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Renormalization in the light-front Tamm-Dancoff approach to field theory

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It has recently been proposed that light-front field theory provides a powerful tool for the analysis of relativistic bound states when one makes a Tamm-Dancoff truncation. Such a truncation introduces nonlocalities that require an unfamiliar nonlocal renormalization procedure, in which counterterms are allowed to depend on the sectors of Fock space within which or between which they act. In this paper we illustrate the simplest features of the light-front Tamm-Dancoff approach using the Yukawa model in $1+1$ dimensions.

I. INTRODUCTION

Since the inception of relativistic field theory, physicists have sought nonperturbative tools for its study. Ultraviolet infinities plagued attempts to borrow Hamiltonian and other methods developed for the study of nonrelativistic quantum mechanics, and little progress was made outside of perturbation theory until the 1970s. With the recognition that gauge theories are asymptotically free and the development of lattice gauge theory, it was hoped that rapid progress might be made in understanding hadronic physics using a fundamental theory, quantum chromodynamics (QCD). Although progress has been made, lattice calculations are still far from able to provide accurate information about light-quark bound states, and the possibility of doing nuclear physics on the lattice is not foreseeable. One must either wait for more powerful computers, resort to phenomenological approximations to QCD, or seek alternatives to lattice gauge theory.[1] The light-front Tamm-Dancoff (LFTD) approximation has been proposed[2] both as a promising alternative to the lattice and as a fertile ground for the development of phenomenology.

The purpose of this article is to illustrate some of the principal features of the LFTD approximation, in particular, the simplest features of the new renormalization program required by this approach. Little work has addressed renormalization in light-front field theory, and most of it is devoted to establishing a connection with covariant perturbation theory.[3–6] Tang is the only author who has discussed renormalization after a Tamm-Dancoff truncation.[7] Although renormalization of time-ordered perturbation theory in the infinite-momentum frame,[8,9] or in covariant perturbation theory using light-front coordinates,[10] is related to direct renormalization in light-front perturbation theory, there are important differences.[11,12]

The Tamm-Dancoff approximation was originally proposed by Tamm[13] in 1945, and independently discovered by Dancoff[14] in 1950. Simply stated, one tries to solve a second-quantized relativistic Schrödinger equation in a truncated Fock space. Tamm and Dancoff considered the possibility of studying the deuteron by keeping only states with up to one pion added to a neutron and proton. By making severe approximations they showed that, in the nonrelativistic limit, this approach leads to a nonrelativistic Schrödinger equation with a Yukawa potential. Unfortunately, further study in the 1950s (for a historical review, see Ref. 15) led to the conclusion that ultraviolet problems prevent this method from being of any practical use.[16–20] The ultraviolet problems are of two types. First, there are divergences associated with the vacuum. Second, the truncation of Fock space seriously violates Lorentz invariance and, in particular, boost invariance. This, in turn, leads to nested noncovariant divergences which, at the time, were believed to be insurmountable. The use of momentum cutoffs further violates Lorentz invariance.

The LFTD approximation, in its simplest form, is just the Tamm-Dancoff approximation applied to light-front field theory instead of equal-time (i.e., instant-form) field theory. Light-front field theory originated from the work of Dirac[21] on the forms of relativistic dynamics. It is most familiar from its use in the parton model, where it allows one to utilize intuitive constituent pictures[22] that cannot be justified in instant-form field theory. In his pioneering work on the infinite-momentum frame, Weinberg[23] recognized the potential of such an approach and the possible need for a Tamm-Dancoff truncation. Since that time many physicists have recognized the virtues of the infinite-momentum frame and/or the light-front[24–28] field theory but only a small fraction of work in this area has dealt with problems beyond the tree level of field theory.[29]

The use of light-front field theory apparently removes the most severe problems of the original Tamm-Dancoff approximation. As has long been recognized,[23,25] the free vacuum is an eigenstate of the full light-front Hamiltonian; therefore, the vacuum divergences that plagued the original program seem to disappear.[30] Furthermore, the light-front boost operators do not contain interactions,[21,31] so the truncation of Fock space according to particle number will only violate rotational invariance.[32]

After making a Tamm-Dancoff truncation, there are only a finite number of degrees of freedom; however, there are still an infinite number of momentum scales. Large transverse-momentum scales lead to ultraviolet divergences, while small longitudinal momenta lead to a new class of "spurious" infrared divergences.[11] These divergences must be regulated in some fashion. There are a number of methods on the market, and we will briefly discuss the use of Pauli-Villars[33] regularization, dimensional regularization,[34] and momentum cutoffs. Clearly when no other considerations exist, one would like to regulate the theory in a manner that preserves all symmetries. Unfortunately, in the LFTD approach the aim is to employ Hamiltonian diagonalization *on a computer,* and this introduces practical constraints that favor the use of cutoffs. Even when one employs Pauli-Villars or dimensional regularization, an additional truncation of the few-body Hilbert space is required to generate a finite-dimensional Hamiltonian matrix. This additional truncation is inevitably equivalent to the introduction of momentum cutoffs, although there are a large number of ways that such cutoffs can be introduced. It is possible to introduce cutoffs *after* using Pauli-Villars or dimensional regularization, and this might minimize the effects from symmetry breaking introduced by the cutoffs. However, in the Pauli-Villars method one must introduce extra degrees of freedom and imaginary coupling constants, and in dimensional regularization one must compute Hamiltonian matrix elements in a noninteger number of dimensions. We believe that the numerical costs associated with these complications outweigh any advantage initially gained by their employment. Certainly the study of these regularization schemes should be pursued, especially for problems where the counterterms they yield can be cataloged and included by hand, as in quantum electrodynamics (QED). However, throughout this paper we will use cutoffs on the invariant mass of states to provide regularization.

There are obviously many possible momentum cutoffs that can be employed. One is free to cut off longitudinal and transverse momenta separately, and to employ unrelated cutoffs in different sectors of Fock space. If feasible, one wants to employ cutoffs that violate a minimal number of symmetries. Lorentz invariance is of particular importance in the renormalization program and will occupy most of our attention. It is easy to select cutoffs that preserve Lorentz symmetries associated with kinematic generators of the Poincaré group.[27,31] This is accomplished by choosing cutoffs that are functions of invariant momentum variables. All Poincaré generators except the Hamiltonian are kinematic in $1+1$ dimensions, so it is easy to find cutoffs that maintain Lorentz invariance. While we believe there may be good reasons to employ noncovariant cutoffs even in $1+1$ dimensions, the considerations leading us to this conclusion are beyond the scope of this article; therefore, we will employ a covariant cutoff throughout this work. In $3+1$ dimensions there are boost-invariant cutoffs, but there is no covariant cutoff because the Poincaré generators of macroscopic rotations about transverse axes contain interactions. In particular, one can easily see that the covariant cutoff on the invariant mass of intermediate states,[35] which we employ here, violates rotational invariance even though it preserves boost invariance. This is most easily seen by looking at a process such as Compton scattering where one can readily find examples in which observers in frames related by a rotation about a transverse axis do not agree on whether the cutoff removes scattering for a given set of external momenta. On the other hand, when one considers only those states which contain a single particle, such as isolated stable bound states, the equations of motion are completely independent of the total momentum when one uses this cutoff.

The goal of nonperturbative renormalization is to identify counterterms that allow the truncated Hamiltonian to yield correct results. If the Tamm-Dancoff truncation or the additional cutoffs spoil locality and covariance, the counterterms must be nonlocal and noncovariant to restore these properties to observables.

The LFTD approach closely resembles an approach advocated by Susskind[25] and his collaborators, and also discretized light-cone quantization (DLCQ). DLCQ was originally applied to the study of the Yukawa model in $1+1$ dimensions by Pauli and Brodsky[36] and has subsequently been employed to study the Schwinger model,[37,38] $(1+1)$-dimensional (QCD) (Refs. 39 and 40) and QED_{3+1}.[7] While a Tamm-Dancoff truncation is not employed in the original work on the $(1+1)$-dimensional Yukawa model, it is almost inevitably employed in all work on gauge theories. Gauge particles in the light-cone gauge in $1+1$ dimensions are not dynamical, so the Tamm-Dancoff truncation affects only fermion-antifermion pairs. There are no divergences in these theories, and it has been shown that composite ground-state wave functions are dominated by the lowest Fock-space component.[41,39] As a result, there has been little need to discuss many aspects of renormalization in these works. In QED_{3+1}, Tang has studied the lowest-order Tamm-Dancoff approximation.[7] He computes the electron-mass counterterm required at this order using the DLCQ approach in the charge-one sector, and discusses the corresponding counterterm in bound-state calculations. Many of the general principles involved in nonperturbative renormalization, including several examples of the effects of renormalization on wave functions, are given by Lepage, Brodsky, Huang, and Mackenzie.[35]

Let us turn to an outline of the paper. We have relegated details concerning light-front variables, quantization, the $(1+1)$-dimensional Yukawa Hamiltonian, etc., to Appendix A. In order to set the stage for the LFTD approach, we compute the second- and fourth-order shifts in the mass of the fermion (ignoring antifermions) in the second section. The natural equation to solve on the light front is Einstein's equation $P^2=2P^+P^-=M^2$. Stationary-state perturbation theory is better suited to the problems in which we are interested than X^+-ordered perturbation theory is, and we utilize both the light-front version of Brillouin-Wigner (LFBW) perturbation theory and the light-front version of Rayleigh-Schrödinger (LFRS), or standard, perturbation theory. The instant-form version of the $(1+1)$-dimensional Yukawa model indicates that it should be possible to avoid all divergences in quenched perturbation theory, but we encounter several divergences in the light-front version of the theory. We show that these "spurious" divergences are removed by the proper treatment of the fermion self-inertia and various instantaneous interactions. The need for nonlocal counterterms whose action depends on the sector of Fock space becomes clear at this point.

A sequence of approximations for the physical fermion in which an increasing number of virtual bosons are allowed is used to illustrate the LFTD approach. We start in Sec. III with the first Tamm-Dancoff approximation and demonstrate Hamiltonian diagonalization, which is equivalent to solving a set of coupled integral equations.[13,14] Mass renormalization is implemented by fixing scattering thresholds, and clarified by the relation between divergences in the LFTD approach and perturbation theory. The LFTD equation for the physical fermion state is used to compute the first fermion mass counterterm. In Sec. IV we turn to the problem of two fermions of different flavor interacting via the exchange of one boson. We derive the LFTD integral equations, and show that the boson exchange is accompanied by the "dressing" of the fermions. We show that the counterterms computed in the charge-one sector remove spurious divergences and properly fix the two-fermion scattering threshold below the boson production threshold when the cutoff is taken to its limit. After removing cutoffs we obtain finite, covariant, unitary equations for two-fermion–one-boson scattering, and for the two-fermion scattering, or bound-state, problem. In Appendix C we show how one can take the nonrelativistic limit of such equations, in particular, demonstrating that when the fermions become infinitely massive, the two-fermion problem reduces to a Schrödinger equation for a nonrelativistic particle moving in a Yukawa potential.

In Sec. V we return to the charge-one sector of the theory and allow up to two virtual bosons to dress the fermion. This is the first point at which one begins to see the complete nonperturbative nature of the LFTD theory. Renormalization of the one-fermion–two-boson sector is trivial, and renormalization of the one-fermion–one-boson sector is accomplished by the mass counterterm originally discovered in Sec. III and now ported to a new sector of Fock space. It is not possible to eliminate both sectors containing bosons and write a simple equation for the physical fermion because such an elimination requires the inversion of several integral operators. However, we are able to demonstrate that all spurious divergences are canceled and that the final eigenvalue equation is finite and covariant after cutoffs are removed.

Section VI contains our conclusions. We speculate without proof that the above procedure can be generalized to any order of the LFTD approach. In each new order, additional sectors of Fock space are added, and counterterms computed in lower orders are moved into new "descendant" sectors. After this first step, one computes a new set of counterterms in the lowest sectors of Fock space by satisfying a specified set of renormalization conditions. After these steps are completed, the LFTD Hamiltonian is completely specified and one can proceed to compute additional eigenvalues and eigenstates. We do not discuss how additional counterterms (e.g., coupling constant renormalization) come into play, although it should be clear that it is necessary to allow such counterterms to depend on the sectors of Fock space within which or between which they act. A complete local, covariant field theory is recovered by a double limiting procedure in which one includes additional particles and removes cutoffs. In most of this paper we act as if such limits are going to be taken directly, but a nonperturbative calculation will require us to "condition" these limits by using renormalization-group techniques.[42,43]

II. FERMION MASS IN PERTURBATION THEORY

Consider the quantized version of Einstein's equation for states with charge one:

$$2P^+(P_M^- + P_I^-)|\Psi\rangle = M^2|\Psi\rangle \ . \tag{2.1}$$

Here P^+ is the momentum operator and we have separated the free part of the Hamiltonian P_M^- from the interactions P_I^-. If we work with momentum eigenstates we can always replace the momentum operator by its expectation value $\mathcal{P}$. Unless otherwise specified, we will do this throughout the rest of the paper. It is relatively straightforward to develop the LFBW perturbation theory for this equation. We begin with the LFBW theory rather than the LFRS expansion in the coupling constant because the LFBW theory is close to what naturally arises in the Hamiltonian matrix diagonalization. We can easily switch to the LFRS theory at any point by making a second expansion of each term in the LFBW expansion. As a specific example, we will consider the charge-one sector, although the only state in this sector directly amenable to analysis with the LFBW theory is the physical fermion. All other states are scattering states that require a Lippmann-Schwinger treatment. It should be straightforward to develop such a treatment but, for our present purposes, we will need only scattering thresholds.

Following a standard textbook derivation one finds that

$$m_F^2 - m_{F0}^2 = \left\langle \Psi_0 \left| 2\mathcal{P}P_I^- \sum_{n=0}^{\infty} \left[\frac{Q}{m_F^2 - 2\mathcal{P}P_M^-} 2\mathcal{P}P_I^- \right]^n \right| \Psi_0 \right\rangle \ . \tag{2.2}$$

The physical fermion mass is m_F while the bare value is m_{F0}. At this point it is assumed that there is a single fermion mass counterterm, so that it is sufficient to specify m_{F0}^2. The fermion is assumed to have a momentum $\mathcal{P}$. The approximate eigenstate will be that of a free bare fermion $|\Psi_0\rangle$, and the projection operator Q projects onto states orthogonal to $|\Psi_0\rangle$. At any order in the sum one can derive the perturbative result by inserting a complete set of free states (i.e., eigenstates of P^+ and P_M^-) between every interaction operator P_I^-. This is equivalent to drawing all X^+-ordered diagrams that begin and end with a single bare fermion line, but do not contain a lone fermion line in any intermediate state. A complete set of diagrammatic rules is listed in Appendix B.

