q^2);\quad \sum \delta(P)P. \tag{17}$$

Note that eigenvectors belonging to the representation $(2,1)$ occur with two different eigenvalues. The association of eigenvalues and eigenvectors for q and $-q$ proved above is illustrated here.

To go to $n = 4$ requires dealing with a 24-dimensional group algebra or with 24×24 matrices (of course these can be reduced to irreducible components which are smaller). It is useful to have a recursion which goes from S_{n-1} to S_n. An obvious recursion is to multiply S_{n-1} on either the left or right by the set of permutations $P_n = \{e, (n-1,n), (n-2,n), \ldots, (1n)\}$. Since each transposition can be written as a product of transpositions on neighboring places,

$$(n-2,n) = (n-2,n-1)(n-1,n)(n-1,n-2),\ldots,$$

$$(1n) = (1,2)(2,3)\cdots(n-2,n-1)(n-1,n)(n-1,n-2)\cdots(3,2)(2,1),$$

the recursion $S_n = P_n S_{n-1}$ can be written $S_n = \{e, (n-1,n), (n-2,n-1,n), \ldots, (12\cdots n-1,n)\}S_{n-1}$, where I used the fact that the transpositions which do not depend on n leave S_n invariant, and I have combined the successive transpositions into cycles. The analogous result holds for $S_n = S_{n-1}P_n : S_n = S_{n-1}\{e, (n,n-1), (n,n-1,n-2), \ldots, (n,n-1,\ldots,21)\}$. Since the inversion numbers of the cycles are $(0,1,2,\ldots,n-1)$ in each case, these recursions lead to two types of factorizations of M_n in terms of M_{n-1} :

$$M_n(q) = [e + (n-1,n) + \cdots + (12\cdots n)]M_{n-1}(q), \tag{18}$$

$$M_n(q) = M_{n-1}(q)[e + (n,n-1) + \cdots + (n,n-1,\ldots,1)]. \tag{19}$$

Thus starting from M_2 there are 2^{n-2} such factorizations of M_n. Factorizations of this kind were given by Zagier[17] and Fivel.[19]

Since the matrix $M_n(q)$ is positive definite for $q = 0$,[15] it will remain positive definite in $-1 < q < 1$ if $\det M_n(q)$ has no zeros there.

In an elegant paper, Zagier[17] has proven that

$$\det M_n(q) = \prod_{k=1}^{n-1} (1 - q^{k(k+1)})^{(n-k)n!/k(k+1)}. \tag{20}$$

Since the only zeros in (20) are roots of unity, Zagier's result proves the positive definiteness in $-1 < q < 1$. (Fivel's paper seems to imply that the zeros occur for $q^{2j} = 1$, j integral. It is not clear whether there is a conflict between the results of Zagier and Fivel.)

Both Zagier and Fivel have given arguments that the proof of positivity in the general case in which any number of particles is in the same state follows from positivity in the case in which all particles are distinct. One can go further and give an explicit formula for the case of more than one particle in the same state in terms of matrix elements in which all particles are different. The scalar product between states in which the operators $a_m^\dagger$ in any positions occur n_m times is $(\Pi_m n_m!)^{-1}$ times the scalar product in which each set of operators which occur more than once is replaced by a symmetric sum of operators with distinct labels. This follows directly from the argument given above for $i(P)$ as the number of crossings of lines connecting $(1,\ldots,n)$ with $(P1,\ldots,Pn)$.

To summarize, all irreducible representations of S_n have positive $(\text{norm})^2$ in this interval. As $q \to \pm 1$ the more symmetric (antisymmetric) irreducibles occur with higher weight. At the end points, $q = \pm 1$, only the symmetric (antisymmetric) representation survives.

For the case $q = 0$, Eq. (7) is an explicit formula for the number operator n_k which has the usual commutation relations $[n_k, a_l]_- = -\delta_{k,l} a_l$ with the annihilation (and creation) operators. It is no surprise that the number operator remains an infinite series in the a's and $a^\dagger$'s for all q in $-1 < q < 1$. Away from $q = 0$, I do not know a closed form for the number operator; however, the first terms in the series are

$$n_{kl} = a_k^\dagger a_l + (1-q^2)^{-1} \sum_t (a_t^\dagger a_k^\dagger - q a_k^\dagger a_t^\dagger)(a_l a_t - q a_t a_l) + \cdots . \tag{21}$$

Here I gave the transition number operator n_{kl} for $k \to l$ since this takes no extra effort. Zagier[17] has conjectured a general formula for n_k.

