PHYSICAL REVIEW D VOLUME 43, NUMBER 12 15 JUNE 1991

Particle fields at finite temperature from string field theory

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(Received 15 February 1991)

An investigation is made into the finite-temperature behavior of a class of scalar field theories derived from string field theory. The fields $\tilde{\phi}$ are the lowest-order fields in a level truncation of the open bosonic string and have momentum-dependent couplings. Expressions for the one-loop effective potential for ϕ^3 and $\tilde{\phi}^3$ theories are derived, and the finite-temperature mass corrections are examined for indications of a second-order phase transition. Questions concerning the stability of the vacuum, the number of degrees of freedom, and the Hagedorn temperature are addressed.

I. INTRODUCTION

One of the primary areas in which string theory is of relevance and should be able to make predictions that go beyond those of particle field theories is in the study of the early Universe. According to classical general relativity, there is a singularity at the initial big bang and a breakdown in the ability to make physical predictions. Since string theory is a candidate for a consistent theory of quantum gravity, it is important to study what effects the extended nature of strings would have on the evolution of the early Universe and in particular how strings behave at high temperature. The effects of temperature on string theory have been examined quite extensively in the context of first-quantized string theory in an ideal-gas approximation.[1–4] One of the striking features is the existence of a maximum temperature, or Hagedorn temperature,[5] at which the canonical ensemble picture of a string gas breaks down. To address properly questions such as what happens near or above the Hagedorn temperature, one clearly must go beyond the ideal-gas approximation and take into account string interactions.

While the thermodynamics of interacting strings have not been formulated in a Hamiltonian formulation, because of the nonlocal nature of the interactions, a perturbative expansion for the thermal partition function has been shown to exist,[6] and the effects of interactions have been shown to imply the existence of a first-order phase transition near the Hagedorn temperature.[7] This phase transition is brought about by certain winding modes in the theory becoming tachyonic, and the divergence of the thermal partition function then arises because the perturbative expansion is being carried out around the wrong vacuum. It was argued in Ref. 7 that in the string phase above the Hagedorn temperature, when the fields are shifted to a stable vacuum, strings will have far fewer degrees of freedom than for any known relativistic quantum field theory.

These results are reminiscent of the results found in Refs. 8–10 for the open bosonic string in the case of zero temperature. The bosonic string has a tachyon in its spectrum, and the canonical 26-dimensional vacuum of the string is unstable. The fact that there is a tachyon in the spectrum indicates that the wrong vacuum for the theory has been chosen. In Refs. 8–10 a candidate nonperturbative vacuum was found and analyzed. It was shown to be perturbatively stable and that the mass spectrum of states is quite different in the new vacuum—in particular, the number of degrees of freedom in the nonperturbative vacuum is smaller than in the canonical vacuum, and the tachyon disappears. Furthermore, it was shown that the coupling in momentum space includes a factor $\exp[-2\alpha' p^2 \ln(3\sqrt{3}/4)]$, due to the extended nature of the string, which smears interactions over the scale $\sqrt{\alpha'}$. Thus the string coupling runs even at the tree level, and the theory is asymptotically free.[8–10]

A natural question to ask is whether this behavior holds in the finite-temperature case as well. Does the nonperturbative vacuum remain stable at high temperature, or is there a phase transition back to the nonstable canonical vacuum? Does the tachyon field reappear at finite temperature and acquire a negative mass squared, or do all the states retain positive masses even at high temperatures?

The approach used to study the structure of the vacuum in Refs. 8-10 was with functional methods and string field theory. Starting from the covariant field theory for the open bosonic string,[11] an expansion of the string field Ψ was obtained in terms of ordinary particle fields. A truncation scheme was used in which the Fock-space expansion of the string field is terminated at a particular level, and this method was shown to be systematic in that successive orders approach a point of convergence.[9] Numerical calculations were then performed on the effective potential that permitted the identification of a candidate nonperturbative vacuum.[9]

The aim of the present work is to begin examining the finite-temperature behavior of the particle fields coming from string field theory. To lowest order in the expansion of Ψ, we obtain a scalar field theory with a momentum-dependent cubic interaction, which we denote as $\tilde{\phi}^3$ theory. The full effective potential for $\tilde{\phi}$ in the string theory would involve integrating out all the other fields in the infinite expansion of Ψ. This is too complicated an expression for us to generalize to finite temperature. Instead, we will work at the level where the theory has been truncated completely to the tachyon field. Thus the only field in the truncated $\tilde{\phi}$ model is a scalar field with $m^2<0$ and a momentum-dependent interaction. It is for this model that we find an expression for the effective potential. We then explore the vacuum structure of this model and see what effect turning on a nonzero temperature has. In particular, we will look at the mass terms and determine whether there is a phase transition coming from m^2 changing sign.

Our aim here is not so much to obtain concrete results for string theory, but rather to investigate a new kind of scalar field theory that is derived as the lowest-order particle field in interacting bosonic string theory. The $\tilde{\phi}$ theory can serve as a toy model, and its behavior may ultimately have implications for string theory. Since the coupling in this model has an unusual momentum dependence,[9] the calculations involved will be fundamentally different from those of conventional finite-temperature field theory. Because the Hagedorn temperature in a free string theory is a consequence of there being an infinite number of degrees of freedom, we do not expect to see any indications of a Hagedorn temperature in a theory that has been truncated to just one field. Nonetheless, the $\tilde{\phi}$ model does exhibit vacuum structure, and the presence of a negative-mass-squared state indicates that we are initially in the wrong vacuum. We investigate the vacuum structure of the $\tilde{\phi}^3$ theory to see if there is a critical temperature at which a phase transition takes place.