The problem we want to address is how a fermion dresses itself with bosons in the quenched (i.e., no antifermions) approximation. Although this is a nonperturbative problem, we can begin by looking at second-order perturbation theory. The only diagrams that occur to this order are shown in Fig. 1. The first diagram represents a self-inertia, which we have chosen to single out as an interaction rather than include in the fermion propagator for reasons that will become obvious. In earlier work[11] we evaluated these diagrams using cutoffs that restrict fermion momenta to lie between ϵ_F and Λ_F, while boson momenta lie between ϵ_B and Λ_B. This type of cutoff is perfectly adequate in $1+1$ dimensions, but in $3+1$ dimensions a cutoff on the invariant mass of intermediate states has practical advantages,[35] so we adopt this cutoff here. The resultant regularized expression in the LFBW theory is

$$m_F^2-m_{F0}^2=\lambda^2 \wp \int_\epsilon^\infty \frac{dx}{4\pi x}\left[\frac{1}{1-x}\right] +\lambda^2 m_F^2 \int_0^1 \frac{dx}{4\pi x}\left[1+\frac{1}{1-x}\right]^2 \times\left[m_F^2-\frac{m_{F0}^2}{1-x}-\frac{m_B^2}{x}\right]^{-1} \times\Theta(\Lambda;1-x;x)\ , \qquad (2.3)$$

where

$$\Theta(\Lambda;1-x;x)=\theta\left[\Lambda^2-\frac{m_F^2}{1-x}-\frac{m_B^2}{x}\right] . \qquad (2.4)$$

We use $\wp$ to denote the Cauchy principal value. We will usually write all momenta as fractions of the total momentum. As discussed in the first section, this cutoff is Lorentz invariant, as is any cutoff in $1+1$ dimensions that depends only on momentum fractions. Since all momenta are positive and conserved, the momentum of the state under study automatically imposes upper cutoffs on the momenta of intermediate states, in particular, preventing any momentum fraction from becoming greater than one. The further constituents of a virtual particle have their momentum even more narrowly constrained to be less than the parental momentum, and so on. Since $P_M^-=m^2/(2P^+)$, the energy of successive layers of virtual particles climbs. This suppresses amplitudes with large numbers of virtual particles, in competition with the transition strength of the interactions. These statements should be clear from an examination of Eq. (2.2). The step function regulator imposes the limits of integration:

$$x_\pm=\frac{1}{2}+\frac{m_B^2-m_F^2}{2\Lambda^2} \pm\frac{1}{2}\left[1-\frac{2(m_B^2+m_F^2)}{\Lambda^2}+\frac{(m_B^2-m_F^2)^2}{\Lambda^4}\right]^{1/2} . \qquad (2.5)$$

The dynamical cutoffs provided by momentum conservation are not sufficient to regulate even the $(1+1)$-dimensional Yukawa model, as can be seen from the fact that both integrals in Eq. (2.3) would diverge without further cutoffs. The first integral represents the self-inertia, and we are free to regulate it in any fashion we please. In fact, we can simply replace the integral with whatever function of $\mathcal{P}$, λ, and m_F we want to use. The point is that this is a term in the Hamiltonian, and we are free to choose the Hamiltonian. For example, a particularly useful choice for calculations in $3+1$ dimensions is to allow the self-inertia to exactly cancel the lowest-order self-energy diagram "on shell" (i.e., when $M^2=m_F^2$). Here we have chosen to retain the functional form of the self-inertia resulting from normal ordering the Hamiltonian, and have regulated it with a cutoff on the boson momentum. We can adjust ϵ to obtain the Feynman result for the second-order mass shift. In the second integral the function $\Theta(\Lambda;1-x;x)$ cuts off the lower momentum of the boson when $x\approx 0$ and the lower momentum of the fermion when $x\approx 1$.

There are well-known problems with nonrelativistic Brillouin-Wigner perturbation theory, and they remain in the LFBW theory. They are manifest in this calculation through the presence of both m_F^2 and m_{F0}^2 in the propagator of states containing fermions. The problem is very clear in $3+1$ dimensions, where m_{F0}^2 is forced to diverge to keep m_F^2 finite. This problem is normally averted by switching to the LFRS theory, which, to second order, allows one to replace m_{F0}^2 with m_F^2 in the propagators. In higher orders one can use the perturbative relationship between m_{F0}^2 and m_F^2 to eliminate either mass in all propagators. In general, one wants to eliminate the bare mass in favor of the physical mass. This step has no effect, but one is then able to make a second perturbative expansion of the resultant propagators and drop all terms beyond the order of the original LFBW calculation. The first term resulting from this expansion of propagators is the same as the original LFBW expression, but with m_{F0}^2 replaced by m_F^2 in every propagator. The remaining terms in the expansion are "counterterms," and only those counterterms required at the given order are retained, while the counterterms dropped appear in higher-order calculations. The reason that we belabor these well-known points is that it is necessary to somehow achieve such a nesting of counterterms in the full Tamm-Dancoff approximation in order to recover perturbative results when physical couplings are small but bare parameters diverge. We will see below how nonlocal counterterms allow one to replace bare masses by physical masses in propagators, without abandoning the nonperturbative advantages of the LFBW theory. For now we simply turn

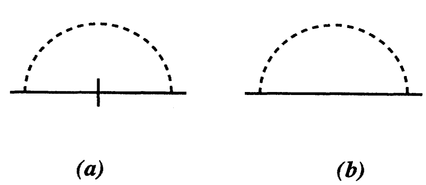

FIG. 1. Contributions to fermion mass in second-order perturbation theory.

to the LFRS theory by noting that, to zeroth order in the coupling constant, m_{F0}^2 and m_F^2 are the same; therefore, to second order in the coupling we find

$$m_F^2 - m_{F0}^2 = \frac{\lambda^2}{4\pi}\wp\int_{x_-}^{\infty}\frac{dx}{x(1-x)} - \frac{\lambda^2}{4\pi}\int_{x_-}^{x_+}\frac{dx}{1-x}\,\frac{(2-x)^2 m_F^2}{x^2 m_F^2+(1-x)m_B^2} . \quad (2.6)$$

We choose to regulate the self-inertia by setting

$$\epsilon = x_- . \quad (2.7)$$

The divergences in both integrals now exactly cancel one another and, as Λ approaches infinity, one finds

$$m_F^2 - m_{F0}^2 = \frac{\lambda^2}{4\pi}\ln\left[\frac{m_F^2}{m_B^2}\right] - \frac{\lambda^2}{4\pi}\int_0^1 dx\,\frac{4m_F^2 - m_B^2}{x^2 m_F^2+(1-x)m_B^2} , \quad (2.8)$$

which is identical to the result one obtains in covariant perturbation theory.

What is the significance of this result? It provides us with a relationship between m_{F0} and m_F. The bare mass has no physical interpretation, so we must regard it as a free parameter and determine its value by fitting an observable. The most natural observable to fit in this model is the physical fermion mass. This is exactly what one does in instant-form, covariant perturbation theory. The difference is that the diagrams confronted to second order require only the lowest sector of Fock space and the one-fermion–one-boson sector. In instant-form perturbation theory one also encounters a Z diagram, and without this diagram the result is not covariant. In fact, in $3+1$ dimensions the second-order result is both noncovariant and linearly divergent (rather than logarithmically divergent) if one drops antifermions in the instant-form calculation, whereas the light-front result exactly reproduces the covariant result when properly regulated.

In the fourth order of the LFBW theory one encounters the diagrams shown in Fig. 2. Although the diagrams containing a fermion loop are easily evaluated, we will not study them here. Once again, if one were to drop antifermions in instant-form field theory, it would not be possible to recover a covariant result. Here we will find that the instantaneous terms provide exactly the portion of the instant-form Z diagrams required by covariance, without requiring us to retain explicit antifermions. This does not happen in $3+1$ dimensions, where renormalization of the coupling constant is not covariant if antifermions are dropped. We have divided the fourth-order diagrams into four categories that depend on the Fock-space content of intermediate states and/or the topology of the diagram, and we begin with a discussion of the diagrams in Fig. 2(a). In ordinary perturbation theory we obtain the corresponding fourth-order diagrams simply by replacing m_{F0} with m_F in the intermediate propagators, just as we did above. We list the resultant value of each diagram separately:

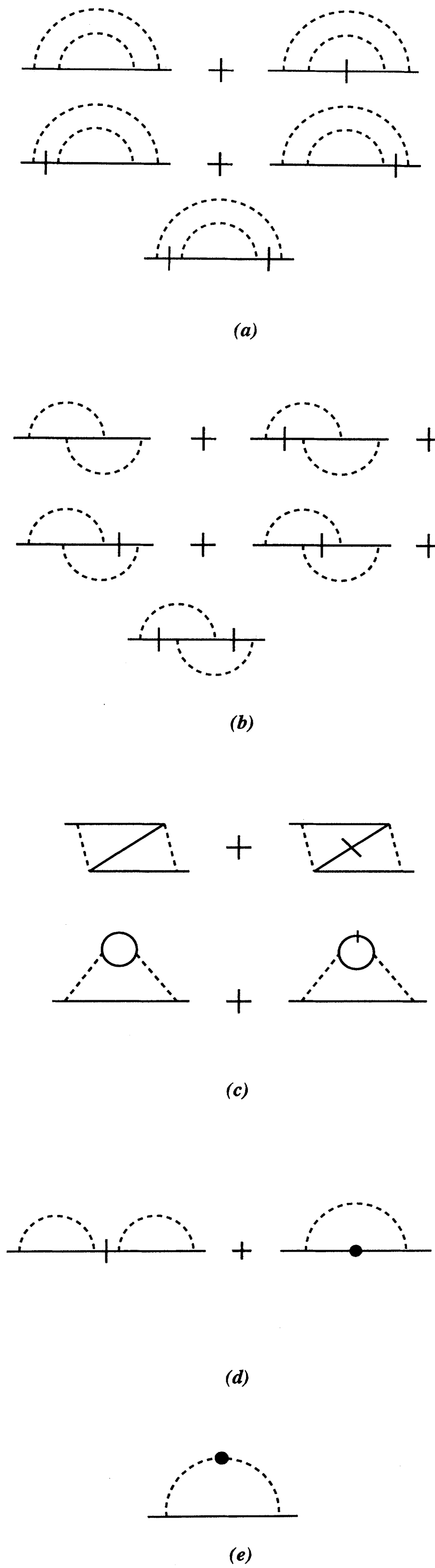

FIG. 2. Contributions to fermion mass in fourth-order perturbation theory.

$$\delta m_{a1}^2=\lambda^4 m_F^4\int_0^1 \frac{dy}{4\pi y}\frac{[1+1/(1-y)]^2}{[m_F^2-m_F^2/(1-y)-m_B^2/y]^2}\Theta(\Lambda;1-y;y)$$
$$\times\int_0^{1-y}\frac{dz}{4\pi z}\frac{[1/(1-y)+1/(1-y-z)]^2}{m_F^2-m_F^2/(1-y-z)-m_B^2/y-m_B^2/z}\Theta(\Lambda;1-y-z;y,z)\ , \quad (2.9)$$

$$\delta m_{a2}^2=\lambda^4 m_F^2\int_0^1 \frac{dy}{4\pi y}\frac{[1+1/(1-y)]^2}{[m_F^2-m_F^2/(1-y)-m_B^2/y]^2}\Theta(\Lambda;1-y;y)\wp\int_{z_-(y)}^{\infty}\frac{dz}{4\pi z}\left[\frac{1}{1-y-z}\right]\ , \quad (2.10)$$

$$\delta m_{a3}^2=\lambda^4 m_F^2\int_0^1 \frac{dy}{4\pi y}\frac{[1/(1-y)][1+1/(1-y)]}{m_F^2-m_F^2/1-y-m_B^2/y}\Theta(\Lambda;1-y;y)$$
$$\times\int_0^{1-y}\frac{dz}{4\pi z}\frac{1/(1-y)+1/(1-y-z)}{m_F^2-m_F^2/(1-y-z)-m_B^2/y-m_B^2/z}\Theta(\Lambda;1-y-z;y,z)\ , \quad (2.11)$$

$$\delta m_{a4}^2=\delta m_{a3}^2\ , \quad (2.12)$$

$$\delta m_{a5}^2=\lambda^4\int_0^1 \frac{dy}{4\pi y}\left[\frac{1}{1-y}\right]^2\int_0^{1-y}\frac{dz}{4\pi z}\frac{1}{m_F^2-m_F^2/(1-y-z)-m_B^2/y-m_B^2/z}\Theta(\Lambda;1-y-z;y,z)\ . \quad (2.13)$$

We have introduced a regulator for the one-fermion–two-boson sector:

$$\Theta(\Lambda;1-y-z;y,z)=\theta\left[\Lambda^2-\frac{m_F^2}{1-y-z}-\frac{m_B^2}{y}-\frac{m_B^2}{z}\right]. \quad (2.14)$$

The lower limit of integration in the self-inertia $z_-(y)$ is defined as the smaller of two roots satisfying

$$\Lambda^2-\frac{m_F^2}{1-y-z}-\frac{m_B^2}{y}-\frac{m_B^2}{z}=0\ . \quad (2.15)$$

This is equal to the lower limit of integration for the nested loop in Eq. (2.9), which is a root of the argument of $\Theta(\Lambda;1-y-z;y,z)$, and with this choice the nested divergence in Eq. (2.9) is canceled by the nested divergence in Eq. (2.10).

Although we will not display further manipulations of these expressions, we will outline the steps required to show that their sum is finite. As noted, $\delta m_{a1}^2+\delta m_{a2}^2$ is free of divergences from the inner-loop integration, and it is clear from inspection that the inner loops of the remaining terms are finite. It is easy to analytically complete all of the inner-loop integrations and substitute the limits of integration appropriate when $\Lambda\to\infty$. If the final result is finite, we only need to keep the leading terms in an expansion in powers of Λ^{-1}. This leads to the cutoffs

$$1-y-\frac{m_F^2}{\Lambda^2}<z<\frac{m_B^2}{\Lambda^2}\ , \quad (2.16)$$

$$1-\frac{m_F^2}{\Lambda^2}<y<\frac{m_B^2}{\Lambda^2}\ . \quad (2.17)$$

We are free to employ this simpler type of cutoff at all orders of perturbation theory in the (1+1)-dimensional Yukawa model, because this theory lacks severe divergences.