V. OPERATORS WHICH CREATE OR ANNIHILATE A PARTICLE IN A GIVEN PLACE

For Bose and Fermi statistics, the place in which a creation operator stands in a state can at most produce a change in sign; however for quon statistics, as pointed out above, there is no bilinear relation between two creation (or two annihilation) operators, so the place in which a creation operator stands is significant. This reflects the fact that quons can belong to many-dimensional representations of the symmetric group. To construct operators which create or annihilate particles in a given place for general q is complicated. Here I discuss only the $q = 0$ case for which explicit formulas are simple. The general case is qualitatively the same.

Clearly, $a_k^\dagger$ creates a particle in quantum state k in the leftmost position in the state. One can insert $a_k^\dagger$ in the jth place from the left using

$$A_k^{j,\dagger} = \sum_{t_1\cdots t_{j-1}} a_{t_1}^\dagger \cdots a_{t_{j-1}}^\dagger a_k^\dagger a_{t_{j-1}} \cdots a_{t_1}. \tag{22}$$

One can annihilate a particle in quantum state k (if there is one) in the jth position from the left in a state using

$$A_k^j = \sum_{t_1\cdots t_{j-1}} a_{t_1}^\dagger \cdots a_{t_{j-1}}^\dagger a_k a_{t_{j-1}} \cdots a_{t_1}. \tag{23}$$

One can also annihilate whatever particle happens to stand in the jth position from the left using

$$\sum_k A_k^j = \sum_{t_1\cdots t_j} a_{t_1}^\dagger \cdots a_{t_{j-1}}^\dagger a_{t_j} a_{t_{j-1}} \cdots a_{t_1}. \tag{24}$$

For general q in $-1 < q < 1$ there are analogous formulas for all of these, but they are complicated; I do not know a closed form for them.

VI. USE OF THE $q=0$ OPERATORS AS STANDARD BUILDING BLOCKS FOR THE GENERAL-q OPERATORS

One can use the $q = 0$ operators as building blocks to construct a representation of the quon operators in $-1 < q < 1$. This can lead to an alternative way to prove the positivity of the norms, since the $q = 0$ operators are known to make states which always have positive norm. Let $|0\rangle_0$ be the vacuum of the $q = 0$ case, b_i, $b_i^\dagger$ be the $q = 0$ operators and $T(q)$ be a map from the Hilbert space of the $q = 0$ operators to the Hilbert space of the operators a_i, $a_i^\dagger$ for some value of q in $-1 \leq q \leq 1$. The representation I conjecture has the form

$$a_i = T(q)\left(b_i + \sum_l \sum_P c^{(2)}(P) b_l^\dagger P b_l b_i + \cdots + \sum_{l_1,l_2,\cdots,l_{n-1}} b_{l_{n-1}}^\dagger \cdots b_{l_1}^\dagger \sum_P c^{(n)}(P) P b_{l_1} \cdots b_{l_{n-1}} b_i + \cdots\right) T^{-1}(q), \tag{25}$$

where the sum goes over the $n!$ permutations of $b_{l_1} \cdots b_{l_{n-1}} b_i$. The coefficients, I conjecture, are independent of the labels of the states and are sums of square roots of polynomials in q. I have worked out the first nontrivial case of this formula:

$$a_k^\dagger = T\left(b_k^\dagger \Lambda_0 + \sum_t \{c(2)[b_k^\dagger, b_t^\dagger]_+ + c(1,1)[b_k^\dagger, b_t^\dagger]_-\}\Lambda_0 b_t + \cdots\right) T^{-1}, \tag{26}$$

where Λ_0 is the projection onto the vacuum of the $q = 0$ operators which can be expressed as $\sin\pi N_0/\pi N_0$, where N_0 is the $q = 0$ number operator, and there is a corresponding formula for the annihilation operators. The coefficients are $c^2(2) = (1+q)/4$ and $c^2(1,1) = (1-q)/4$, which suggest that the coefficients of higher terms will be related to the square roots of the eigenvalues of $M_n(q)$. The fact that the coefficients are independent of the labels of the $b^\dagger$'s and b's is a great simplification. The following similar representation results if one does not use the vacuum projector:

$$a_k^\dagger = T\left(b_k^\dagger + \tfrac{1}{2}(\sqrt{1+q} - 1)\sum_t [b_k^\dagger, b_t^\dagger]_+ b_t + \tfrac{1}{2}(\sqrt{1-q} - 1)\sum_t [b_k^\dagger, b_t^\dagger]_- b_t + \cdots\right) T^{-1}. \tag{27}$$