String field theory at finite temperature has been previously examined and tachyon amplitudes have been calculated for $T>0$.[12] While these results take into account the full spectrum of states in string theory, they do not account for the vacuum structure of the theory—the perturbative calculations are performed about the unstable canonical vacuum. The hope of the present work is to gain insight into how nonperturbative calculations might be performed at finite temperature. One way would be to include additional fields in the truncation scheme and to look for a better approximation to the effective potential. As another approach, since the off-shell extensions of the N-point tachyon amplitudes in Witten's covariant string field theory have been obtained,[13–16] one could try to derive an expression for the full static effective potential with tachyons as external states. But clearly some approximation scheme would be required here as well to obtain meaningful results.

Although the work presented here is for open bosonic strings and the particle fields derived from them, the same analysis may be applied to the closed bosonic string. It has been shown in Ref. 17 that a nonperturbative extremum in the effective potential exists for closed strings. The closed string theory exhibits the same features as open strings: The fields are smeared out, the coupling runs at the tree level, and the number of degrees of freedom is reduced in the nonperturbative vacuum.

In the next section, we present some background material on the level-truncation method of Refs. 8–10 and show how the momentum dependence enters into the $\tilde{\phi}^3$ model. In Sec. III, for the sake of comparison and as a limiting calculation, we work out the effective potential for conventional scalar ϕ^3 field theory and its finite-temperature extension. These results should correspond to the $\alpha'\to 0$ limit of the $\tilde{\phi}^3$ model. We work in four-dimensional space-time, where ϕ^3 theory is renormalizable, and we use the Euclidean formulation of finite-temperature field theory. The renormalizability of the $\tilde{\phi}^3$ theory in 4 and 26 dimensions is discussed as well. In Sec. IV we consider the finite-temperature behavior of the $\tilde{\phi}^3$ model. Our results and conclusions are summarized in Sec. V.

II. BACKGROUND

In covariant string field theory, the action is a Chern-Simons form:[11]

$$S=\frac{1}{2\alpha'}\int \Psi * Q\Psi+\frac{g}{3}\int \Psi*\Psi*\Psi\ . \tag{2.1}$$

The string field Ψ can be expanded in terms of ordinary particle fields whose coefficients are solutions of the first-quantized theory:

$$\Psi=(\phi+A_\mu\alpha^\mu_{-1}+i\alpha b_{-}c_0+\cdots)|0\rangle . \tag{2.2}$$

Here ϕ is the tachyon field, A_μ is the massless vector field, and α is an auxiliary field. The form of the three-string vertex in Eq. (2.1) introduces the momentum dependence of the particle fields. The particle fields f in the interaction Lagrangian become smeared over a distance $\sqrt{\alpha'}$:[9]

$$\tilde{f}=\exp\left[\alpha'\ln\left[\frac{3\sqrt{3}}{4}\right]\partial_\mu\partial^\mu\right]f . \tag{2.3}$$

A consequence of this is that the tree-level string coupling runs, which induces asymptotic freedom.[9,18] For the tachyon cubic coupling

$$\mathcal{L}_{\text{int}}=\frac{1}{3!}\lambda\tilde{\phi}^3, \tag{2.4}$$

where λ is the three-tachyon coupling (as well as some numerical factors),[18] and $\tilde{\phi}$ is defined as in Eq. (2.3). Going to momentum space, we find that this is a usual ϕ^3 interaction, but with a momentum-dependent coupling $(\lambda e^{-2\alpha'\ln(3\sqrt{3}/4)k^2})$. The additional momentum-dependent factors result from the nonlocal nature of the string and are what account for the good ultraviolet behavior of the theory.[18]

The level-truncation scheme of Refs. 8–10 consists of truncating the Fock-space expansion of the string field Ψ to the lowest-lying states. A combination of truncating both the expansion Ψ at a certain point and the level number of the interaction Lagrangian is used. The method is then utilized to find the dominant tree-level contributions to the static tachyon potential at all orders, and a candidate nonperturbative vacuum is obtained.

In the present work, we consider the case in which all the fields have been truncated to the level of the tachyon field. In particular, we consider a scalar field theory with a Euclidean Lagrangian

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2+\frac{1}{2}m^2\phi^2+\frac{\lambda}{3!}\tilde{\phi}^3, \tag{2.5}$$

where

$$\tilde{\phi}=e^{\alpha\partial^2}\phi \tag{2.6}$$

and

$$\alpha\equiv\alpha'\ln\frac{3\sqrt{3}}{4}. \tag{2.7}$$

The motivation here is to study the effects of the string-induced momentum dependence in the coupling and to go to the one-loop level in the effective potential and to finite temperature.

We define the field theory entirely in Euclidean space to facilitate the transition to finite temperature. In the imaginary-time formulation of bosonic finite-temperature field theory,[19-21] the time direction becomes periodic and the energy is discrete:

$$k_4=\frac{2\pi n}{\beta}, \tag{2.8}$$

where $\beta=1/T$ and T is the temperature. Integrals over the energy in one-loop calculations are replaced by sums

$$\int\frac{dk_4}{(2\pi)}\rightarrow\frac{1}{\beta}\sum_{n=-\infty}^{\infty}. \tag{2.9}$$

We work in four dimensions of space-time for convenience. For string theory, the fields are 26 dimensional, and a ϕ^3 scalar field theory in 26 dimensions is nonrenormalizable. However, the $\tilde{\phi}^3$ theory has the additional momentum-dependent factor $\exp(-2\alpha k^2)$ appearing in loop integrals, which gives the $\tilde{\phi}^3$ theory good ultraviolet behavior. Since, for comparison, we want to consider the case of ϕ^3 theory at finite temperature (for which $\alpha'=0$), we work in four dimensions where the ϕ^3 theory is renormalizable. The extension of the results presented here for $\tilde{\phi}^3$ to 26 dimensions is straightforward.