After completing the inner-loop integrations analytically, any remaining divergences occur as poles in the outer-loop integration. With the above cutoffs, the remaining integrand diverges only logarithmically, so that no divergences remain after integration. Thus, one finds a finite result when $\Lambda\to\infty$. We conclude that when cutoffs are removed, δm_a^2 is both finite and covariant. We note that if one places arbitrary cutoffs on the lower range of boson and fermion momenta that are not related to their masses as in Eqs. (2.16) and (2.17), divergences will remain in δm_a^2.

Before proceeding, we want to note several points whose significance might not be obvious. All "spurious" divergences appear as poles in momentum fraction integrations. These poles occur in phase-space factors, but in this form are usually canceled by companion poles in propagators. However, further poles appear in instantaneous interactions, and the removal of "spurious" divergences is performed by properly pairing instantaneous interactions with other terms.

Instantaneous interactions involve the instantaneous "exchange" of fermions, so the Fock-space content of the "intermediate" state is ambiguous. Since we intend to make a Tamm-Dancoff approximation, some choice must be made, so we use perturbation theory as a guide. The example we are about to encounter is shown in Fig. 8(e) of Appendix B. The instantaneous interaction contributes a factor $\lambda^2\wp 1/(x_1-y_1)$. We split it into two pieces, acting as if the intermediate state is a fermion when $x_1-y_1>0$ and an antifermion when $x_1-y_1<0$, corresponding to the last two diagrams in Fig. 8(e). Furthermore, we link the limiting procedure associated with the principal value with our cutoff limiting procedure. With the above division of the instantaneous interaction, which puts part of it in δm_b^2 and part in δm_c^2, δm_b^2 and δm_c^2 are *separately* finite. Again, this is not true in 3+1 dimensions where the ultraviolet divergence associated with coupling-constant renormalization is split between these two pieces of the fourth-order self-mass correction.

We will not list the individual expressions for the diagrams in Fig. 2(b). When combined, some manipulation leads to

$$\delta m_b^2 = -\lambda^4 \int_0^1 \frac{dy}{4\pi} \int_0^{1-y} \frac{dz}{4\pi} [y^2 m_F^2 + (1-y) m_B^2]^{-1} [yz m_F^2 + (1-y-z) m_B^2]^{-1} [z^2 m_F^2 + (1-z) m_B^2]^{-1} \left[\frac{1}{y+z} \right]$$
$$\times [(16 - 8y - 8z + yz) yz m_F^4 - (8y + 8z - 5y^2 - 5z^2 - 8yz + y^2 z + yz^2) m_B^2 m_F^2 + (1-y-z) m_B^2]$$
$$\times \Theta(\Lambda; 1-y; y) \Theta(\Lambda; 1-y-z; y,z) \Theta(\Lambda; 1-z; z) . \quad (2.18)$$

No poles remain in this expression, so δm_b^2 is manifestly finite.

The only other fourth-order diagrams encountered if one ignores antifermions are shown in Fig. 2(d), and it is easy to show that both are finite when $\Lambda \to \infty$. The first two diagrams in Fig. 2(c) can be analyzed exactly as above, and their sum is finite. The last two diagrams in Fig. 2(c) are more interesting because they contain the only "true" perturbative divergence found in the (1+1)-dimensional Yukawa model. The second-order shift in the boson mass is infinite and requires a mass counterterm. This counterterm leads to the diagram in Fig. 2(e), and the sum of this with the last two diagrams in Fig. 2(c) is also finite.

III. FIRST TAMM-DANCOFF APPROXIMATION FOR A SINGLE FERMION

In the LFTD approximation we again want to solve the quantized version of Einstein's equation (2.1); however, now we want to solve the equation "exactly" in a truncated Fock space. There are two types of truncation to consider. First, the Tamm-Dancoff truncation eliminates sectors of Fock space according to particle number, and second, within the remaining sectors, momentum cutoffs are used. In 1+1 dimensions, the above perturbative analysis leads one to suspect that spurious infrared divergences must first be removed by infrared cutoffs before finite calculations can be performed.

We want to study the charge-one sector of the (1+1)-dimensional Yukawa model, beginning with the most drastic Tamm-Dancoff truncation we can make and still have an interesting problem. We will retain only the sectors of Fock space with one fermion and with a fermion-boson pair. We expect that there will be a physical fermion and a complete set of fermion-boson scattering states. The physical fermion will be dressed, whereas the asymptotic fermion in the scattering states will not be dressed. We will begin by assuming that renormalization involves only the adjustment of parameters occurring in the original Hamiltonian, and return to this issue when we face the problem of satisfying renormalization conditions. Any eigenstate within this approximation can be written

$$|\Psi(\mathcal{P})\rangle = \frac{c_0(\mathcal{P})}{\sqrt{2\pi\mathcal{P}}} b^\dagger(\mathcal{P})|0\rangle + \int_0^\infty \frac{dp}{\sqrt{2\pi p}} \int_0^\infty \frac{dk}{\sqrt{2\pi k}} \delta(\mathcal{P}-p-k)\theta\left[\frac{\Lambda^2}{2\mathcal{P}} - \frac{m_F^2}{2p} - \frac{m_B^2}{2k}\right] c_1(p;k) b^\dagger(p) a^\dagger(k)|0\rangle . \quad (3.1)$$

The step function cutoff in the last integral is the result of our constraint that the invariant mass be less than Λ.

The first LFTD approximation results from putting this eigenstate into Eq. (2.1), and then projecting the resultant equation onto noninteracting one-fermion and one-fermion–one-boson states. This leads to

$$\langle p'|M^2 - 2\mathcal{P}P_M^-|\Psi\rangle = \langle p'|2\mathcal{P}P_I^-|\Psi\rangle , \quad (3.2)$$

$$\langle p',k'|M^2 - 2\mathcal{P}P_M^-|\Psi\rangle = \langle p',k'|2\mathcal{P}P_I^-|\Psi\rangle , \quad (3.3)$$

where

$$|p'\rangle = \frac{1}{\sqrt{2\pi p'}} b^\dagger(p')|0\rangle , \quad (3.4)$$

$$|p',k'\rangle = \frac{1}{\sqrt{2\pi p'}} \frac{1}{\sqrt{4\pi k'}} b^\dagger(p') a^\dagger(k')|0\rangle . \quad (3.5)$$

It is convenient to rewrite all momenta as fractions of the total momentum $\mathcal{P}$ and to redistribute this momentum through Einstein's equation. We represent these equations diagrammatically in Fig. 3, where boxes are used to represent matrix elements involving the eigenstate. At this point we simply drop the last two instantaneous interactions on the right-hand side of Fig. 3(b). The choice to keep or drop such interactions in the highest sectors of Fock space is completely arbitrary, as long as they are eventually included in every sector of Fock space so that the effective Hamiltonian becomes local in the limit where Fock space is filled. At this point we keep the fermion self-inertia in the one-fermion–one-boson sector. This term is divergent, and we will soon see that it must be canceled; however, it is instructive to see how one can use renormalization conditions to eliminate such terms. Note that the manner in which this term is drawn is suggestive, since it involves an intermediate state containing two bosons.

Before writing out the full expressions, let us discuss the physical content of the diagrams. At later stages the LFTD equations become sufficiently complicated that simple underlying physics can become obscured, so our discussion relies heavily on the use of diagrams. The first diagram, Fig. 3(a), relates the amplitude for finding only a bare fermion in either a physical fermion or a scattering state to the amplitude for producing a boson and finding the resultant fermion-boson pair in the physical state. Figure 3(b) represents the Hermitian-conjugate process in which a boson is absorbed. The solution of the second equation corresponds to replacing the two-particle amplitude by an operator (in this case a simple operator) acting on the one-particle amplitude, and substituting the result back into the first diagrammatic equation. After the self-inertia in Fig. 3(b) is canceled, we will see that the result is a single diagram, found in Fig. 3(c). As suggested by the diagram, the lowest-order LFTD equation for the fermion mass corresponds to the second-order LFRS equation. In higher-order approximations, we will be forced to invert integral operators that lead to infinite sums of diagrams conveniently summarized by Dyson equations.

It is straightforward to work through all contractions and convert Eqs. (3.2) and (3.3) into coupled integral equations for the amplitudes c_0 and c_1. The result is

$$\left[M^2-m_{F0}^2-\frac{\lambda^2}{4\pi}\beta(1)\right]c_0 =\lambda m_F\int_0^1\frac{dx}{\sqrt{4\pi x}}\left[1+\frac{1}{1-x}\right] \times\Theta(\Lambda;1-x;x)c_1(1-x;x)\,, \tag{3.6}$$

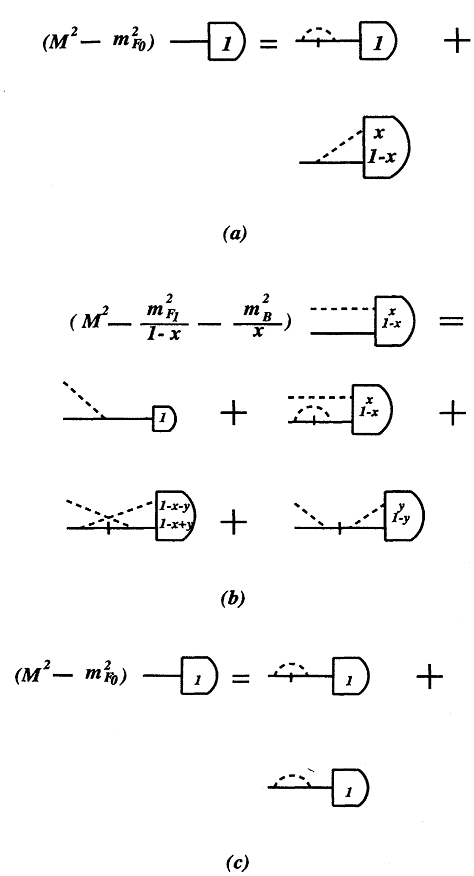

FIG. 3. First LFTD approximation for single fermion.

$$\left[M^2-\frac{1}{1-x}\left[m_{F1}^2+\frac{\lambda^2}{4\pi}\beta(1-x)\right]-\frac{m_B^2}{x}\right]c_1(1-x;x) =\frac{\lambda m_F}{\sqrt{4\pi x}}\left[1+\frac{1}{1-x}\right]c_0\,. \tag{3.7}$$

We have rewritten momenta as fractions of total momentum, and used the cutoff function defined in the preceding section. There is no dependence on the total momentum $\mathcal{P}$ because of the choice to use a covariant cutoff. We are allowing the fermion mass to depend on the sector of Fock space, with the mass appearing in P_M^- being m_{F0}^2 in the lowest sector and m_{F1}^2 in the sector containing a boson. The eigenvalue M^2 is left arbitrary because these equations are valid for the physical fermion and for fermion-boson scattering states.

For scattering states we expect c_1 to have a δ-function part, corresponding to outgoing plane-wave fermions and bosons. For this to be a solution of Eq. (3.7), the coefficient of c_1 must vanish where the δ function has support. This condition allows us to determine the scattering threshold, which should be $(m_F+m_B)^2$. If m_{F1}^2 is a finite constant, the coefficient of c_1 is divergent because of the presence of $\beta(1-x)$. To remove this divergence and to obtain the correct scattering threshold, we require

$$m_{F1}^2+\frac{\lambda^2}{4\pi}\beta(1-x)=m_F^2\,. \tag{3.8}$$

Using this result, Eq. (3.7) becomes

$$\left[M^2-\frac{m_F^2}{1-x}-\frac{m_B^2}{x}\right]c_1(1-x;x) =\frac{\lambda m_F}{\sqrt{4\pi x}}\left[1+\frac{1}{1-x}\right]c_0\,. \tag{3.9}$$

As indicated above, one can use this equation as a starting point to investigate scattering states; however, we will concentrate on the physical fermion. In this case, $M^2=m_F^2$, and the coefficient of c_1 in Eq. (3.9) cannot vanish. This allows us to simply invert the equation and replace c_1 in Eq. (3.6), leading to

$$\left[m_F^2-m_{F0}^2-\frac{\lambda^2}{4\pi}\beta(1)\right]c_0 =\lambda^2 m_F^2\int_0^1\frac{dx}{4\pi x}\,\frac{[1+1/(1-x)]^2}{m_F^2-m_F^2/(1-x)-m_B^2/x} \times\Theta(\Lambda;1-x;x)c_0\,. \tag{3.10}$$

Both $\beta(1)$ and the integral on the right-hand side of Eq. (3.10) diverge; however, the divergences cancel precisely as they did in perturbation theory. In fact, this eigenvalue condition is exactly the same as Eq. (2.6) obtained in the LFRS theory and we can see that our use of different masses in each sector takes us from the LFBW to the LFRS results. c_0 must be determined using a normalization condition. As discussed in the preceding section, we can now adjust m_{F0}^2 to obtain the correct physical mass m_F^2.

IV. FIRST TAMM-DANCOFF APPROXIMATION FOR TWO FERMIONS

Most interesting relativistic bound-state problems involve two or more fermions interacting via boson exchange. Probably the simplest example of such a system is provided by the (1+1)-dimensional Yukawa model. This example will provide us with some insight into the light-front bound-state problem, and it will also illustrate the effect of "spectators" on renormalization. In this context we note that several authors[44,45] have previously discussed the two-body bound-state equation in the first LFTD approximation (without self-energy corrections) and its nonrelativistic limit in the context of the Wick-Cutkosky model.

To avoid the need for antisymmetric wave functions, we choose to consider fermions of two different "flavors." To delineate the two flavors we will use upper-case letters for the creation operators and parameters of one flavor and lower-case letters for the other. Again for simplicity we truncate Fock space by excluding all antifermions and by allowing only one virtual or real boson. With these restrictions we can write an arbitrary eigenstate as

$$|\Psi(\mathcal{P})\rangle = \int_0^1 \frac{dx_1}{\sqrt{2\pi x_1}} \frac{dx_2}{\sqrt{2\pi x_2}} \Upsilon(\Lambda;x_1,x_2) c_0(x_1,x_2) B^\dagger(x_1) b^\dagger(x_2)|0\rangle + \int_0^1 \frac{dx_1}{\sqrt{2\pi x_1}} \frac{dx_2}{\sqrt{2\pi x_2}} \frac{dy}{\sqrt{4\pi y}} \Upsilon(\Lambda;x_1,x_2;y) c_1(x_1,x_2;y) B^\dagger(x_1) b^\dagger(x_2) a^\dagger(y)|0\rangle \, . \tag{4.1}$$

We have introduced the functions $\Upsilon(\Lambda;x_1;x_2)$ and $\Upsilon(\Lambda;x_1;x_2;y)$ both to enforce the constraint that the invariant mass be less than the cutoff Λ and to enforce momentum conservation. In other words,

$$\Upsilon(\Lambda;x_1;x_2)=\Theta(\Lambda;x_1,x_2)\delta(1-x_1-x_2) \, , \tag{4.2}$$

$$\Upsilon(\Lambda;x_1;x_2;y)=\Theta(\Lambda;x_1;x_2;y)\delta(1-x_1-x_2-y) \, , \tag{4.3}$$

where

$$\Theta(\Lambda;x_1;x_2)=\theta\left[\Lambda^2-\frac{M_F^2}{x_1}-\frac{m_F^2}{x_2}\right] , \tag{4.4}$$

$$\Theta(\Lambda;x_1;x_2;y)=\theta\left[\Lambda^2-\frac{M_F^2}{x_1}-\frac{m_F^2}{x_2}-\frac{m_B^2}{y}\right] . \tag{4.5}$$

Note that we are free to choose different cutoffs in different sectors of Fock space. For convenience, we will occasionally assume that these cutoffs are implied by the amplitudes c_0 and c_1 themselves, and will not always explicitly show them below.