These expressions are normal ordered expansions in the $q = 0$ b and $b^\dagger$ operators. The expansions (26) and (27) can be used to calculate the normal-ordered expansion of the product of two $a^\dagger$'s (from now on, I suppress the operators T, T^{-1}):

$$a_k^\dagger a_l^\dagger = \tfrac{1}{2}\sqrt{1+q}[b_k^\dagger, b_l^\dagger]_+ + \tfrac{1}{2}\sqrt{1-q}[b_k^\dagger, b_l^\dagger]_- + \cdots . \tag{28}$$

The two terms in (28) exhibit the expected symmetry or antisymmetry of the product of two $a^\dagger$'s (in the two-particle sector) when $q \to 1$ or $q \to -1$. The inversion of the formula for $a^\dagger$ in terms of $b^\dagger$ and b:

$$b_k^\dagger = a_k^\dagger + \tfrac{1}{2}[(1+q)^{-1/2} - 1]\sum_t [a_k^\dagger, a_t^\dagger]_+ a_t + \tfrac{1}{2}[(1-q)^{-1/2} - 1]\sum_t [a_k^\dagger, a_t^\dagger]_- a_t + \cdots \tag{29}$$

shows that the inversion fails at $q = \pm 1$.

One can also use the $q = 0$ operators to construct the matrix $M_n(q)$ as follows: define

$$A_i^\dagger = b_i^\dagger + q\sum_t b_t^\dagger b_i^\dagger b_t + q^2 \sum_{t_1,t_2} b_{t_2}^\dagger b_{t_1}^\dagger b_i^\dagger b_{t_1} b_{t_2} + \cdots = \sum_{n=0}^{\infty} q^n \sum_{t_1,\cdots,t_n} b_{t_n}^\dagger \cdots b_{t_1}^\dagger b_i^\dagger b_{t_1} \cdots b_{t_n}. \tag{30}$$

Then

$$A_{Pn}^\dagger \cdots A_{P1}^\dagger|0\rangle = \sum_Q [M_n(q)]_{P,Q} b_{Qn}^\dagger \cdots b_{Q1}^\dagger|0\rangle. \tag{31}$$

The matrix which constructs $A_k^\dagger$ in terms of the $b^\dagger$'s and b's is essentially the square root of M_n. The norm squared of the $A^\dagger$ states (which is $[M_n(q)]^2$) is clearly non-negative, since the $b^\dagger$'s live in a positive metric space.

VII. THE TCP THEOREM AND CLUSTERING HOLD DESPITE THE FAILURE OF LOCALITY

Fredenhagen[35] showed that a theory which has neither para-Bose (including Bose) nor para-Fermi (including Fermi) statistics cannot be local, where local means both (a) that the observables are pointlike functionals of the fields and (b) that the observables commute at spacelike separation [i.e., satisfy local commutativity (LC)]. For the $q = 0$ case, I showed, in agreement with Fredenhagen's result, that the fields associated with infinite statistics are not local.[15] There is no reason to expect that the situation is more favorable for $q \neq 0$, and, indeed, Fredenhagen's result rules out this possibility. It is worthwhile to remark that the locality requirement (b) can hold when requirement (a) fails. For example, in a theory with a charged field the total charge $Q = \int d^3x\, j^0(x)$ does not satisfy (a) and neither does the product $J^\mu(x) = Qj^\mu(x)$, but the product $J^\mu(x)$ still satisfies (b).