III. ϕ^3 THEORY

First, we will consider the case of ϕ^3 theory. This is the $\alpha'\rightarrow 0$ limit of the $\tilde{\phi}^3$ theory. The Euclidean Lagrangian is given by

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2+\frac{1}{2}m^2\phi^2+\frac{\lambda}{3!}\phi^3, \tag{3.1}$$

where $m^2<0$ for a theory describing tachyons. The tree-level contribution to the effective potential

$$V^{(0)}(\phi)=\frac{1}{2}m^2\phi^2+\frac{\lambda}{3!}\phi^3 \tag{3.2}$$

(with $\lambda>0$) has a local maximum at $\phi=0$ and a local minimum at $\phi=-2m^2/\lambda$. The canonical vacuum $\langle\phi\rangle=0$ is unstable because of the tachyon, and perturbation theory is ill defined about such a vacuum. The presence of the tachyon indicates that an incorrect choice of vacuum has been made. Shifting to the $\phi=-2m^2/\lambda$ vacuum, we obtain a physical mass for the scalar field equal to $-m^2$, which is positive. The theory remains unstable because of tunneling effects, as V_{eff} is unbounded from below as $\phi\rightarrow-\infty$; nonetheless, perturbative solutions in the new vacuum are well defined.

In open bosonic string theory, the effective potential is also that of a cubic interaction, which is unbounded from below as $\phi\rightarrow-\infty$. The canonical vacuum, in which all of the fields take zero expectation values, is unstable because of the tachyon. In Ref. 9 a candidate nonperturbative vacuum was found, and perturbation theory was shown to be well defined about the new vacuum. It was argued in Ref. 9 that barrier penetration of the full string out of the local minimum of the potential is inhibited relative to particle tunneling, since a string field consists of an infinite number of fields, and there is a suppression factor for each one. Although the cubic-interaction field theories considered here are unstable because of quantum tunneling effects, the full (untruncated) effective potential for $\tilde{\phi}$, when all the other fields in the string theory are included, should have an infinite number of suppression factors, which would combine to stabilize the nonperturbative vacuum. For the (truncated) $\tilde{\phi}$ theory, we are effectively examining a toy model, and we leave aside the question of the ultimate stability of the theory against tunneling until a fuller treatment of the string theory can be made. Instead, we will examine the effective potential at finite temperature for indications of a second-order phase transition. A second-order phase transition occurs when the minimum of the effective potential gradually merges with the maximum point, and there is no barrier penetration or tunneling (as in a first-order phase transition).

The zero-temperature one-loop contribution to the ϕ^3 theory effective potential is given by [22]

$$V^{(1)}(\phi)=\int\frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left[\frac{-\lambda\phi}{k^2+m^2}\right]^n. \tag{3.3}$$

The momenta k are in Euclidean space, and the mass-shell condition in $k^2=-m^2$.

The first two terms in this series are divergent and require counterterms. We add a counterterm in ϕ^3 as well for a finite renormalization of λ. We then sum the series and impose the conditions

$$\left.\frac{dV_{\text{eff}}}{d\phi}\right|_{\phi=0}=0,\quad\left.\frac{d^2V_{\text{eff}}}{d\phi^2}\right|_{\phi=0}=m^2,\quad\left.\frac{d^3V_{\text{eff}}}{d\phi^3}\right|_{\phi=0}=\lambda. \tag{3.4}$$

We find that the minimum of V_{eff} at $\phi=-2m^2/\lambda$ persists even after the one-loop effects have been included.

The mass in the minimum of V_{eff} is given by

$$\left.\frac{d^2V_{\text{eff}}}{d\phi^2}\right|_{\phi=-2m^2/\lambda}=-m^2+\frac{\lambda^2}{16\pi^2}+\text{imaginary part}. \tag{3.5}$$

The real part of this solution corresponds to a positive physical mass. The imaginary part indicates that the solution $\phi=-2m^2/\lambda$ has gone beyond the full range of validity of the one-loop approximation and that higher-order terms are required. Also, the ϕ^3 theory is intrinsically unstable—it is only when all the fields in the string theory are taken into account that the minimum of V_{eff}

becomes stable. For the model considered here, we will ignore the imaginary part and take the real part as an approximation to the behavior in the context of a full string theory.

We examine $V_{\rm eff}$ to see where its inflection points are. These are the values of ϕ for which $V_{\rm eff}''(\phi)=0$. At the tree level, with $V_{\rm eff}=\frac{1}{2}m^2\phi^2+(\lambda/3!)\phi^3$, the inflection point is at $\phi=-m^2/\lambda$, which is midway between the maximum at $\phi=0$ and the minimum at $\phi=-2m^2/\lambda$. When the one-loop effects are included, we find that the inflection point has moved to a new position at

$$\phi_{\rm inflection}=-\frac{m^2}{\lambda}-\frac{\lambda m^2/32\pi^2}{m^2-\lambda^2/32\pi^2} . \tag{3.6}$$

In a second-order phase transition at finite temperature, we expect the minimum point to meet up with the inflection point at the critical temperature. We find that this is indeed what happens in ϕ^3 theory at finite temperature.