The LFTD equations are derived exactly as they were in the preceding section, by projecting Einstein's equations for this state onto a complete set of free states within the truncated Fock space. Solving the resultant coupled integral equations is equivalent to diagonalizing the charge-two block of the Hamiltonian. We drop the same instantaneous interactions here that were dropped in the one-boson sector of the charge-one problem, leading to equations diagrammatically represented in Fig. 4. Note that we have already dropped the fermion self-inertia in Fig. 4(b). Here again one can view this as the result of choosing the fermion-mass counterterms in the sector containing a boson, M_{F1}^2 and m_{F1}^2; to reproduce the correct three-body scattering threshold. An examination of the equations below will reveal that this threshold is $(M_F+m_F+m_B)^2$ after this cancellation. Before writing out these equations, let us again use the diagrams to illustrate the physical content of the LFTD theory at this order. When we are below the three-body or boson production threshold, we can eliminate the three-body amplitude simply by replacing it with the diagrams on the right-hand side of Fig. 4(b). In the process we generate internal lines that correspond to propagators, which are simply the inverse of the factor multiplying the three-body amplitude on the left-hand side of Fig. 4(b). The result is a set of six diagrams shown in Fig. 5. When we eliminate the sector with a boson we obtain separate self-energy corrections for each fermion, each of which seems to correspond to the mass shift studied in the charge-one problem, and we obtain a one-boson-exchange interaction.

To be more precise, let us turn to the equations themselves. After some simple operator algebra the LFTD equations are found to be

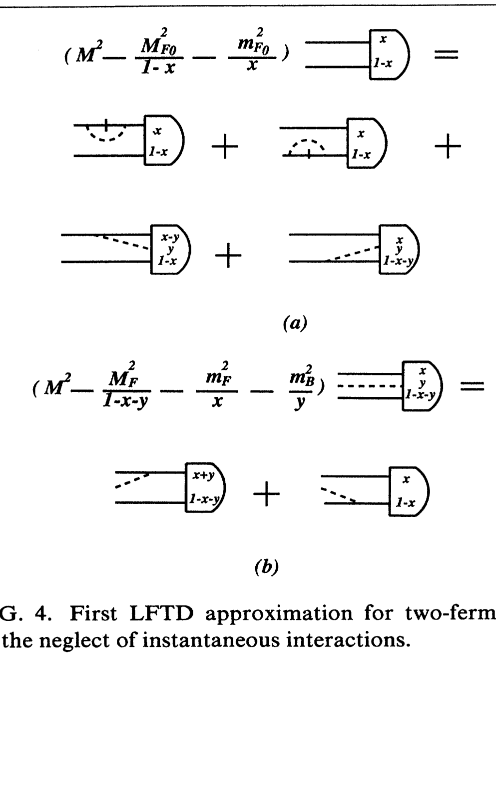

FIG. 4. First LFTD approximation for two-fermion states with the neglect of instantaneous interactions.

$$\left[M^2-\frac{M_{F0}^2+(\lambda^2/4\pi)\beta(1-x)}{1-x}-\frac{m_{F0}^2+(\lambda^2/4\pi)\beta(x)}{x}\right]c_0(1-x;x)$$
$$=\lambda M_F\int_0^{1-x}\frac{dy}{\sqrt{4\pi y}}\left[\frac{1}{1-x}+\frac{1}{1-x-y}\right]c_1(1-x-y;x;y)+\lambda m_F\int_0^x\frac{dy}{\sqrt{4\pi y}}\left[\frac{1}{x}+\frac{1}{x-y}\right]c_1(1-x;x-y;y)\,, \tag{4.6}$$

$$\left[M^2-\frac{M_F^2}{1-x-y}-\frac{m_F^2}{x}-\frac{m_B^2}{y}\right]c_1(1-x-y;x;y)=\frac{\lambda M_F}{\sqrt{4\pi y}}\left[\frac{1}{1-x-y}+\frac{1}{1-x}\right]c_0(1-x;x)$$
$$+\frac{\lambda m_F}{\sqrt{4\pi y}}\left[\frac{1}{x}+\frac{1}{x+y}\right]c_0(1-x-y;x+y)\,. \tag{4.7}$$

We have already substituted the physical masses for M_{F1}^2 and m_{F1}^2 after canceling the appropriate self-inertias. It should be clear from Eq. (4.7) that the three-body scattering threshold is correct after these renormalizations.

When the invariant mass is below the boson production threshold, the second LFTD equation is easily inverted. In the diagrammatic analysis the elimination of the three-body amplitude led to Fig. 5, which is now seen to be

$$\left[M^2-\frac{M_{F0}^2+(\lambda^2/4\pi)\beta(1-x)}{1-x}-\frac{m_{F0}^2+(\lambda^2/4\pi)\beta(x)}{x}\right]c_0(1-x;x)$$
$$=\frac{\lambda^2M_F^2}{1-x}\int_0^1\frac{dz}{4\pi z}\frac{[1+1/(1-z)]^2}{(1-x)(M^2-m_F^2/x)-M_F^2/(1-z)-m_B^2/z}\Theta(\Lambda;(1-x)(1-z);x;(1-x)z)c_0(1-x;x)$$
$$+\frac{\lambda^2m_F^2}{x}\int_0^1\frac{dz}{4\pi z}\frac{[1+1/(1-z)]^2}{x[M^2-M_F^2/(1-x)]-m_F^2/(1-z)-m_B^2/z}\Theta(\Lambda;1-x;x(1-z);xz)c_0(1-x;x)$$
$$+\lambda^2M_Fm_F\int_0^{1-x}\frac{dy}{4\pi y}\frac{[1/(1-x)+1/(1-x-y)][1/x+1/(x+y)]}{M^2-M_F^2/(1-x-y)-m_F^2/x-m_B^2/y}$$
$$\times\Theta(\Lambda;1-x-y;x;y)c_0(1-x-y;x+y)$$
$$+\lambda^2m_FM_F\int_0^x\frac{dy}{4\pi y}\frac{[1/x+1/(x-y)][1/(1-x)+1/(1-x+y)]}{M^2-M_F^2/(1-x)-m_F^2/(x-y)-m_B^2/y}$$
$$\times\Theta(\Lambda;1-x;x-y;y)c_0(1-x+y;x-y)\,. \tag{4.8}$$

Although this equation is rather long, its structure is fairly simple. The first two terms on the right-hand side of this equation correspond to self-energy diagrams, and we have made a change of variables in each of them to facilitate comparison with the single fermion result in Eq. (3.10). Both of these terms should be moved to the left-hand side of the equation and regrouped with the individual fermion-mass terms. The final two terms correspond to the two X^+ orderings of single-boson exchange. It is not obvious that these terms lead to the Fourier transform of a Yukawa potential in the nonrelativistic limit, but this is shown to be the case in Appendix C.

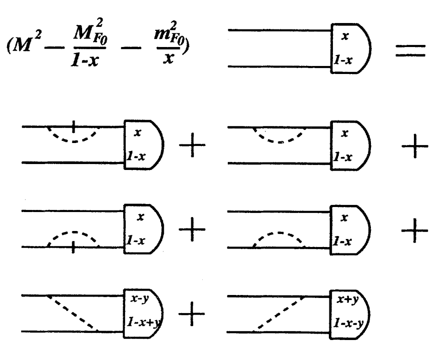

FIG. 5. Two-fermion bound-state equation in the first LFTD approximation after the elimination of the three-body amplitude, with the neglect of instantaneous interactions.

The final step in developing an equation for two-fermion scattering and bound states is to complete the mass renormalization by specifying the bare masses in the lowest sector of Fock space, M_{F0} and m_{F0}. We again use the renormalization condition that above the scattering threshold, the free asymptotic fermions should propagate with their physical masses; i.e., that the two-fermion scattering threshold should be correctly reproduced. This threshold is given by the zeros of the complete coefficient of $c_0(1-x;x)$ using the fact that above threshold we should have

$$M^2=\frac{M_F^2}{1-\chi}+\frac{m_F^2}{\chi}\,. \tag{4.9}$$

Here χ determines how the total momentum is shared between the asymptotic fermions.

To find the necessary counterterms, adjust M_{F0} and m_{F0} so that the eigenvalue in Eq. (4.9) is a solution to Eq. (4.8). This leads to

$$M_{F0}^2=M_F^2-\frac{\lambda^2}{4\pi}\beta(1-\chi)-\lambda^2 M_F^2\int_0^1\frac{dz}{4\pi z}\frac{[1+1/(1-z)]^2}{M_F^2-M_F^2/(1-z)-m_B^2/z}\times\Theta(\Lambda;(1-\chi)(1-z);\chi;(1-\chi)z)\ , \tag{4.10}$$

$$m_{F0}^2=m_F^2-\frac{\lambda^2}{4\pi}\beta(\chi)-\lambda^2 m_F^2\int_0^1\frac{dz}{4\pi z}\frac{[1+1/(1-z)]^2}{m_F^2-m_F^2/(1-z)-m_B^2/z}\times\Theta(\Lambda;1-\chi;\chi(1-z);\chi z)\ . \tag{4.11}$$

It is important to note that, in these expressions, the cutoff used to regulate β depends on the momentum fraction of the fermion line χ. As always, this cutoff should be adjusted to equal the lower zero of the relevant Θ. If we compare these counterterms with the mass counterterm found in the charge-one sector, Eq. (3.10), we see that the only difference comes from the dependence of the cutoffs on χ. Furthermore, since χ occurs only in the cutoffs, it is easy to see that this difference disappears when $\Lambda\to\infty$. If we use the mass counterterm from the charge-one sector in the charge-two sector, we obtain the correct two-fermion scattering threshold when the cutoff is taken to its limit.

When Λ is finite, we can obtain the correct scattering threshold exactly only if we allow the mass counterterms to depend on the fermion momentum fraction. To understand the nature of this dependence, consider the cutoff function that occurs in Eq. (4.10):

$$\Theta(\Lambda;(1-\chi)(1-z);\chi;(1-\chi)z)=\theta\left[(1-\chi)\left[\Lambda^2-\frac{m_F^2}{\chi}\right]-\frac{M_F^2}{1-z}-\frac{m_B^2}{z}\right]\ . \tag{4.12}$$

Comparing this with the cutoff found in the charge-one sector, Eq. (2.4), we see that we obtain the correct result if we change the cutoff used in the charge-one sector:

$$\Lambda^2\to(1-\chi)\left[\Lambda^2-\frac{m_F^2}{\chi}\right]\ . \tag{4.13}$$

In other words, the cutoff on the invariant mass of the single fermion must be decreased because the spectator carries part of the total invariant mass and part of the momentum. Except when $\chi\approx 0$ or 1, if Λ is large, one makes a small error in the scattering threshold by using a constant-mass counterterm.

Having determined a counterterm that cancels self-energy corrections on mass shell, we have not completely eliminated the corrections. The cancellation we have been discussing occurs only when Eq. (4.9) is satisfied *and* $x=\chi$. These conditions are never met in a bound state. The remaining corrections would form part of the Lamb shift in hydrogen.[7] The final renormalized two-fermion equation is

$$\begin{aligned}\Bigg[M^2&-\frac{M_F^2}{1-x}-\frac{m_F^2}{x}-\frac{\lambda^2M_F^2}{1-x}\int_0^1\frac{dz}{4\pi z}\left[\frac{[1+1/(1-z)]^2}{(1-x)(M^2-m_F^2/x)-M_F^2/(1-z)-m_B^2/z}-\frac{[1+1/(1-z)]^2}{M_F^2-M_F^2/(1-z)-m_B^2/z}\right]\\&\times\Theta(\Lambda;(1-x)(1-z);x;(1-x)z)\\&-\frac{\lambda^2m_F^2}{x}\int_0^1\frac{dz}{4\pi z}\left[\frac{[1+1/(1-z)]^2}{x[M^2-M_F^2/(1-x)]-M_F^2/(1-z)-m_B^2/z}-\frac{[1+1/(1-z)]^2}{m_F^2-m_F^2/(1-z)-m_B^2/z}\right]\\&\times\Theta(\Lambda;1-x;x(1-z);xz)\Bigg]c_0(1-x;x)\\&=\lambda^2M_Fm_F\int_0^{1-x}\frac{dy}{4\pi y}\frac{[1/(1-x)+1/(1-x-y)][1/x+1/(x+y)]}{M^2-M_F^2/(1-x-y)-m_F^2/x-m_B^2/y}\Theta(\Lambda;1-x-y;x;y)c_0(1-x-y;x+y)\\&+\lambda^2m_FM_F\int_0^{x}\frac{dy}{4\pi y}\frac{[1/x+1/(x-y)][1/(1-x)+1/(1-x+y)]}{M^2-M_F^2/(1-x)-m_F^2/(x-y)-m_B^2/y}\Theta(\Lambda;1-x;x-y;y)c_0(1-x+y;x-y)\ .\end{aligned} \tag{4.14}$$

We are not interested in solving this equation in this paper, but we want to mention some of its properties. We also study its nonrelativistic limit in Appendix C in order to begin the process of building physical intuition in terms of light-front variables and the LFTD theory. In the limit that $\Lambda\to\infty$, Eq. (4.14) is a finite, covariant, unitary, two-fermion equation of motion. It is valid both for bound and scattering states below the boson production threshold. Since this equation has a well-defined limit when $\Lambda\to\infty$, it is tempting to simply remove the cutoff. This is no doubt the first thing that should be done in any numerical study of Eq. (4.14); however, more complicated problems will not allow us the luxury of easily removing cutoffs, so the study of cutoff effects in this simple example is worthwhile.