It is amusing to note that, despite the failure of LC, the TCP theorem is valid for free fields which obey q-mutator relations.[20] Let the q-mutator relations for annihilation and creation operators for particles, b and $b^\dagger$, and antiparticles, d and $d^\dagger$, be $b_k b_l^\dagger - q b_l^\dagger b_k = \delta_{kl}$, $b_k d_l^\dagger - q d_l^\dagger b_k = 0$ and the same for b and d interchanged. Again no commutation relations are assumed between two annihilation or between two creation operators. As before, assume that the vacuum is annihilated by the annihilation operators. Construct free, charged, scalar, quon fields from these annihilation and creation operators in the same way free, charged, scalar, Bose fields are constructed from the Bose annihilation and creation operators. In momentum space, the quon fields are

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} (b_k e^{-ik\cdot x} + d_k^\dagger e^{ik\cdot x}), \qquad (32)$$

$$\phi^\dagger(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} (d_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}), \qquad (33)$$

$\omega_k = k^0 = \sqrt{\mathbf{k}^2 + m^2}$. In terms of charged scalar fields, these relations for the annihilation and creation operators translate into

$$\phi^{(+)}(x)\phi^{\dagger\,(-)}(y) - q\phi^{\dagger\,(-)}(y)\phi^{(+)}(x) = \Delta^{(+)}(x-y), \qquad (34)$$

$$\phi^{\dagger\,(+)}(x)\phi^{(-)}(y) - q\phi^{(-)}(y)\phi^{\dagger\,(+)}(x) = \Delta^{(+)}(x-y). \qquad (35)$$

The vacuum conditions are

$$\phi^{(+)}(x)|0\rangle = 0, \; \phi^{\dagger\,(+)}(x)|0\rangle = 0. \qquad (36)$$

Before discussing general properties of the vacuum matrix elements of products of these field, I give a few simple examples. For a vacuum matrix element to be nonvanishing, the product must have the same number of ϕ and $\phi^\dagger$ fields. The nonvanishing two-point functions are $\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \Delta^{(+)}(x-y)$, $\langle 0|\phi^\dagger(x)\phi(y)|0\rangle = \Delta^{(+)}(x-y)$, $\Delta^{(+)}(x-y) = (2\pi)^{-3}\int d^3k(2\omega_k)^{-1}\exp[-k\cdot(x-y)]$. The nonvanishing four-point functions are

$$\begin{aligned}\langle 0|\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\phi^\dagger(x_4)|0\rangle &= \langle 0|\phi(x_1)\phi^\dagger(x_4)|0\rangle\langle 0|\phi(x_2)\phi^\dagger(x_3)|0\rangle \\ &\quad + q\langle 0|\phi(x_1)\phi^\dagger(x_3)|0\rangle\langle 0|\phi(x_2)\phi^\dagger(x_4)|0\rangle, \end{aligned}\qquad (37)$$

$$\begin{aligned}\langle 0|\phi(x_1)\phi(x_2)^\dagger\phi(x_3)\phi^\dagger(x_4)|0\rangle &= \langle 0|\phi(x_1)\phi^\dagger(x_2)|0\rangle\langle 0|\phi(x_3)\phi^\dagger(x_4)|0\rangle \\ &\quad + \langle 0|\phi(x_1)\phi^\dagger(x_4)|0\rangle\langle 0|\phi^\dagger(x_2)\phi(x_3)|0\rangle, \end{aligned}\qquad (38)$$

$$\begin{aligned}\langle 0|\phi(x_1)\phi^\dagger(x_2)\phi^\dagger(x_3)\phi(x_4)|0\rangle &= \langle 0|\phi(x_1)\phi^\dagger(x_2)|0\rangle\langle 0|\phi^\dagger(x_3)\phi(x_4)|0\rangle \\ &\quad + q\langle 0|\phi(x_1)\phi^\dagger(x_3)|0\rangle\langle 0|\phi^\dagger(x_2)\phi(x_4)|0\rangle, \end{aligned}\qquad (39)$$

together with these which follow by taking the adjoint. The two-point function satisfies not only weak local commutativity (WLC) but also LC from the Källén-Lehmann representation which does not depend on commutation relations. For a general $2n$-point function, WLC (and, by Jost's theorem,[36] also TCP) follows from a combinatorial argument. The $2n$-point function is a sum of products of n two-point functions. Aside from powers of q, this sum is the same as for the usual scalar field theory which obeys WLC. To calculate the power of q belonging to a given term in the sum, draw the points $x_1, x_2, \ldots, x_{2n}$ on a line. Above the line, draw directed paths for each contraction $i\Delta(x_i - x_j)$ which occurs. For charged fields, of course, contractions can only occur between a field and its adjoint. The number of crossings of these paths give the power of q. WLC holds because the same contractions occur and, since the number of crossings remains the same when all the paths are reversed, the same powers of q occur. Thus TCP holds. As usual, this argument extends directly to arbitrary spin fields. Note that there is no spin-TCP constraint analogous to the spin-statistics theorem.