Now we want to consider the case of finite temperature and reexamine the mass terms. At finite temperature, the effective potential V_β splits into two terms:

$$V_\beta=V_{T=0}+\frac{1}{2\pi^2\beta^4}\times\int_0^\infty dx\, x^2\ln(1-e^{-[x^2+\beta^2(m^2+\lambda\phi)]^{1/2}}) , \tag{3.7}$$

where a change of integration variable has been made from $|\mathbf{k}|$ to x. All the temperature dependence resides in the second term.

For small temperatures ($T \ll |m|$), we can show that, to lowest order, the minimum of this expression occurs for the value

$$\phi_{\rm min}(T)=-\frac{2m^2}{\lambda}+\frac{\lambda T^2}{4\pi^2}\left[\frac{\pi}{2}\right]^{1/2}\times\frac{1}{(m^2-\lambda^2/32\pi^2)}(\beta|m|)^{1/2}e^{-\beta|m|}, \tag{3.8}$$

where $|m|=(-m^2)^{1/2}$ and $\beta=1/T$. For small T the minimum is displaced toward the origin $\phi=0$.

At higher temperatures ($T \gg |m|$ and for $\lambda \ll |m|$), we find that the minimum in V_β is at

$$\phi_{\rm min}(T)=-\frac{2m^2}{\lambda}-\frac{\lambda T^2}{24(\lambda^2/32\pi^2-m^2)} . \tag{3.9}$$

The solution is displaced toward the origin as well, away from the $T=0$ solution at $\phi=-2m^2/\lambda$. Furthermore, the depth of the potential well at this minimum, $V_\beta(\phi_{\rm min}(T))$, decreases as T gets large.

We have examined the maximum in the effective potential as well—the $\phi=0$ solution at $T=0$. We find that this maximum moves toward positive values of ϕ as the temperature increases, and therefore that the maximum and minimum points move toward each other as the temperature increases. In fact, we find that the minimum point meets up with the zero-temperature inflection point $\phi_{\rm inflection}$, given in Eq. (3.6), when

$$T^2=\frac{24m^2(m^2-\lambda^2/32\pi^2)}{\lambda^2} . \tag{3.10}$$

This suggests that there is a second-order phase transition at this temperature.

In $\lambda\phi^4$ field theory, with spontaneous symmetry breaking, there can be either a first- or second-order phase transition, depending on the shape of V_β.[23] In a first-order phase transition, there is tunneling through a barrier from one vacuum to the other, whereas in a second-order phase transition, the extremum points move together at the critical temperature. In a $\lambda\phi^4$ theory, it suffices to look at $V_{\rm eff}$ at $\phi=0$ regardless of the temperature, and we need not work in the temperature-dependent local minimum of $V_{\rm eff}$. The symmetry in the theory prohibits the extremum at $\phi=0$ from wandering from this position.

For ϕ^3 theory, on the other hand, the potential has no symmetry, and the situation is different. The theory is unstable against quantum or thermal tunneling, and so there is no point in considering a first-order phase transition. It does, however, appear that there is a critical temperature for a second-order phase transition in ϕ^3 theory. To verify this more explicitly, we look at the second derivative of V_β, namely, the mass term at finite temperature.

We define the temperature-dependent mass squared by $m_T^2(\phi=\langle\phi\rangle)=V_\beta''(\phi=\langle\phi\rangle)$. Taking the second derivative V_β and choosing $\langle\phi\rangle=0$ as the vacuum, we find

$$m_T^2(\phi=0)=m^2-\frac{\lambda^2}{8\pi^2}\int\frac{dx\, x^2}{(x^2+\beta^2m^2)}\frac{1}{(e^{(x^2+\beta^2m^2)^{1/2}}-1)}\left[\frac{1}{(x^2+\beta^2m^2)^{1/2}}+\frac{e^{(x^2+\beta^2m^2)^{1/2}}}{e^{(x^2+\beta^2m^2)^{1/2}}-1}\right] . \tag{3.11}$$

Here we have to be concerned with the fact that $m^2<0$ and that the integral is complex. Following the procedure of Ref. 19, we find that the lowest-order temperature correction is imaginary, and therefore this expression is not useful.

Instead, we concentrate on the temperature-dependent mass as defined in the physical vacuum, $\phi=-2m^2/\lambda$, which corresponds to a field of positive mass squared at the tree level. In this case, we get real contributions at finite temperature given by

$$V_\beta''\left[\phi=-\frac{2m^2}{\lambda}\right]=-m^2+\frac{\lambda^2}{16\pi^2}-\frac{\lambda^2}{4}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2-m^2}\frac{1}{e^{\beta(\mathbf{k}^2-m^2)^{1/2}}-1}\left[\frac{1}{(\mathbf{k}^2-m^2)^{1/2}}+\frac{\beta e^{\beta(\mathbf{k}^2-m^2)^{1/2}}}{e^{\beta(\mathbf{k}^2-m^2)^{1/2}-1}}\right], \tag{3.12}$$

where now we can evaluate the integral in the small-β (high-temperature) limit. Keeping only the lowest-order real contributions, we get

$$V''_\beta\left[\phi=-\frac{2m^2}{\lambda}\right]=-m^2+\frac{\lambda^2}{16\pi^2}\left[1-\frac{\pi}{\beta|m|}\right]+\cdots . \tag{3.13}$$

This would appear to have a critical temperature at $T=16\pi|m|^3/\lambda^2+|m|/\pi$, but this conclusion is incorrect since it does not take into account the fact that the minimum of V_β moves away from $-2m^2/\lambda$ as T increases. To find m_T^2 at the minimum, what we need to use is $\phi_{\min}(T)$, given in Eq. (3.9) for large T, and to evaluate $V_\beta''(\phi_{\min}(T))$.