This is a convenient point to mention numerical problems posed by the LFTD theory. The first step required by numerical computation is the discretization of the above equations. This can be accomplished by expanding the amplitudes in Eq. (4.1) using basis functions or finite element methods. After this is done, the functions $c_0(x_1,x_2)$ and $c_1(x_1,x_2;y)$ are replaced by a finite number of coefficients, and the coupled integral equations (4.6) and (4.7) become a single matrix equation. The diagonalization of this matrix can be accomplished by first eliminating the coefficients found in the expansion of c_1, although this is *not* necessarily the best procedure. For purposes of discussion, we will act as if diagonalization is accomplished in this manner. The elimination of these coefficients generates a nonlinear matrix equation corresponding to Eq. (4.8). In the process, divergent self-energies are generated that cancel the divergent self-inertias which occur as matrix elements of the original Hamiltonian. Just as mass counterterms depend on cutoffs, they must now depend on the set of basis functions chosen to discretize the problem if one insists that the physical fermion masses and scattering thresholds be correctly reproduced for all relevant momenta. It becomes a question of numerical efficiency whether it is best to use large cutoffs and simple counterterms, or best to use smaller cutoffs and fewer basis functions with a more complicated Hamiltonian to achieve a given level of accuracy.

The two-fermion problem has served to illustrate some of the renormalization problems associated with cutoffs imposed after a Tamm-Dancoff truncation. We now turn to problems associated with the Tamm-Dancoff truncation itself.

V. SECOND TAMM-DANCOFF APPROXIMATION FOR A SINGLE FERMION

A single example will be used to illustrate some of the issues that arise when one considers higher-order LFTD approximations. We return to the problem of a single fermion, and simply allow up to two virtual bosons. Allowed eigenstates are then of the form

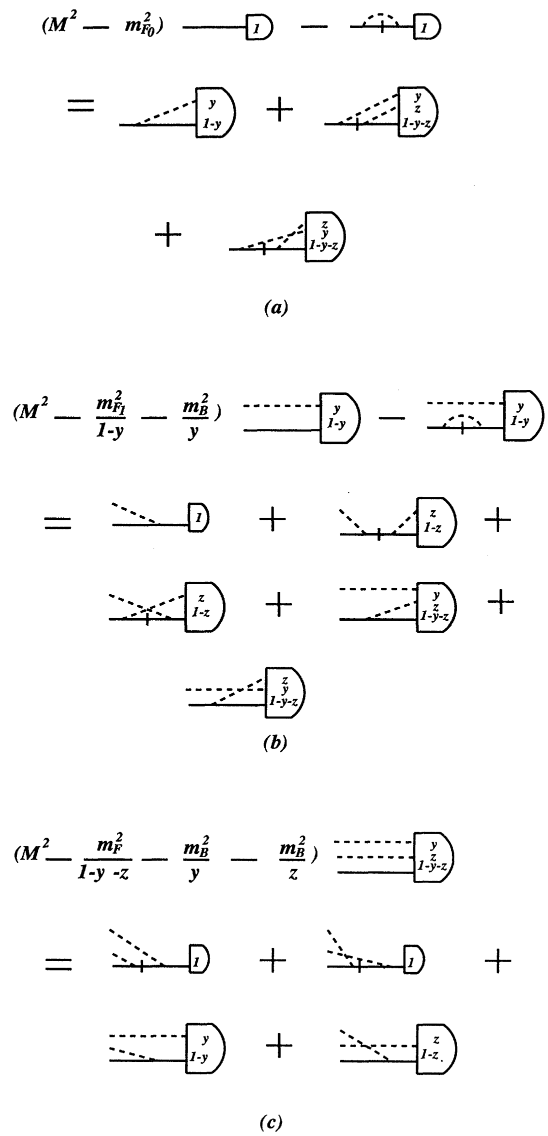

FIG. 6. Second LFTD approximation for a single fermion.

$$|\Psi(\mathcal{P})\rangle=\frac{c_0(1)}{\sqrt{2\pi}}b^\dagger(1)|0\rangle+\int_0^1\frac{dx}{\sqrt{2\pi x}}\int_0^1\frac{dy}{\sqrt{2\pi y}}\delta(1-x-y)\Theta(\Lambda;x;y)c_1(x;y)b^\dagger(x)a^\dagger(y)|0\rangle$$
$$+\int_0^1\frac{dx}{\sqrt{2\pi x}}\int_0^1\frac{dy}{\sqrt{2\pi y}}\int_0^1\frac{dz}{\sqrt{2\pi z}}\delta(1-x-y-z)\Theta(\Lambda;x;y,z)c_2(x;y,z)b^\dagger(x)a^\dagger(y)a^\dagger(z)|0\rangle\ . \tag{5.1}$$

We have chosen to immediately write all momenta as fractions of the fixed total momentum. Beyond this, the only change from Sec. III is the presence of the last term. The two cutoff functions are defined in close analogy with those in previous sections.

Figure 6 shows the diagrammatic LFTD equations for this system. These are complicated both by the extra sector and the fact that we must include instantaneous interactions. It is convenient to define a simple notation to delineate various Fock-space sectors. We use $\{1,0\}$ for the one-fermion–zero-boson sector, $\{1,1\}$ for the one-fermion–one-boson sector, etc. We have again dropped instantaneous interactions and the fermion self-inertia in the highest allowed sector of Fock space, $\{1,2\}$, as indicated in Fig. 6(c). All of these terms are retained in the $\{1,1\}$ sector to obtain a finite limit when cutoffs are removed, as is seen in Fig. 6(b). To understand this necessity, consider the diagrams generated when one solves these equations perturbatively. All of the fourth-order diagrams discussed in Sec. II are generated with this Hamiltonian. Dropping any instantaneous interaction in the $\{1,1\}$ sector eliminates one of the diagrams containing an instantaneous interaction, all of which diverge. This leads to an uncanceled divergence in the remaining diagrams. This argument should be clarified below when we eliminate the amplitudes containing bosons and generate the perturbative diagrams in the process.

To simplify the LFTD equations, we will again allow the amplitudes to imply the cutoffs, so that these need not be separately listed. The LFTD equations corresponding to Fig. 6 are then found to be

$$\left[M^2-m_{F0}^2-\frac{\lambda^2}{4\pi}\beta(1)\right]c_0=\lambda m_F\int_0^1\frac{dy}{\sqrt{4\pi y}}\left[1+\frac{1}{1-y}\right]c_1(1-y;y)+\sqrt{2}\lambda^2\int_0^1\frac{dy}{\sqrt{4\pi y}}\int_0^{1-y}\frac{dz}{\sqrt{4\pi z}}\left[\frac{1}{1-y}\right]c_2(1-y-z;y,z)\,, \tag{5.2}$$

$$\begin{aligned}\left[M^2-\frac{1}{1-y}\left[m_{F1}^2+\frac{\lambda^2}{4\pi}\beta(1-y)\right]-\frac{m_B^2}{y}\right]c_1(1-y;y)={}&\frac{\lambda m_F}{\sqrt{4\pi y}}\left[1+\frac{1}{1-y}\right]c_0\\&+\frac{\lambda^2}{\sqrt{4\pi y}}\int_0^1\frac{dz}{\sqrt{4\pi z}}\left[1+\frac{\Theta(\Lambda;1-y-z;y,z)}{1-y-z}\right]c_1(1-z;z)\\&+\sqrt{2}\lambda m_F\int_0^{1-y}\frac{dz}{\sqrt{4\pi z}}\left[\frac{1}{1-y}+\frac{1}{1-y-z}\right]\\&\times c_2(1-y-z;y,z)\,,\end{aligned} \tag{5.3}$$

$$\begin{aligned}\left[M^2-\frac{m_F^2}{1-y-z}-\frac{m_B^2}{y}-\frac{m_B^2}{z}\right]c_2(1-y-z;y,z)={}&\frac{\lambda^2}{\sqrt{2}}\frac{1}{\sqrt{4\pi y}}\frac{1}{\sqrt{4\pi z}}\left[\frac{1}{1-y}+\frac{1}{1-z}\right]c_0\\&+\frac{\lambda m_F}{\sqrt{2}}\left[\frac{1}{\sqrt{4\pi z}}\left[\frac{1}{1-y-z}+\frac{1}{1-y}\right]c_1(1-y;y)\right.\\&\left.+\frac{1}{\sqrt{4\pi y}}\left[\frac{1}{1-y-z}+\frac{1}{1-z}\right]c_1(1-z;z)\right]\,.\end{aligned} \tag{5.4}$$

In several places we have used the fact that c_2 is symmetric under the exchange of boson momenta to combine terms. The path followed in previous sections begins with the analytic elimination of the highest amplitude. In the case at hand we can use Eq. (5.4) to eliminate c_2, but it is extremely tedious to follow the results in detail. We prefer to perform this and subsequent steps diagrammatically and with a simplified algebraic notation. When c_2 is eliminated below the three-body threshold, the resultant equations are shown diagrammatically in Fig. 7.

It is easy to see that the elimination of the $\{1,2\}$ sector has generated both self-energies and effective interactions in each of the remaining sectors. The renormalization-group approach to the Tamm-Dancoff truncation involves cataloging such induced terms and determining which are relevant. Here we will simply retain everything and show how spurious divergences generated in the diagonalization procedure cancel against self-inertias and instantaneous interactions. Figure 7 has been drawn to emphasize the intermediate cancellations. The complete set of cancellations cannot be seen until one eliminates the $\{1,1\}$ sector also. First, consider the diagrams of Fig. 7(a). The self-energy diagrams are the self-inertia and a subset of the fourth-order diagrams discussed in Sec. II. Every one of these diverge, and none of them are directly canceled. Cancellations must involve the two-particle amplitudes on the right-hand side of the

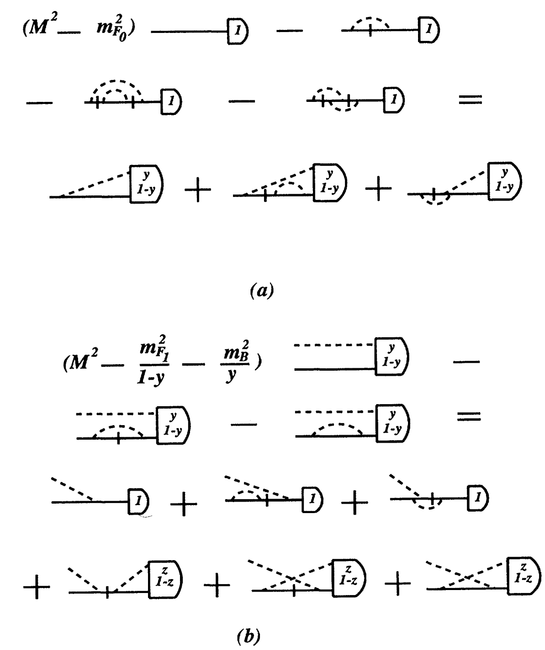

FIG. 7. Diagrams in the second LFTD approximation for a single fermion that illustrate the intermediate cancellation of divergences after the elimination of one-fermion–two-boson intermediate states.

equation.

In Fig. 7(b) we see that the self-inertia in the $\{1,1\}$ sector has become paired with the second-order self-energy. The result is not the same as the mass shift studied in Sec. II, because of the spectator boson which shifts the energy denominators. However, the divergences in the two diagrams cancel one another. Furthermore, if we use the mass counterterm found in the first Tamm-Dancoff approximation of Sec. III, we obtain the correct fermion-boson scattering threshold. The demonstration of this proceeds exactly as the discussion of the two-fermion scattering threshold in Sec. IV. In summary, the left-hand side of Fig. 7(b) is finite and with the proper choice of m_{F1}^2 it displays the correct scattering thresholds even with a finite cutoff. The right-hand side of the diagrammatic equation in Eq. 7(b) shows that the coupling between the $\{1,1\}$ and $\{1,0\}$ sectors is modified by two-particle irreducible (2PI) vertex pieces. To see that these vertex corrections are 2PI, remember that it is not possible to cut through instantaneous fermion lines, because they do not correspond to particles. The full effective vertex connecting these sectors is

$$\tilde{\mathcal{V}}_{1,0}(y)=\frac{\lambda m_F}{\sqrt{4\pi y}}\left[1+\frac{1}{1-y}\right] + \frac{\lambda^3 m_F}{\sqrt{4\pi y}}\int_0^{1-y}\frac{dz}{4\pi z}\frac{[1/(1-y)+1/(1-y-z)][1/(1-y)+1/(1-z)]}{M^2-m_F^2/(1-y-z)-m_B^2/y-m_B^2/z}\Theta(\Lambda;1-y-z;y;z)\ . \tag{5.5}$$

The cutoff function in the last integral is inherited from the $\{1,2\}$ sector. The vertex depends not only on how the momentum is shared between the boson and fermion, but also on the eigenvalue itself.

The fourth diagram on the right-hand side of the equation in Fig. 7(b) is always finite because the momentum flowing through the instantaneous fermion line is always equal to the total momentum and the diagram is proportional to the inverse of this momentum. On the other hand, the momentum flowing through the instantaneous fermion in the fifth diagram and the intermediate fermion in the sixth diagram on the right-hand side of Fig. 7(b) can vanish, and each of these terms generates a divergent interaction. However, the sum of these diagrams is always finite and no spurious divergences occur as long as they are properly paired. This fact plays an essential role in the cancellation of infinities in the fourth-order diagrams in Fig. 2(b). Thus, the last three diagrams in Fig. 7(b) do not introduce any spurious divergences and the whole equation is finite. These last three diagrams prevent one from eliminating the $\{1,1\}$ amplitude analytically because they introduce momentum-dependent self-interactions in this sector. (It is possible to solve the equation analytically if the last two diagrams are dropped but the fourth is retained.[7])

In order to facilitate further discussion, we will introduce drastically simplified algebraic notation to represent the amplitudes, propagators, and integral operators occurring in the LFTD equations. Before detailing the notation, we rewrite Eqs. (5.2)–(5.4) in the new notation. They become

$$(M^2-m_F^2-\delta m_{F0}^2)c_0=\lambda\mathcal{V}_{0,1}(y)\otimes c_1(y)+\lambda^2\mathcal{V}_{0,2}(y,z)\otimes c_2(y,z)+\lambda^2\mathcal{V}_{0,2}(y,z)\otimes c_2(z,y)\ , \tag{5.6}$$

$$\left[G_1^{-1}(y)-\frac{\delta m_{F1}^2(y)}{1-y}\right]c_1(y)=\lambda\mathcal{V}_{1,0}(y)\otimes c_0+\lambda^2\mathcal{V}_{1,1}(y,z)\otimes c_1(z)+\lambda\mathcal{V}_{1,2}(y,z)\otimes c_2(y,z)+\lambda\mathcal{V}_{1,2}(y,z)\otimes c_2(z,y)\ , \tag{5.7}$$

$$G_2^{-1}(y,z)c_2(y,z)=\lambda^2\mathcal{V}_{2,0}(y,z)\otimes c_0+\lambda^2\mathcal{V}_{2,0}(z,y)\otimes c_0+\lambda\mathcal{V}_{2,1}(y,z)\otimes c_1(y)+\lambda\mathcal{V}_{2,1}(z,y)\otimes c_1(z)\ . \tag{5.8}$$

The new equations resemble tensors equations, with both interactions and amplitudes represented by tensors and integral equations resulting from tensor multiplication. The "bare" propagators in each sector are represented by G, with a subscript indicating the number of bosons. We have defined new mass counterterms that contain the self-inertias, so that

$$G^{-1}(y,z,\ldots)=M^2-\frac{m_F^2}{1-y-z-\cdots}-\frac{m_B^2}{y}-\frac{m_B^2}{z}-\cdots\ . \tag{5.9}$$

Note that only physical masses occur in these propagators. The subscript on the interactions show the sectors between which they act, and we have separated the coupling constant for later convenience. This notation takes advantage of the fact that we only consider sectors of Fock space with one fermion and no antifermions, and a more general notation would probably be far more cumbersome. We have also suppressed the fermion momentum argument in the amplitudes, since this momentum is fixed by momentum conservation.