To prove clustering, consider a vacuum matrix element with n fields and n adjoint fields. Translate all the fields and adjoint fields which occur to the right of the jth field by the same translation:

$$\begin{aligned}&\langle 0|\phi^{()}(x_1)\cdots\phi^{()}(x_j)U(a)\phi^{()}(x_{j+1})\ldots\phi^{()}(x_{2n})|0\rangle \\ &= \langle 0|\phi^{()}(x_1)\cdots\phi^{()}(x_j)\phi^{()}(x_{j+1}+a)\ldots\phi^{()}(x_{2n}+a)|0\rangle, \end{aligned}\qquad (40)$$

where the open parentheses stand for an adjoint sign or for no adjoint sign. Clustering requires that for large spacelike a the expression in (40) should approach the product of two vacuum matrix elements with the extra $|0\rangle\langle 0|$ inserted between the translated and untranslated fields. The graphical argument just given shows that the expansion of (40) in sums of products of two-point functions can be separated into two types of terms: (a) those with factors containing a contraction between a translated and an untranslated field and (b) those without such factors. Terms of type (a) vanish in the limit, because $\Delta^{(+)}(x)$ vanishes for large spacelike a. (This limit is taken in the sense of distribution theory.) It remains to show that the sum of terms of type (b) factors into the product of the two vacuum matrix elements which are the desired limit. To see that this is true, consider reducing the vacuum matrix element of (40) by separating all the fields into annihilation (positive-frequency) and creation (negative-frequency) parts. The creation part of the leftmost field annihilates the vacuum acting to the left. The annihilation part of the leftmost field passes through the creation part of its immediate neighbor, possibly producing a contraction, and gets a factor of q according to the commutation rules (34) and (35). The contraction term, if nonzero, multiplies a vacuum matrix element with $2n-2$ fields. The remaining vacuum matrix element of $2n$ fields has the product of the annihilation parts of the first two fields from the left in their original order. Continue this procedure, always moving annihilation parts of fields to the right, possibly generating contraction terms, and, where possible, annihilating the vacuum on the left with creation parts until all creation parts of untranslated fields have annihilated the vacuum on the left or have contracted with annihilation parts. The sums of products of two-point functions generated in this way are precisely those which would occur in calculating the vacuum matrix element of the untranslated fields and this entire sum will multiply the vacuum matrix element of the translated fields. The remaining $2n$-point vacuum matrix element consists of the annihilation parts of the untranslated fields in their original order and the translated fields, also in their original order. Because $\Delta^{(+)}(x_i - x_j) \to 0$, for x_i belonging to an untranslated field and x_j belonging to a translated field (as remarked above), any contractions of the annihilation parts of the untranslated fields with the translated fields will vanish in the spacelike limit, so this remaining $2n$-point matrix element will vanish in the limit. Thus clustering holds. This property is important, since it is necessary and sufficient for the vacuum of the theory to be unique.

As mentioned above, Wick's theorem holds for quon fields, provided proper powers of q are supplied for the terms. The normal-ordered terms have all creation operators to the right and all annihilation operators to the left, just as in the usual case; however, for quon fields the original relative order of the creation operators (and separately, of the annihilation operators) among themselves must be preserved. For a term with no contractions, the power of q is the inversion number of the permutation from the original order to the normal-ordered form. For a term with contractions, the power of q associated with a given contraction is the inversion number of the permutation which brings the contracted pair next to each other (with the annihilation operator to the left of the creation operator). The contracted pair is then removed from the product and the procedure is iterated. The power of q associated with contraction of a given pair depends on the order in which the pairs are contracted; however the total power of q in the normal-ordered expansion does not depend on the order in which the contractions are carried out.

It is straightforward to generalize these results about the TCP theorem, clustering, and Wick's theorem to a collection of fields of arbitrary spin.