For $\beta|m|\ll 1$ and $|m|^2\gg\lambda$, we find an additional contribution to m_T^2:

$$m_T^2=-m^2+\frac{\lambda^2}{16\pi^2}\left[1+\frac{2\pi^2T^2}{3m^2}-\frac{\pi T}{|m|}\right]+\cdots . \tag{3.14}$$

This mass squared becomes zero at the temperature

$$T^2=\frac{24m^2(m^2-\lambda^2/32\pi^2)}{\lambda^2}\approx\frac{24m^4}{\lambda^2}, \tag{3.15}$$

which agrees with the temperature found earlier in Eq. (3.10) for which $\phi_{\min}$ meets up with the inflection point in V_{eff}. Thus we find that there is a second-order phase transition. The minimum in V_{eff} disappears at this critical temperature, and we no longer have a stable vacuum.

IV. $\tilde{\phi}^3$ THEORY

Now we want to consider the case of the $\tilde{\phi}^3$ theory defined by the Euclidean Lagrangian

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2+\frac{1}{2}m^2\phi^2+\frac{\lambda}{3!}\tilde{\phi}^3, \tag{4.1}$$

with

$$\tilde{\phi}=e^{\alpha\partial^2}\phi \tag{4.2}$$

and

$$\alpha=\alpha'\ln\frac{3\sqrt{3}}{4}. \tag{4.3}$$

The Feynman rules for this theory are the same as for the ϕ^3 theory except that now the coupling in the interaction has an additional momentum dependence $\lambda\to\lambda e^{-2\alpha k^2}$. The effective potential up to the one-loop level for this theory is given by

$$V_{\text{eff}}=\frac{1}{2}m^2\phi^2+\frac{\lambda}{3!}\phi^3+\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\{\ln[k^2+m^2+(\lambda\phi)e^{-2\alpha k^2}]-\ln(k^2+m^2)\}+\text{counterterms} . \tag{4.4}$$

By expanding the logarithm, we see that this expression consists of the sum of diagrams

$$V_{\text{eff}}=\frac{1}{2}m^2\phi^2+\frac{\lambda}{3!}\phi^3-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{-(\lambda\phi)e^{-2\alpha k^2}}{k^2+m^2}\right]^n \tag{4.5}$$

The low-level diagrams are ultra violet finite because of the additional exponential factor in the interaction.

For example, the static tadpole diagram for $\tilde{\phi}$ theory is given by

$$\Gamma_E^{(1)}(0)=\int\frac{d^4k}{(2\pi)^4}\frac{-\lambda e^{-2\alpha k^2}}{k^2+m^2}. \tag{4.6}$$

This can be reexpressed as

$$\Gamma_E^{(1)}(0)=-\frac{\lambda}{16\pi^2}\left[\frac{1}{2\alpha}+m^2e^{2\alpha m^2}\Gamma(0,2\alpha m^2)\right] \tag{4.7}$$

where $\Gamma(0,x)$ is the incomplete gamma function

$$\Gamma(0,x)=e^{-x}\int_0^\infty\frac{e^{-t}}{x+t}dt, \tag{4.8}$$

which is finite for $x\neq 0$. For $m^2<0$, $\Gamma(0,2\alpha m^2)$ has a pole, and we take the principal part of the integral. Then, as long as $\alpha\neq 0$, the integral is ultraviolet finite.

For the two-point function, we get

$$\Gamma_E^{(2)}(0,0)=\int \frac{d^4k}{(2\pi)^4}\frac{\lambda^2 e^{-4\alpha k^2}}{(k^2+m^2)^2}=\frac{\lambda^2}{16\pi^2}[(1-4\alpha m^2)e^{4\alpha m^2}\Gamma(0,4\alpha m^2)-1], \tag{4.9}$$

which is also ultraviolet finite for $\alpha\neq 0$.

Since the tadpole and two-point diagrams for $\tilde{\phi}^3$ theory are ultraviolet finite, the counterterms provide only a finite renormalization of λ and m, and are determined by imposing Eq. (3.4). We find

$$V_{\rm eff}^{(\alpha)}=\frac{1}{2}m^2\phi^2+\frac{\lambda}{3!}\phi^3+\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left[\ln\left[1+\frac{(\lambda\phi)e^{-2\alpha k^2}}{k^2+m^2}\right]-\frac{(\lambda\phi)e^{-2\alpha k^2}}{k^2+m^2}+\frac{\frac{1}{2}(\lambda\phi)^2e^{-4\alpha k^2}}{(k^2+m^2)^2}-\frac{\frac{1}{3}(\lambda\phi)^3e^{-6\alpha k^2}}{(k^2+m^2)^3}\right]. \tag{4.10}$$