The algebraic solution of Eq. (5.8) below the three-body scattering threshold is simple, leading to the equations represented by Fig. 7:

$$[M^2-m_F^2-\delta m_{F0}^2-2\lambda^4\mathcal{V}_{0,2}(y,z)\otimes G_2(y,z)\mathcal{V}_{2,0}(y,z)-2\lambda^4\mathcal{V}_{0,2}(y,z)\otimes G_2(y,z)\mathcal{V}_{2,0}(z,y)]c_0$$
$$=[\lambda\mathcal{V}_{0,1}(y)+2\lambda^3\mathcal{V}_{0,2}(y,z)\otimes G_2(y,z)\mathcal{V}_{2,1}(y,z)+2\lambda^3\mathcal{V}_{0,2}(z,y)\otimes G_2(y,z)\mathcal{V}_{2,1}(y,z)]\otimes c_1(y)\ , \quad (5.10)$$

$$\left[G_1^{-1}(y)-\frac{\delta m_{F1}^2(y)}{1-y}-2\lambda^2\mathcal{V}_{1,2}(y,z)\otimes G_2(y,z)\mathcal{V}_{2,1}(y,z)\right]c_1(y)$$
$$=[\lambda\mathcal{V}_{1,0}(y)+2\lambda^3\mathcal{V}_{1,2}(y,z)\otimes G_2(y,z)\mathcal{V}_{2,0}(y,z)+2\lambda^3\mathcal{V}_{1,2}(y,z)\otimes G_2(y,z)\mathcal{V}_{2,0}(z,y)]c_0$$
$$+[\lambda^2\mathcal{V}_{1,1}(y,z)+2\lambda^2\mathcal{V}_{1,2}(y,z)\otimes G_2(y,z)\mathcal{V}_{2,1}(z,y)]\otimes c_1(z)\ , \quad (5.11)$$

We have written these equations to closely correspond to Fig. 7 so that the reader can readily follow our notation. The symmetry of G_2 under exchange of boson momenta has been used to combine terms.

Equation (5.10) contains several uncanceled divergences, but all terms in Eq. (5.11) remain finite when the cutoff approaches infinity. We will not perform a rigorous analysis of the existence of a solution to Eq. (5.11). Instead, we assume a solution exists and again simplify our notation to rewrite Eq. (5.11) as

$$G_1^{-1}c_1=(\lambda\mathcal{V}_{1,0}+\lambda^3\tilde{\mathcal{V}}_{1,0})c_0+\lambda^2\tilde{\mathcal{V}}_{1,1}\otimes c_1\ . \quad (5.12)$$

This equation defines $\tilde{\mathcal{V}}_{1,0}$ and $\tilde{\mathcal{V}}_{1,1}$. The solution of Eq. (5.12) below boson production threshold can be written

$$c_1=\left[\sum_{n=0}^{\infty}(\lambda^2G_1\tilde{\mathcal{V}}_{1,1})^n\right]G_1(\lambda\mathcal{V}_{1,0}+\lambda^3\tilde{\mathcal{V}}_{1,0})c_0\ . \quad (5.13)$$

When this solution is used in a notationaly simplified version of Eq. (5.10), we obtain

$$(M^2-m_F^2-\delta m_{F0}^2-2\lambda^4\mathcal{V}_{0,2}G_2\mathcal{V}_{2,0}-2\lambda^4\mathcal{V}_{0,2}G_2\mathcal{V}_{2,0})c_0=(\lambda\mathcal{V}_{0,1}+\lambda^3\tilde{\mathcal{V}}_{0,1})\left[\sum_{n=0}^{\infty}(\lambda^2G_1\tilde{\mathcal{V}}_{1,1})^n\right]G_1(\lambda\mathcal{V}_{1,0}+\lambda^3\tilde{\mathcal{V}}_{1,0})c_0\ . \quad (5.14)$$

This last equation is actually a finite eigenvalue equation for the physical fermion mass, but this is not obvious because we have not yet paired all divergent terms to make cancellations between them obvious. This can be done by first determining how and where the divergences arise in this equation, a task that is simplified by considering the relationship between this equation and perturbation theory. All divergences arise either from a self-inertia that is not properly paired with a second-order self-energy, or from the fact that denominators arising from instantaneous terms can vanish and must be properly paired with products of vertex operators. This latter task has been accomplished in $\tilde{\mathcal{V}}_{1,1}$, which has no poles remaining. The problems occur because the second- and fourth-order self-energy corrections have not been isolated. After some simple algebraic manipulation, we rewrite Eq. (5.14) as

$$M^2-m_F^2-\delta m_{F0}^2-\lambda^2\mathcal{V}_{0,1}G_1\mathcal{V}_{1,0}-\lambda^4\tilde{\mathcal{V}}_{0,1}G_1\mathcal{V}_{1,0}-\lambda^4\mathcal{V}_{0,1}G_1\tilde{\mathcal{V}}_{1,0}-\lambda^4\mathcal{V}_{0,1}G_1\tilde{\mathcal{V}}_{1,1}G_1\mathcal{V}_{1,0}$$
$$-2\lambda^4\mathcal{V}_{0,2}G_2\mathcal{V}_{2,0}-2\lambda^4\mathcal{V}_{0,2}G_2\mathcal{V}_{2,0}=\lambda^6(\tilde{\mathcal{V}}_{0,1}+\mathcal{V}_{0,1}G_1\tilde{\mathcal{V}}_{1,1})\left[\sum_{n=0}^{\infty}(\lambda^2G_1\tilde{\mathcal{V}}_{1,1})^n\right]G_1(\tilde{\mathcal{V}}_{1,0}+\tilde{\mathcal{V}}_{1,1}G_1\mathcal{V}_{1,0})\ . \quad (5.15)$$

The right-hand side of this equation begins at sixth order in the coupling constant, with all second- and fourth-order terms that occur in this order of the LFTD theory explicitly isolated and moved to the left-hand side. With some work, the reader should be able to convince herself that the second-order term is exactly the second-order self-energy evaluated in Sec. II when $M^2=m_F^2$. δm_{F0}^2 contains the second-order self-inertia, the second-order contribution to the finite part of the mass counterterm, and a new term which is adjusted to exactly satisfy Eq. (5.15) when $M^2=m_F^2$. This means that every piece of the second-order mass shift studied in Sec. II is present, and the sum of these terms has been shown to be finite. Furthermore, the fourth-order terms in Eq. (5.15) are exactly those terms represented in Figs. 2(a), 2(b), and 2(d). As claimed above, all fourth-order diagrams with intermediate states that are members of the Fock subspace retained in this order of the LFTD theory are generated in the process of Hamiltonian diagonalization.

Perturbation theory is sufficient to show that all terms on the left-hand side of Eq. (5.15) are finite when $\Lambda\to\infty$. The right-hand side of the equation is nonperturbative, and we will only argue that it is finite when the coupling constant is sufficiently small that the sum converges. When the sum converges it is sufficient to show that each term is finite, which is done by again noting that, in 1+1 dimensions, all divergences arise as poles in a momentum fraction. For every momentum fraction in a diagram, there is at least one pole in a propagator, G_1. Since this pole occurs in a denominator, a divergence will occur only if a second-order pole arises in another part of the diagram. In our simplified notation, we have absorbed phase-space factors associated with bosons into the $\mathcal{V}$'s. There is one pole associated with each phase-space factor, and these are exactly canceled by the poles in the propagators. The only other poles occur in self-inertias and instantaneous interactions. These types of poles can be seen in the fourth-order diagrams analyzed in Sec. II, and in all orders their cancellation is of the same origin. The self-inertias in Eqs. (5.15) occur inside of $\tilde{\mathcal{V}}_{1,1}$, where they are paired with a second-order self-energy that contains an identical pole of opposite sign. Both instantaneous interactions are also found in $\tilde{\mathcal{V}}_{1,1}$. As mentioned above, one instantaneous term does not give rise to any poles, while the second is paired with a second-order interaction that contains an identical pole of opposite sign. Thus, all poles are identically canceled in every term in Eq. (5.15).

We will not provide any estimate of the magnitude of the terms in Eq. (5.15), although it should be fairly easy to see that, for small coupling (i.e., $\lambda \ll m_F$, $\lambda \ll m_B$), the sum converges and perturbation theory through fourth order is accurate. This is a drastically simpler situation than found in $3+1$ dimensions. Our main intention is to show that higher orders of the LFTD theory are finite if interactions are properly paired to allow the *exact* cancellation of all spurious divergences. It is also important to note that such divergences are canceled as soon as they arise in our simplified diagonalization procedure. Nested divergences can pose difficult numerical problems if they remain uncanceled through a high order in perturbation theory.

Analyses of the type presented in this section are probably more complicated than actual numerical calculations, and become rapidly impossible as the number of allowed intermediate particles grows beyond three. However, the numerical analysis, which proceeds with a discretized version of the initial integral equations does not grow rapidly in algebraic complexity. There the chief problem is to find suitable discretization procedures in constrained multidimensional spaces.

VI. CONCLUSION

The basis of the LFTD theory is Hamiltonian diagonalization, with the Tamm-Dancoff truncation of Fock space and the employment of light-front coordinates being forced upon us as computational necessities. We have provided several simple examples of the equations of motion resulting from Hamiltonian diagonalization within severely truncated Fock spaces, and derived nonlocal (i.e., projected onto a single sector of Fock space) counterterms that effect mass renormalization. This procedure is required if one insists on maintaining both the physical masses of the fundamental particles and the thresholds of states containing only fundamental particles in asymptotic scattering states.

There are two limits involved in filling Fock space, and there is considerable freedom in the manner in which these limits are taken. The number of constituent particles must be taken to infinity (Tamm-Dancoff limit), and each constituent must be allowed to have any momentum (cutoff limit). In covariant perturbation theory, only the latter limiting procedure is of interest, so little experience exists in examining the former. Although these limiting procedures can be connected, as they are in DLCQ,[36] we believe it is profitable to decouple them.

If one imposes a cutoff on the invariant mass of intermediate states and then starts to take the Tamm-Dancoff limit, the cutoff will eventually constrain the number of possible constituents in any massive theory. This is simply because the minimum invariant mass in any sector is the sum of the constituent masses. In this article we have discussed the alternative procedure of imposing a Tamm-Dancoff truncation and then taking the cutoff limit within the sectors of Fock space remaining. We already know that, in a superrenormalizable theory, the cutoff limit is trivial in any order of perturbation theory, and we expect from the analysis of Sec. V that a similar result will hold for the cutoff limit after a Tamm-Dancoff truncation; however, performing a power-counting analysis to show that no naive divergences result in the cutoff limit is not sufficient. The Tamm-Dancoff approximation gives rise to all perturbative diagrams up to a given order, and to an additional infinite class of diagrams. These diagrams result generically from inverting integral operators, and such inverses need not exist even when every term in their Taylor-series expansion exists. We have no intention of pursuing this issue in any detailed fashion here, but it is important to realize that renormalization of the LFTD theory requires us to address this type of problem.

At several points we have mentioned that a naive procedure in which one simply projects a local Hamiltonian onto a truncated Fock space, introduces a few counterterms motivated by perturbation theory, and then directly takes the Tamm-Dancoff and cutoff limits, may not converge rapidly or at all. We expect that such a procedure will fail in all but the simplest cases; in particular, it will almost certainly fail whenever a coupling constant becomes large. In this case a more sophisticated approach is required in which the limits are somehow conditioned; e.g., by resumming a particular set of diagrams to all orders. This leads to a renormalization-group approach to the problem, where one seeks to anticipate the most important effects arising in the two limits and tries to modify the effective Hamiltonian so that these effects already occur early in a modified limiting procedure.

One of the chief advantages of models in $1+1$ dimensions is that they should allow us to focus on the unfamiliar (within the context of relativistic field theory) Tamm-Dancoff limiting procedure. The lack of any serious divergences in the cutoff limit makes it possible to simply impose "large" cutoffs and make small errors even in nonperturbative calculations, because we can study problems where the coupling constant is kept small. In $3+1$ dimensions, the coupling constant is forced to run and every problem is potentially a nonperturbative, strong-coupling problem. When coupling constants become sufficiently large, there is every reason to expect that bare and dressed particles have very different properties and interactions, and that new dynamical degrees of freedom arise. While it may be possible to study such effects in the few-body sectors of Fock space, it can be extremely difficult to accurately represent such effects in the many-body sectors whose "elimination" generates the interactions in the few-many sectors, and it is highly improbable that one can build such effects into the many-body sectors by naively eliminating even higher sectors of Fock space. In this article we have ignored all interactions in the highest sector of Fock space except those which arise through mixing with lower sectors; however, this is clearly inconsistent with the resultant picture of the one-body and two-body sectors where interactions are generated by particle exchange. Ideally one wants to put interactions into the highest sectors of Fock space by hand that are consistent with the interactions that then result from eliminating these highest sectors to compute the effective Hamiltonian governing the few-body sectors.