The reader may wonder what happens to some of the theorems of local relativistic quantum field theory. Many of these theorems are evaded, because the quon fields are neither local or antilocal at spacelike separation. For example, the theorem of Federbush and Johnson,[37] Jost,[38] and Schroer[39](see also Refs. 40 and 41) that if a field has a Källén-Lehmann weight concentrated at a single mass (in other words, if the field has a free two-point function), then this field is a free field of that mass, requires that the field be local or antilocal. Thus this theorem does not hold for quon fields.

VIII. EXPERIMENTS

The simplest way to detect small violations of statistics is to find a state which either Fermi or Bose statistics would not allow. For Fermi (Bose) statistics, this would be a state in which identical particles are not totally antisymmetric (symmetric). The path-breaking high-precision experiment of Ramberg and Snow[42] searches for transitions to a state in which the electrons of the copper atom are not totally antisymmetric. The failure to detect such transitions (above background) leads to the following upper bound on violation of the exclusion principle:

$$\rho_2 = \tfrac{1}{2}(1-\beta^2)\rho_a + \tfrac{1}{2}\beta^2\rho_s, \quad \tfrac{1}{2}\beta^2 \le 1.7\times 10^{-26}, \qquad (41)$$

ρ_2 is the two-electron density matrix, $\rho_{a(s)}$ is the antisymmetric (symmetric) two-electron density matrix. For two electrons in different states ρ_2 can be expressed in terms of q of the q mutator as

$$\rho_2 = \tfrac{1}{2}(1-q)\rho_a + \tfrac{1}{2}(1+q)\rho_s, \qquad (42)$$

so the Ramberg-Snow bound is

$$0 \le (1+q)/2 \le 1.7\times 10^{-26}. \qquad (43)$$

A high-precision experiment to detect or bound violations of the exclusion principle for electrons in helium is being conducted by Kelleher *et al.*[43] The analysis[30] of the search for β decays to nuclear states which violate the exclusion principle as providing bounds on violations of the exclusion principle seems to assume violation of conservation statistics (as remarked in Ref. 29). The absence of $A=5$ nuclei provides evidence for the validity of the exclusion principle for nucleons. For bosons, one looks for transitions to states in which the bosons are not totally symmetric. For pions, the decay $K_L^0 \to \pi^+\pi^-$ which is usually interpreted as due to CP violation could occur without CP violation if there is a small violation of generalized Bose statistics for the pions.[5] The assumption that CP violation in K^0 decay comes from violation of statistics implies $(1-q_B)/2 \le 0.5\times 10^{-6}$. All the experimental bounds on violation of statistics given in Ref. 5 translate into bounds on q: for small violations of Fermi statistics,

$$\beta^2/2 = (1+q)/2; \tag{44}$$

for small violations of Bose statistics,

$$\beta_B^2/2 = (1-q_B)/2. \tag{45}$$

I conclude this brief discussion of experimental bounds on small violations of statistics by remarking that there are three types of such experiments: (1) to detect an accumulation of particles in anomalous states, (2) to detect transitions to anomalous states, and (3) to detect deviations from the usual statistical properties of many-particle systems. Here and in Ref. 5 type (2) experiments are discussed, because they allow detection of single transitions to anomalous states. Type (1) experiments require detection of a small concentration of anomalous states in a macroscopic system; for that reason they are generally less sensitive than type (2) experiments. I have not analyzed type (3) experiments; however it seems likely that they will fail to provide high-precision tests for the same reason that type (1) experiments fail: it will be difficult to detect the modification of the statistical properties of a macroscopic sample due to a small concentration of anomalous states.

IX. SUMMARY AND OUTLOOK

Quantum field theory based on q-mutators in the range $-1 < q < 1$ is the first theory that allows small violations of the exclusion principle (i.e., of Fermi statistics) or of Bose statistics. This theory is a valid nonrelativistic field theory. The theory can have relativistic kinematics and at least the free field theories obey the TCP theorem and clustering; however the theory is not local in the sense that the observables are pointlike functionals of the fields and that they obey spacelike commutativity. Thus its status as a relativistic field theory is in doubt.

ACKNOWLEDGMENTS

I thank Don B. Zagier for stimulating discussions and for informing me of his results prior to publication. I have benefited from conversations with I. Bakas, D.I. Fivel, P.G.O. Freund, R.N. Mohapatra, S. Nussinov, Y. Shamir, G.A. Snow, J. Sucher, L.J. Swank, and C.-H. Woo. The author was supported in part by the National Science Foundation.

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