As $\alpha\to 0$, corresponding to the particle limit of the string-related field, this expression reduces back to the result for $V_{\rm eff}$ for ϕ^3 theory. We are interested in the behavior of the effective potential near the minimum point $\phi=-2m^2/\lambda$. The momentum dependence of $V_{\rm eff}^{(\alpha)}$ makes it impossible to evaluate the integrals explicitly. Instead, we look at the near-particle limit, where we replace $\mathcal{L}_{\rm int}$ for small α by

$$\mathcal{L}_{\rm int}=\frac{\lambda}{3!}e^{-2\alpha k^2}\phi^3\approx\frac{\lambda}{3!}(1-2\alpha k^2)\phi^3 \tag{4.11}$$

in the momentum-space integrals. The resulting expression for $V_{\rm eff}^{(\alpha)''}(-2m^2/\lambda)$, the mass in the minimum of the effective potential, can then be evaluated in this approximation. We find

$$V_{\rm eff}^{(\alpha)''}\left[-\frac{2m^2}{\lambda}\right]=-m^2-\frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}(1-4\alpha k^2)\left[\frac{1}{(k^2-m^2)^2}-\frac{1}{(k^2+m^2)^2}-\frac{8\alpha m^2k^2}{(k^2-m^2)^3}-\frac{4m^2(1-2\alpha k^2)}{(k^2+m^2)^3}\right]. \tag{4.12}$$

This contains the $\alpha=0$ part $V_{\rm eff}''$ plus a correction

$$V_{\rm eff}^{(\alpha)''}\left[-\frac{2m^2}{\lambda}\right]=V_{\rm eff}^{(\alpha=0)''}\left[-\frac{2m^2}{\lambda}\right]+2\alpha\lambda^2\int\frac{d^4k}{(2\pi)^4}k^2\left[\frac{1}{(k^2-m^2)^2}-\frac{1}{(k^2+m^2)^2}-\frac{4m^2}{(k^2+m^2)^3}\right]. \tag{4.13}$$

The remaining integrals can be regularized by introducing a cutoff and can be evaluated. Using Eq. (3.5) for $V_{\rm eff}^{(\alpha=0)''}(-2m^2/\lambda)$, we find

$$V_{\rm eff}^{(\alpha)''}\left[-\frac{2m^2}{\lambda}\right]=-m^2+\frac{\lambda^2}{16\pi^2}(1-16\alpha m^2). \tag{4.14}$$

This is the mass in the minimum of the potential including the lowest-order one-loop effects coming from the momentum dependence of the coupling. For $m^2<0$, the correction is positive and adds to the mass, stabilizing the vacuum even more.

For the $\tilde{\phi}^3$ theory at finite temperature, we have to evaluate the sum in the one-loop contribution

$$V_\beta^{(1)}=\frac{1}{2\beta}\int\frac{d^3k}{(2\pi)^3}\sum_{n=-\infty}^{\infty}\ln\left[1+\frac{(\lambda\phi)e^{2\alpha m^2}\exp[-2\alpha(4\pi^2n^2/\beta^2+\omega_k^2)]}{4\pi^2n^2/\beta^2+\omega_k^2}\right], \tag{4.15}$$

where Eqs. (2.8) and (2.9) have been used. Since what we are interested in is the mass in the minimum of the potential at finite temperature, we take the second derivative of $V_\beta^{(1)}$ and set $\phi=-2m^2/\lambda$:

$$V_\beta^{(1)''}\left[-\frac{2m^2}{\lambda}\right]=\frac{-\lambda^2}{2\beta}\int\frac{d^3k}{(2\pi)^3}\sum_{n=-\infty}^{\infty}\frac{e^{4\alpha m^2}\exp[-4\alpha(4\pi^2n^2/\beta^2+\omega_k^2)]}{\{4\pi^2n^2/\beta^2+\omega_k^2-2m^2e^{2\alpha m^2}\exp[-2\alpha(4\pi^2n^2/\beta^2+\omega_k^2)]\}^2}. \tag{4.16}$$

This sum cannot be evaluated explicitly, and again we consider the approximation in which α is small—the near-particle limit—and replace $\mathcal{L}_{\rm int}$ by

$$\mathcal{L}_{\rm int}=\frac{\lambda}{3!}(1-2\alpha k^2)\phi^3=\frac{\lambda}{3!}\left[1-2\alpha\left[\frac{4\pi^2n^2}{\beta^2}+\mathbf{k}^2\right]\right]\phi^3. \tag{4.17}$$

Then we obtain

$$V_\beta^{(1)''}\left[-\frac{2m^2}{\lambda}\right]=\frac{-\lambda^2}{2\beta}\int\frac{d^3k}{(2\pi)^3}\sum_{n=-\infty}^{\infty}\frac{1-4\alpha(4\pi^2n^2/\beta^2+\mathbf{k}^2)}{4\pi^2n^2/\beta^2+\mathbf{k}^2-m^2+8\alpha m^2(4\pi^2n^2/\beta^2+\mathbf{k}^2)}. \tag{4.18}$$

We expand the denominator, keeping only the lowest-order terms in α, and use the summation formulas given in the Appendix. We obtain

$$V_{\beta}^{(1)''}\left[-\frac{2m^2}{\lambda}\right]=\frac{-\lambda^2}{4}\int\frac{d^3k}{(2\pi)^3}\left[\left[\frac{1}{2}+\frac{1}{e^{\beta(\mathbf{k}^2-m^2)^{1/2}}-1}+\frac{\beta(\mathbf{k}^2-m^2)^{1/2}e^{\beta(\mathbf{k}^2-m^2)^{1/2}}}{(e^{\beta(\mathbf{k}^2-m^2)^{1/2}}-1)^2}\right]\right.$$
$$\times\left[\frac{1-4\alpha\mathbf{k}^2}{(\mathbf{k}^2-m^2)^{3/2}}-\frac{4\alpha}{(\mathbf{k}^2-m^2)^{1/2}}-\frac{4\alpha m^2}{(\mathbf{k}^2-m^2)^{3/2}}\right]$$
$$\left.-\frac{\alpha m^2}{(\mathbf{k}^2-m^2)^{3/2}}\left[\frac{\beta^2(\mathbf{k}^2-m^2)e^{\beta(\mathbf{k}^2-m^2)^{1/2}}}{(e^{\beta(\mathbf{k}^2-m^2)^{1/2}}-1)^2}-\frac{2\beta^2(\mathbf{k}^2-m^2)e^{\beta(\mathbf{k}^2-m^2)^{1/2}}}{(e^{\beta(\mathbf{k}^2-m^2)^{1/2}}-1)^3}\right]\right]. \quad (4.19)$$