As the Tamm-Dancoff limit is taken, the number of nonlocal interactions grows rapidly, and the renormalization procedure may become impractical unless one has some method of relating new counterterms at each order with counterterms computed in lower orders. We have suggested that counterterms are computed in the lowest relevant sector (e.g., the single bare fermion sector for the fermion mass counterterm) at each order, and moved to immediate descendants at the next order. What is actually moved are parameters governing the strength of local interactions, whose matrix elements are separately computed in each sector of Fock space. We have allowed the possibility that these parameters depend on momentum cutoffs, although there may be little need for this freedom in $1+1$ dimensions. A choice relating the cutoff dependence in the parent sector at a given order of the LFTD theory to the cutoff dependence in the descendant at the next order must be made. Since descendants contain additional degrees of freedom, this relationship can be somewhat complicated, as we have discussed in Sec. III. In particular we have allowed this relationship to depend explicitly on spectator momenta and energies. This inheritance scheme is fairly rich but it still severely limits possible counterterms, and it is suggested by a renormalization-group approach to the problem. The Tamm-Dancoff and cutoff limits will only exist if the few-body effective Hamiltonian reach a fixed point. If a fixed point is reached, the counterterms will cease to evolve. In our scheme, when the effective Hamiltonian ceases to change in the lowest sector, the counterterms in immediate descendants both cease to evolve and approach the counterterm in the lowest sector. This is reasonable because locality prevents us from arbitrarily varying the strength of counterterms throughout Fock space, and we expect to explicitly recover locality when the Tamm-Dancoff limit is reached. The renormalization scheme we have chosen conditions the Tamm-Dancoff limit to produce local Hamiltonians.

We have provided little discussion of the numerical issues facing the LFTD theory. At any order of a calculation, one is faced with a set of coupled multidimensional integral equations. The integrals are regulated in the cutoff limiting procedure, and must be discretized for numerical analysis. This discretization can be accomplished in several schemes, and a principal criterion will be the possibility of using the scheme in multidimensional spaces. The use of nonorthogonal basis functions (e.g., Gaussians) and/or finite element methods is suggested. A well-known problem with finding convenient bases is that momentum conservation produces boundaries within many-body phase space that are not easily maintained. Any serious discussion of these issues must await actual numerical work.

In closing, we believe that the LFTD approach to the difficult nonperturbative problems of field theory is quite promising. The approach is far more complicated than perturbation theory, of course, but it seems highly unlikely that any simple procedure will be adequate. The question of whether the LFTD theory is able to solve any outstanding physical problem is still far from being answered. We believe that all important conceptual problems facing the study of "weakly" interacting theories in $1+1$ dimensions have been solved, and any further progress requires numerical work. Possible conceptual problems facing the study of "strongly" interacting theories have been outlined, but there is much work left to be done on this issue even in $1+1$ dimensions. This work will benefit from initial work on weak coupling, and it will be important to carefully study the results of slowly increasing the coupling strength in simple models such as the $(1+1)$-dimensional Yukawa model. Many of the most interesting challenges facing the LFTD theory are only encountered in $3+1$ dimensions, and future work must rapidly begin to face these challenges.

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APPENDIX A: LIGHT-FRONT VARIABLES, FIELDS, AND HAMILTONIAN

There are two commonly used definitions of light-front "space" and "time." We use the conventions of Bjorken and Drell for all equal-time quantities, unless otherwise stated. We choose the light-front time variable to be

$$x^+=\frac{1}{\sqrt{2}}(x^0+x^1)\ , \tag{A1}$$

and the light-front space variable to be

$$x^-=\frac{1}{\sqrt{2}}(x^0-x^1)\ . \tag{A2}$$

With these definitions, the scalar product becomes

$$x\cdot y=x^0y^0-x^1y^1=x^+y^-+x^-y^+\ ,$$

so that $x^2=2x^+x^-$. The light-front metric tensor $g^{\mu\nu}$ with these conventions is

$$g^{++}=g^{--}=0, g^{+-}=g^{-+}=1 .$$

The light-front temporal and spatial derivatives are, respectively,

$$\partial^- = \partial_+ = \frac{\partial}{\partial x^+}, \quad \partial^+ = \partial_- = \frac{\partial}{\partial x^-} . \tag{A3}$$

The light-front "momentum" and "energy" variables are

$$p^+ = \frac{1}{\sqrt{2}}(p^0+p^1), \quad p^- = \frac{1}{\sqrt{2}}(p^0-p^1) . \tag{A4}$$

As a result, the mass-shell condition for free particles is given by

$$p^2=m^2 \Longrightarrow 2p^+p^-=m^2 .$$

The Lagrangian density for the (1+1)-dimensional Yukawa model is given by

$$\mathcal{L}=\frac{i}{2}(\bar\psi\gamma^\mu\partial_\mu\psi-\partial_\mu\bar\psi\gamma^\mu\psi)-m_F\bar\psi\psi + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}m_B^2\phi^2-\lambda\bar\psi\psi\phi . \tag{A5}$$

We will list the quantization conditions without derivation. The reader is referred to the work of Chang, Root, and Yan[3] to see how these conditions are derived with the use of Schwinger's action principle. Using this principle, the light-front equal-time (i.e., equal x^+) scalar field commutation relation is found to be

$$[\phi(x^+,x^-),\partial^+\phi(x^+,y^-)]_- = \frac{i}{2}\delta(x^- - y^-) . \tag{A6}$$

If we define the antisymmetric step function as $\epsilon(x)=\theta(x)-\theta(-x)$, we also have

$$[\phi(x^+,x^-),\phi(x^+,y^-)]_- = \frac{-i}{4}\epsilon(x^- - y^-) . \tag{A7}$$

For the spin-$\frac{1}{2}$ fermion field we begin by specifying our choice of γ matrices. Here it is algebraically convenient to depart from the conventions of Bjorken and Drell and use a chiral representation for the γ matrices

$$\gamma^0=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \gamma^1=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} . \tag{A8}$$

The light-front γ matrices are then

$$\gamma^+ = \frac{1}{\sqrt{2}}(\gamma^0+\gamma^1) = \begin{bmatrix}0 & 0\\ \sqrt{2} & 0\end{bmatrix}, \tag{A9}$$

$$\gamma^- = \frac{1}{\sqrt{2}}(\gamma^0-\gamma^1) = \begin{bmatrix}0 & \sqrt{2}\\ 0 & 0\end{bmatrix} . \tag{A10}$$

From these definitions it follows that $\gamma^+\gamma^+=0$ and $\gamma^-\gamma^-=0$. We also define the projection operators

$$\Lambda^+ = \tfrac{1}{2}\gamma^-\gamma^+ = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \tag{A11}$$

$$\Lambda^- = \tfrac{1}{2}\gamma^+\gamma^- = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \tag{A12}$$

and the fields

$$\psi^+ = \Lambda^+\psi, \quad \psi^- = \Lambda^-\psi . \tag{A13}$$

With these definitions the free fermion equations of motion are

$$i\gamma^+\partial^-\psi^+ = m_F\psi^- \tag{A14}$$

and

$$i\gamma^-\partial^+\psi^- = m_F\psi^+ . \tag{A15}$$

Since the second equation does not involve a time derivative, ψ^- is a constrained variable and cannot be quantized. We can use the equation of constraint to eliminate it from the energy-momentum tensor when we want to express this tensor entirely in terms of second-quantized field variables.

Finally, we have the commutation relations for ψ^+:

$$\{\psi_\alpha^+(y^-),(\psi_\beta^+)^\dagger(x^-)\}_+ = \frac{\Lambda_{\alpha\beta}^+}{\sqrt{2}}\delta(y^- - x^-) \tag{A16}$$

and

$$\{\psi_\alpha^+(y^-),\psi_\beta^+(x^-)\}_+ = 0 . \tag{A17}$$

In order to derive the light-front momentum and energy operators, one begins with the energy-momentum tensor which, for the (1+1)-dimensional Yukawa model, is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi + \frac{i}{2}\bar\psi\gamma^\mu\partial^\nu\psi - \frac{i}{2}\partial^\nu\bar\psi\gamma^\mu\psi - g^{\mu\nu}\mathcal{L} , \tag{A18}$$

where $\mathcal{L}$ is the Lagrangian. This immediately yields the light-front momentum density

$$\mathcal{P}^+ = T^{++} = \partial^+\phi\partial^+\phi + \frac{i}{2}\bar\psi\gamma^+\partial^+\psi - \frac{i}{2}(\partial^+\bar\psi)\gamma^+\psi , \tag{A19}$$

and the light-front Hamiltonian density

$$\mathcal{P}^- = T^{+-} = \tfrac{1}{2}m_B^2\phi^2 + (m_F+\lambda\phi)\bar\psi\psi - \frac{i}{2}\bar\psi\gamma^-\partial^+\psi + \frac{i}{2}(\partial^+\bar\psi)\gamma^-\psi . \tag{A20}$$

After some tedious algebra, one obtains

$$\mathcal{P}^- = \tfrac{1}{2}m_B^2\phi^2 + \sqrt{2}i\{(\Psi^-)^\dagger\tfrac{1}{2}\partial^+\Psi^- - [\tfrac{1}{2}\partial^+(\Psi^-)^\dagger]\Psi^-\} + \lambda\bar\Psi\Psi\phi + \sqrt{2}i\{(\eta^-)^\dagger\tfrac{1}{2}\partial^+\eta^- - [\tfrac{1}{2}\partial^+(\eta^-)^\dagger]\eta^-\} . \tag{A21}$$

The light-front Hamiltonian is simply $P^- = \int dx^-\mathcal{P}^-$. We have formally written

$$\Psi = \psi^+ + \Psi^- , \quad \Psi^- = \frac{1}{i2}\frac{1}{\partial^+}m_F\gamma^+\psi^+ , \tag{A22}$$

and defined

$$\eta^- = \frac{1}{i2}\frac{1}{\partial^+}\lambda\phi\gamma^+\psi^+ . \tag{A23}$$

It is necessary to provide a definition of the inverse of ∂^+. We do so with the commonly accepted principal-value prescription. The issue is not as severe here as it is in gauge theories, where one encounters $(1/\partial^+)^2$.

The next step towards writing the light-front Hamiltonian in second-quantized form is to provide expansions of the dynamical field operators in terms of a free-particle basis. The boson field is expanded as

$$\phi(x^-)=\int_0^\infty \frac{dk^+}{2\pi 2k^+}[a(k^+)e^{-ik^+x^-}+a(k^+)^\dagger e^{ik^+x^-}] \tag{A24}$$

with

$$[a(k^+),a(k'^+)]_-=0,\quad [a(k^+)^\dagger,a(k'^+)^\dagger]_-=0\ , \tag{A25}$$

and

$$[a(k^+),a(k'^+)^\dagger]_-=2\pi 2k^+\delta(k^+-k'^+)\ . \tag{A26}$$

With these definitions one can readily verify that the original commutation relations are satisfied.

The fermion free spinor $u(k^+)$ obeys

$$(\gamma^-k^++\gamma^+k^--m_F)u(k^+)=0\ , \tag{A27}$$

so that

$$u=\mathcal{N}\begin{bmatrix}\sqrt{2}k^+\\ m_F\end{bmatrix}, \tag{A28}$$

where $\mathcal{N}$ is a normalization factor. Choosing the normalization factor so that $\bar{u}u=2m_F$, we have

$$u=\frac{1}{2^{1/4}(k^+)^{1/2}}\begin{bmatrix}\sqrt{2}k^+\\ m_F\end{bmatrix}, \tag{A29}$$

so that

$$u^+=\frac{1}{2^{1/4}(k^+)^{1/2}}\begin{bmatrix}\sqrt{2}k^+\\ 0\end{bmatrix}. \tag{A30}$$

By charge conjugation we find that the antifermion spinor is

$$v=\frac{1}{2^{1/4}(k^+)^{1/2}}\begin{bmatrix}\sqrt{2}k^+\\ -m_F\end{bmatrix}. \tag{A31}$$

Thus, $v^+=u^+$. Finally, the fermion field operator can be written as

$$\psi^+(x^-)=\frac{1}{2^{1/4}}\begin{bmatrix}1\\0\end{bmatrix}\times\int_0^\infty\frac{dk^+}{2\pi(k^+)^{1/2}}[b(k^+)e^{-ik^+x^-}+d(k^+)^\dagger e^{ik^+x^-}] \tag{A32}$$

with

$$\{b(k^+),b(k'^+)^\dagger\}_+=\{d(k^+)d(k'^+)^\dagger\}_+=2\pi k^+\delta(k^+-k'^+)\ . \tag{A33}$$

Again, it is a straightforward matter to demonstrate that this leads to the original commutation relations. Now,

$$\Psi=\psi^++\Psi^-=\frac{1}{2^{1/2}}\int_0^\infty\frac{dk^+}{2\pi k^+}[b(k^+)u(k^+)e^{-ik^+x^-}+d(k^+)^\dagger v(k^+)e^{ik^+x^-}]\ . \tag{A34}$$

We now have the required elements to write the light-front Hamiltonian

$$P^-=P_M^-+P_V^-+P_F^-+P_S^-\ . \tag{A35}$$

We have followed the conventions of Pauli and Brodsky,[36] splitting the Hamiltonian into four pieces. The first piece is

$$P_M^-=\int\frac{dk_1^+}{2\pi 2k_1^+}a^\dagger(k_1^+)a(k_1^+)\frac{1}{2k_1^+}\left[m_B^2+\frac{\lambda^2}{4\pi}\alpha(k_1^+)\right]+\int\frac{dk_1^+}{2\pi k_1^+}b^\dagger(k_1^+)b(k_1^+)\frac{1}{2k_1^+}\left[m_F^2+\frac{\lambda^2}{4\pi}\beta(k_1^+)\right]+\int\frac{dk_1^+}{2\pi k_1^+}d^\dagger(k_1^+)d(k_1^+)\frac{1}{2k_1^+}\left[m_F^2+\frac{\lambda^2}{4\pi}\gamma(k_1^+)\right] \tag{A36}$$

with

$$\alpha(k_1^+)=\wp\int dk_2^+\left[\frac{1}{k_1^+-k_2^+}-\frac{1}{k_1^++k_2^+}\right], \tag{A37}$$

$$\beta(k_1^+)=\wp\int dk_2^+\frac{k_1^+}{k_2^+}\frac{1}{k_1^+-k_2^+}\ , \tag{A38}$$

$$\gamma(k_1^+)=\int dk_2^+\frac{k_1^+}{k_2^+}\frac{1}{k_1^++k_2^+}\ . \tag{A39}$$

The terms arising from normal ordering, α, β, and γ, are called "self-induced inertias." In each of these, $\wp$ indicates the "principal value." These terms diverge and we regulate them with cutoffs in all calculations. If one drops these "mass" terms after normal ordering the Hamiltonian, perturbation theory produces noncovariant divergences.