By letting $\beta\rightarrow\infty$, we can extract the one-loop $T=0$ piece:

$$V_{T=0}^{(1)''}\left[-\frac{2m^2}{\lambda}\right]=\frac{-\lambda^2}{8}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1-4\alpha\mathbf{k}^2}{(\mathbf{k}^2-m^2)^{3/2}}-\frac{4\alpha}{(\mathbf{k}^2-m^2)^{1/2}}-\frac{4\alpha m^2}{(\mathbf{k}^2-m^2)^{3/2}}\right]. \quad (4.20)$$

For the full effective potential up to one loop, we add the tree-level terms plus the counterterms. All of the renormalizations occur at $T=0$ and are taken into account by the $T=0$ part of V_β. We can extract as well the finite-temperature $\alpha=0$ piece of Eq. (4.19) and use the results of ϕ^3 theory found earlier. In this way, by using Eqs. (4.14) and (3.22), we find that, to lowest order in α and for $T\gg|m|$,

$$m_T^2=m^2+\frac{\lambda^2}{16\pi^2}\left[1-16\alpha m^2+\frac{2\pi^2T^2}{3m^2}\right]+\delta m^2. \quad (4.21)$$

Here δm^2 is the remaining finite-temperature α-dependent piece of Eq. (4.19), which we approximate for high temperature as

$$\delta m^2=\frac{2\alpha\lambda^2}{\pi^2\beta^2}\int_0^\infty y\,dy\left[\frac{1}{e^y-1}+\frac{ye^y}{(e^y-1)^2}\right], \quad (4.22)$$

where we substituted $y=\beta k$. These integrals can be evaluated, and we get

$$\delta m^2=\frac{\alpha\lambda^2}{\beta^2}, \quad (4.23)$$

which we add to m_T^2. The final expression for the temperature-dependent mass to lowest order in α is then

$$m_T^2=-m^2+\frac{\lambda^2}{16\pi^2}\left[1-16\alpha(m^2-\pi^2T^2)+\frac{2\pi^2T^2}{3m^2}\right]. \quad (4.24)$$

The theory has a second-order phase transition, since for high enough temperature $m_T^2=0$, and the minimum of V_{eff} becomes an inflection point. Solving for the critical temperature to lowest order in α and λ/m^2, we get

$$T^2=(1-24\alpha m^2)\frac{24m^2}{\lambda^2}. \quad (4.25)$$

Comparing this with Eq. (3.23), the $\alpha=0$ result obtained for ϕ^3 theory at finite temperature, we see that the effect of the α dependence is to drive the critical temperature higher (since $m^2<0$). Thus for the $\tilde{\phi}^3$ theory, there continues to be a second-order phase transition, but the momentum dependence in the coupling has raised the critical temperature.

This result for the critical temperature clearly depends on the interactions between $\tilde{\phi}$ fields, and it explicitly depends on the coupling λ and the momentum factor α. It is thus very different from the Hagedorn temperature, which results from the exponentially increasing density of states of a string theory in an ideal-gas approximation. The Hagedorn temperature, as originally proposed, does not take interactions into account.[1-5] In Ref. 7 interactions were included in the analysis, and it was shown that there is a phase transition near the Hagedorn temperature associated with winding modes of the finite-temperature string theory becoming tachyonic. These winding modes have a symmetry that forces the effective potential, defined in terms of them, to be a quartic function, and it is argued that the phase transition is of first order.[7] The winding modes become tachyonic because above the Hagedorn temperature the stable vacuum has shifted, and the winding mode fields need to be given nonzero expectation values.[7] The first-order phase transition is then a jump from the wrong vacuum to the stable one.

There are, however, two additional ingredients that need to be considered. First is the fact that, even at zero temperature, on must define the theory in the correct vacuum. The bosonic string, with a cubic interaction, has a tachyon in its spectrum, and the canonical vacuum is intrinsically unstable. As shown in Refs. 8–10, there does exist a stable nonperturbative vacuum, and when the theory is defined about this vacuum, the tachyon disappears and the number of degrees of freedom is reduced.[9] This is exactly analogous to what is happening at finite temperature for the case of the winding modes,[7] only here it is happening at zero temperature and for the tachyon. The second new ingredient that has to be taken into account is that the particle fields coming from a string field theory have momentum-dependent couplings.[9] This arises from the extended nature of the string and accounts for its good ultraviolet behavior. Any discussion of string theory at finite temperature that includes interactions needs to take these two additional ingredients into account: The theory exhibits complicated vacuum structure even at zero temperature, and the coupling runs even at the tree level.

V. CONCLUSIONS

Our goal here has been to study, in the context of a toy model, the effects of vacuum structure and the momentum dependence of the string coupling on the finite-temperature behavior of open bosonic strings. In particular, we wanted to see if there are any phase transitions occuring between the possible vacua of the theory. Since, for the full string theory, this is an extremely complex problem, several approximations have been made. First, we have truncated the string field to its lowest level—that of a tachyonic scalar field $\tilde{\phi}$. This field by itself exhibits both of the properties we are interested in—it has nontrivial vacuum structure, and its coupling contains the momentum-dependent factor $e^{-2\alpha k^2}$. Thus this $\tilde{\phi}$ theory serves well as a toy model in which to study the effects of finite temperature. Next, we have assumed that we are interested in the $\tilde{\phi}$ theory primarily in the context of a full string theory. A cubic $\tilde{\phi}$ theory by itself is unstable to tunneling of the fields off to $\phi \to -\infty$,though we may still define the theory perturbatively in the new vacuum. In the context of a full string theory, however, with an infinite number of fields, tunneling out of the minimum is greatly suppressed.[9] Our approximation here has been to ignore questions concerning the instability of the theory due to tunneling and to restrict our investigation of phase transitions to second-order phase transitions, which do dot involve tunneling. Last, we have worked in a near-particle limit in which the parameter α, coming from the extended nature of the string, is taken to be small. We have also taken λ and β to be small compared to the mass $|m|$.

Within the restrictions set by considering such a toy model, we have been able to determine that there is a second-order phase transition in the $\tilde{\phi}^3$ theory, and that it occurs at a temperature $T^2=(1-24\alpha m^2)24m^2/\lambda^2$. The effect of the string correction (the α terms) is to deepen the potential well at the minimum and to drive up the critical temperature of the phase transition. However, at high enough temperatures (at this level of approximation), the theory becomes unphysical again, and there is no stable vacuum.

In comparing ϕ^3 theory to $\tilde{\phi}^3$ theory, there are several interesting properties to point out. While ϕ^3 in greater than six dimensions is a nonrenormalizable theory, $\tilde{\phi}$ is not. The additional momentum dependence in the coupling makes all the Feynman graphs ultraviolet finite. This is a direct consequence of the fact that the $\tilde{\phi}^3$ theory is coming from a string theory, which has good ultraviolet properties. We have also seen that ϕ^3 theory by itself has a second-order phase transition. At zero temperature, the $\langle\phi\rangle=0$ vacuum is unstable, but there does exist a minimum in the effective potential at $\langle\phi\rangle=-2m^2/\lambda$, in which the field has a physical mass. When the temperature turns on, this minimum point moves, and gradually the minimum and maximum points come together at the inflection point, and there is no longer a locally stable vacuum in the theory. The $\tilde{\phi}$ theory for small α has these same properties, but the minimum is somewhat more pronounced and the critical temperature of the second-order phase transition is higher. Thus, to this level of approximation, the string effects tend to stabilize the vacuum as compared to ϕ^3 theory. To determine whether a stable vacuum exists at temperatures above the critical temperature requires going beyond the lowest level of approximation and taking into account the full suppression factor $e^{-2\alpha k^2}$ in the coupling. This in turn would require doing the sum in Eq. (4.16) and at present does not seem feasible.

The Hagedorn temperature, as originally introduced, did not involve interactions between strings. As interactions are taken into account, it is seen that what really matters is vacuum structure. In Ref. 7 it was shown that winding modes can acquire unphysical masses near the Hagedorn temperature and that a phase transition can occur. Even at zero temperature, the bosonic string exhibits nontrivial vacuum structure, and a stable vacuum can be found that eliminates the tachyon. For a complete understanding of the Hagedorn transitions in a string theory, the vacuum structure of all the particle fields will have to be better understood, and the full momentum dependence of the interactions will have to be included.

ACKNOWLEDGMENT

I would like to thank Alan Kostelecký for many helpful discussions.

APPENDIX

The following three summation formulas are used in the derivation of Eq. (4.19):

$$\sum_{n=-\infty}^{\infty}\frac{1}{(4\pi^2n^2/\beta^2+k^2-m^2)^2}=\frac{\beta}{2(k^2-m^2)^{3/2}}\left[\frac{1}{2}+\frac{1}{e^{\beta(k^2-m^2)^{1/2}-1}}+\frac{\beta(k^2-m^2)^{1/2}e^{\beta(k^2-m^2)^{1/2}}}{(e^{\beta(k^2-m^2)^{1/2}}-1)^2}\right], \tag{A1}$$

$$\sum_{n=-\infty}^{\infty}\frac{4\pi^2n^2/\beta^2}{(4\pi^2n^2/\beta^2+k^2-m^2)^2}=\frac{\beta}{2(k^2-m^2)^{1/2}}\left[\frac{1}{2}+\frac{1}{e^{\beta(k^2-m^2)^{1/2}}-1}-\frac{\beta(k^2-m^2)^{1/2}e^{\beta(k^2-m^2)^{1/2}}}{(e^{\beta(k^2-m^2)^{1/2}}-1)^2}\right], \tag{A2}$$

$$\sum_{n=-\infty}^{\infty}\frac{4\pi^2n^2/\beta^2}{(4\pi^2n^2/\beta^2+k^2-m^2)^3}=\frac{\beta}{8(k^2-m^2)^{3/2}}\left[\frac{1}{2}+\frac{1}{e^{\beta(k^2-m^2)^{1/2}}-1}+\frac{\beta(k^2-m^2)^{1/2}e^{\beta(k^2-m^2)^{1/2}}}{(e^{\beta(k^2-m^2)^{1/2}}-1)^2}\right.$$
$$\left.+\frac{\beta^2(k^2-m^2)e^{\beta(k^2-m^2)^{1/2}}}{(e^{\beta(k^2-m^2)^{1/2}}-1)^2}-\frac{2\beta^2(k^2-m^2)e^{\beta(k^2-m^2)^{1/2}}}{(e^{\beta(k^2-m^2)^{1/2}}-1)^3}\right]. \tag{A3}$$

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