$$P_V^- = \frac{\lambda}{4\pi}\int\frac{dk_1^+}{k_1^+}\int\frac{dk_2^+}{2\pi 2k_2^+}\int\frac{dk_3^+}{k_3^+}[b^\dagger(k_1^+)b(k_3^+)a(k_2^+)\bar{u}(k_1^+)u(k_3^+)\delta(k_1^+-k_2^+-k_3^+)$$
$$+b^\dagger(k_3^+)b(k_1^+)a^\dagger(k_2^+)\bar{u}(k_3^+)u(k_1^+)\delta(k_1^+-k_2^+-k_3^+)$$
$$-d^\dagger(k_3^+)d(k_1^+)a(k_2^+)\bar{v}(k_1^+)v(k_3^+)\delta(k_1^++k_2^+-k_3^+)$$
$$-d^\dagger(k_1^+)d(k_3^+)a^\dagger(k_2^+)\bar{v}(k_3^+)v(k_1^+)\delta(k_1^++k_2^+-k_3^+)$$
$$+b^\dagger(k_1^+)d^\dagger(k_3^+)a(k_2^+)\bar{u}(k_1^+)v(k_3^+)\delta(k_1^++k_3^+-k_2^+)$$
$$-b(k_1^+)d(k_3^+)a^\dagger(k_2^+)\bar{v}(k_3^+)u(k_1^+)\delta(k_1^++k_3^+-k_2^+)] \,. \quad \text{(A40)}$$

Note that P_V^- changes particle number by one.

$$P_F^- = \frac{1}{2}\frac{\lambda^2}{4\pi}\int\frac{dk_1^+}{k_1^+}\int\frac{dk_2^+}{2\pi 2k_2^+}\int\frac{dk_3^+}{2\pi 2k_3^+}\int\frac{dk_4^+}{k_4^+}$$
$$\times\Bigg[b^\dagger(k_1^+)b(k_4^+)a(k_2^+)a(k_3^+)\bar{u}(k_1^+)\gamma^+u(k_4^+)\frac{\delta(k_3^++k_4^+-k_1^++k_2^+)}{k_3^++k_4^+}$$
$$+b^\dagger(k_4^+)b(k_1^+)a^\dagger(k_3^+)a^\dagger(k_2^+)\bar{u}(k_4^+)\gamma^+u(k_1^+)\frac{\delta(k_3^++k_4^+-k_1^++k_2^+)}{k_3^++k_4^+}$$
$$+d^\dagger(k_4^+)d(k_1^+)a(k_2^+)a(k_3^+)\bar{v}(k_1^+)\gamma^+v(k_4^+)\frac{\delta(k_4^+-k_3^+-k_1^+-k_2^+)}{k_4^+-k_3^+}$$
$$+d^\dagger(k_1^+)d(k_4^+)a^\dagger(k_3^+)a^\dagger(k_2^+)\bar{v}(k_4^+)\gamma^+v(k_1^+)\frac{\delta(k_4^+-k_3^+-k_1^+-k_2^+)}{k_4^+-k_3^+}$$
$$+b^\dagger(k_1^+)d^\dagger(k_4^+)a^\dagger(k_3^+)a(k_2^+)\bar{u}(k_1^+)\gamma^+v(k_4^+)\frac{\delta(k_3^++k_4^+-k_2^++k_1^+)}{k_1^+-k_2^+}$$
$$+b^\dagger(k_1^+)d^\dagger(k_4^+)a^\dagger(k_3^+)a(k_2^+)\bar{u}(k_1^+)\gamma^+v(k_4^+)\frac{\delta(k_3^++k_4^+-k_2^++k_1^+)}{k_1^++k_3^+}$$
$$-b(k_1^+)d(k_4^+)a^\dagger(k_2^+)a(k_3^+)\bar{v}(k_4^+)\gamma^+u(k_1^+)\frac{\delta(k_3^++k_4^+-k_2^++k_1^+)}{k_1^+-k_2^+}$$
$$-b(k_1^+)d(k_4^+)a^\dagger(k_2^+)a(k_3^+)\bar{v}(k_4^+)\gamma^+u(k_1^+)\frac{\delta(k_3^++k_4^+-k_2^++k_1^+)}{k_1^++k_3^+}\Bigg] \,. \quad \text{(A41)}$$

P_F^- changes particle number by two.

$$P_S^- = \frac{1}{2}\frac{\lambda^2}{4\pi}\int\frac{dk_1^+}{k_1^+}\int\frac{dk_2^+}{2\pi 2k_2^+}$$
$$\times\int\frac{dk_3^+}{2\pi 2k_3^+}\int\frac{dk_4^+}{k_4^+}\Bigg[b^\dagger(k_1^+)b(k_4^+)a^\dagger(k_2^+)a(k_3^+)\bar{u}(k_1^+)\gamma^+u(k_4^+)\frac{\delta(k_2^+-k_4^+-k_3^++k_1^+)}{k_4^+-k_2^+}$$
$$+b^\dagger(k_1^+)b(k_4^+)a^\dagger(k_2^+)a(k_3^+)\bar{u}(k_1^+)\gamma^+u(k_4^+)\frac{\delta(k_2^+-k_4^+-k_3^++k_1^+)}{k_3^++k_4^+}$$
$$+d^\dagger(k_4^+)d(k_1^+)a^\dagger(k_2^+)a(k_3^+)\bar{v}(k_1^+)\gamma^+v(k_4^+)\frac{\delta(k_4^+-k_3^+-k_1^++k_2^+)}{k_1^+-k_2^+}$$
$$+d^\dagger(k_4^+)d(k_1^+)a^\dagger(k_2^+)a(k_3^+)\bar{v}(k_1^+)\gamma^+v(k_4^+)\frac{\delta(k_4^+-k_3^+-k_1^++k_2^+)}{k_2^++k_4^+}$$
$$+b^\dagger(k_1^+)d^\dagger(k_4^+)a(k_2^+)a(k_3^+)\bar{u}(k_1^+)\gamma^+v(k_4^+)\frac{\delta(k_3^+-k_4^+-k_1^++k_2^+)}{k_3^+-k_4^+}$$
$$+d(k_4^+)b(k_1^+)a^\dagger(k_3^+)a^\dagger(k_2^+)\bar{v}(k_4^+)\gamma^+u(k_1^+)\frac{\delta(k_3^+-k_4^+-k_1^++k_2^+)}{k_3^+-k_4^+}\Bigg] \,. \quad \text{(A42)}$$

P_S^- changes particle number by zero.

APPENDIX B: LIGHT-FRONT PERTURBATION THEORY RULES FOR THE EVALUATION OF M^2/P^+

In this appendix we list the rules for the evaluation of M^2/P^+ in perturbation theory for the (1+1)-dimensional Yukawa model.

(1) Draw all topologically distinct diagrams with beginning and ending fermion (or boson) line(s), where the initial state does not appear as an intermediate state.

(2) Assign a "momentum" to each line.

(3) To each intermediate state, assign an "energy" denominator

$$\frac{1}{2(P^-_{\text{initial}}-P^-_{\text{intermediate}})} ,$$

where $P^-_{\text{intermediate}}=\sum_i m_i^2/2k_i$ and the sum is over the particles in the intermediate state.

(4) The interaction vertices are shown in Fig. 8 and correspond to the following.

(a) Trilinear vertex:

$$\frac{\lambda}{\sqrt{4\pi}}\frac{\bar{u}(k_1)u(k_3)}{\sqrt{k_1}\sqrt{k_2}\sqrt{k_3}} .$$

To change fermion lines into antifermion lines replace $u\rightarrow v$ and/or $\bar{u}\rightarrow -\bar{v}$.

(b) Fermion self-inertia:

$$\frac{\lambda^2}{4\pi}\frac{1}{k_1}\int\frac{dk_2}{k_2}\frac{k_1}{k_1-k_2} .$$

(c) Antifermion self-inertia:

$$\frac{\lambda^2}{4\pi}\frac{1}{k_1}\int\frac{dk_2}{k_2}\frac{k_1}{k_1+k_2} .$$

(d) Boson self-inertia:

$$\frac{\lambda^2}{4\pi}\frac{1}{k_1}\int dk_2\left[\frac{1}{k_1-k_2}-\frac{1}{k_1+k_2}\right] .$$

(e) Instantaneous vertex:

$$\frac{1}{2}\frac{\lambda^2}{4\pi}\frac{\bar{u}(k_4)\gamma^+u(k_1)}{k_1-k_3}\frac{1}{\sqrt{k_1}}\frac{1}{\sqrt{k_2}}\frac{1}{\sqrt{k_3}}\frac{1}{\sqrt{k_4}}$$

$$\times[\theta(k_1-k_3)+\theta(k_3-k_1)]$$

with the separate pieces illustrated in Fig. 8(e).

To see how the remaining instantaneous interactions can be derived from this expression, first rewrite the denominator in a symmetrized form:

$$k_1-k_3\rightarrow\tfrac{1}{2}(k_1-k_3+k_4-k_2) .$$

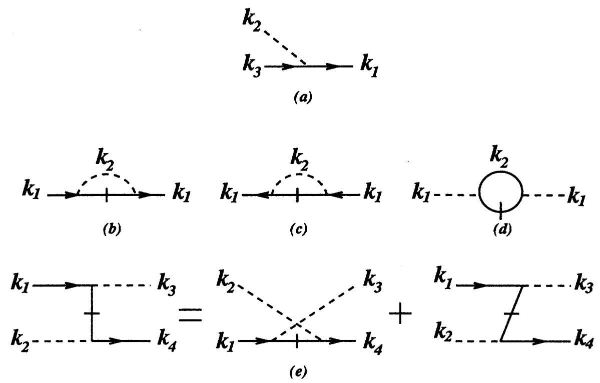

FIG. 8. Diagrams illustrating the vertices of the (1+1)-dimensional Yukawa model in light-front perturbation theory.

Now, to change fermion lines into antifermion lines replace $u\rightarrow v$, and/or $\bar{u}\rightarrow -\bar{v}$ and change the sign of the corresponding fermion momentum in the denominator. To change any outgoing boson line into an incoming boson line, change the sign of the corresponding boson momentum in the denominator.

(5) Conserve momentum at each vertex.

(6) Integrate over independent momenta.

(7) Multiply by $(-1)^n$, where n is the number of closed fermion loops.

We note useful relations:

$$\bar{u}u=-\bar{v}v=2m_F ,$$

$$u(k)\bar{u}(k)=\not{k}+m_F,\quad v(k)\bar{v}(k)=\not{k}-m_F ,$$

$$\bar{u}(k_1)u(k_2)=-\bar{v}(k_1)v(k_2)=m_F\frac{k_1+k_2}{\sqrt{k_1}\sqrt{k_2}} ,$$

$$\bar{u}(k_1)v(k_2)=-\bar{v}(k_1)u(k_2)=-m_F\frac{k_1-k_2}{\sqrt{k_1}\sqrt{k_2}} ,$$

$$\bar{u}(k_1)\gamma^+u(k_2)=\bar{v}(k_1)\gamma^+v(k_2)=\sqrt{2k_1}\sqrt{2k_2} ,$$

$$\bar{u}(k_1)\gamma^+v(k_2)=\bar{v}(k_1)\gamma^+u(k_2)=\sqrt{2k_1}\sqrt{2k_2} .$$

APPENDIX C: A NONRELATIVISTIC LIMIT FOR THE TWO-FERMION LFTD EQUATION

We start from the renormalized light-front bound-state equation for two fermions (of different flavor) in the first Tamm-Dancoff approximation given by Eq. (4.14). For simplicity we assume that the fermion masses are equal. Further, we let $\Lambda\rightarrow\infty$. We can ignore the self-energy corrections when studying the nonrelativistic limit of Eq. (4.14), as is easily justified *a posteriori*. Using the variables $1-x=x_1$, $x=x_2$, $1-x-y=y_1$, $x+y=y_2$, we have

$$\left[M^2-\frac{m_F^2}{x_1}-\frac{m_F^2}{x_2}\right]c_0(x_1,x_2)=\frac{\lambda^2}{4\pi}m_F^2\int dy_1dy_2\delta(1-y_1-y_2)\left[\frac{1}{x_1}+\frac{1}{y_1}\right]\left[\frac{1}{x_2}+\frac{1}{y_2}\right]K(x_1,y_1;x_2,y_2;M^2)c_0(y_1,y_2)\ , \tag{C1}$$

where

$$K(x_1,y_1;x_2,y_2;M^2)=\frac{\theta(1-y_1-x_2)}{1-y_1-x_2}\frac{1}{M^2-m_F^2/y_1-m_F^2/x_2-m_B^2/(1-x_2-y_1)}+\frac{\theta(1-x_1-y_2)}{1-x_1-y_2}\frac{1}{M^2-m_F^2/x_1-m_F^2/y_2-m_B^2/(1-x_1-y_2)}\ . \tag{C2}$$

Introduce the variable q so that

$$x_1=\frac{1}{2}\left[1-\frac{q}{\epsilon(q)}\right],\quad x_2=\frac{1}{2}\left[1+\frac{q}{\epsilon(q)}\right]\ , \tag{C3}$$

where

$$\epsilon=(q^2+m_F^2)^{1/2}$$

and

$$\phi(q)=\frac{4}{\sqrt{x_1x_2}}c_0(x_1,x_2)\ .$$

Then the bound-state equation is

$$\left[q^2+m_F^2-\frac{M^2}{4}\right]\phi(q)=\frac{\lambda^2}{4\pi}m_F^2\int\frac{dq'}{\epsilon(q')}\frac{\phi(q')}{(q-q')^2+m_B^2+R(q,q';M^2)}\frac{4\epsilon(q)\epsilon(q')}{m_F^2}\frac{1}{2}\left[1-\frac{1}{4}\left[\frac{q}{\epsilon(q)}+\frac{q'}{\epsilon(q')}\right]^2\right]\ , \tag{C4}$$

where

$$R(q,q';M^2)=-qq'\frac{[\epsilon(q)-\epsilon(q')]^2}{\epsilon(q)\epsilon(q')}+\left[\epsilon(q)^2+\epsilon(q')^2-\frac{M^2}{2}\right]\times\left|\frac{q}{\epsilon(q)}-\frac{q'}{\epsilon(q')}\right|\ . \tag{C5}$$

Defining $\bar{B}=B-B^2/4m_F$, where $B=2m_F-M$, in the limit where m_F is much greater than q and q', we recover the Schrödinger equation with a Yukawa potential:

$$\left[\frac{q^2}{m_F}+\bar{B}\right]\phi(q)=\frac{\lambda^2}{2\pi}\int dq'\phi(q')\frac{1}{(q-q')^2+m_B^2}\ . \tag{C6}$$

For nonrelativistic systems we expect $\bar{B}\ll m_F$, and we expect the constituents in the nonrelativistic system to have their momentum fraction strongly peaked about $x=\frac{1}{2}$. In this limit it is easy to convince oneself that the self-energy corrections are negligible.

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