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Quantum wormholes

Matt Visser
Physics Department, Washington University, St. Louis, Missouri 63130-4899
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This paper presents an application of quantum-mechanical principles to a microscopic variant of the traversable wormholes recently introduced by Morris and Thorne. The analysis, based on the surgical grafting of two Reissner-Nordström spacetimes, proceeds by using a minisuperspace model to approximate the geometry of these wormholes. The "thin shell" formalism is applied to this minisuperspace model to extract the effective Lagrangian appropriate to this one-degree-of-freedom system. This effective Lagrangian is then quantized and the wave function for the wormhole is explicitly exhibited. A slightly more general class of wormholes—corresponding to the addition of some "dust" to the wormhole throat—is analyzed by recourse to WKB techniques. In all cases discussed in this paper, the expectation value of the wormhole radius is calculated to be of the order of the Planck length. Accordingly, though these quantum wormholes are of considerable theoretical interest they do not appear to be useful as a means for interstellar travel. The results of this paper may also have a bearing on the question of topological fluctuations in quantum gravity. These calculations serve to suggest that topology-changing effects might in fact be *suppressed* by quantum-gravity effects.

I. INTRODUCTION

The past 3 years have seen a massive resurgence of interest in wormhole physics. The interest in Lorentzian wormholes was rekindled by the work of Morris and Thorne.[1] These authors constructed and investigated a class of objects they referred to as "traversable wormholes." Prompted by that work, investigations have been initiated of many other aspects of wormhole physics such as time machines and causality,[2–4] simple examples,[5,6] classical stability,[7] quantum models,[8–11] and energy extraction.[12] This paper is intended to give a fuller account of the present author's work on quantum-mechanical aspects of the stability of Lorentzian wormholes.[9,10]

As a separate issue, the last 3 years have also seen a flurry of interest in Euclidean wormholes. These Euclidean wormholes are alleged to be of interest in attempting to solve the cosmological-constant problem. For a summary of the status of these Euclidean ideas, see Unruh.[13] It is to be emphasized that the Lorentzian wormholes espoused by Morris and co-workers are a topic completely disjoint from the Euclidean ideas just mentioned. We shall have no further need to discuss Euclidean wormholes in this paper.

This paper—concentrating on the quantum-mechanical aspects of Lorentzian wormholes—must immediately address the question of quantum gravity. Now the fundamental principles of quantum gravity are as yet obscure. No satisfactory formulation of the problem exists. When confronted with interpretational and formal problems of this magnitude, one's only hope of being able to calculate is to resort to some (drastic) approximation scheme.

The approximation scheme to be used in this paper is the minisuperspace restriction of the canonical Wheeler-DeWitt formalism. The basic idea of the minisuperspace approach is to separate the three-metric into "modes" and then insist that all but a finite number of these modes (often one) be forced to satisfy the *classical* Einstein field equations. The remaining finite number of modes (*not* satisfying the Einstein field equations) are then quantized by following the standard prescription of canonical quantization.

This approach is most commonly adopted in quantum cosmology calculations.[14] Typically, all the "translational" modes of the three-metric are "frozen out" by using the classical field equations, leaving only the "radius of the Universe" (more precisely, the scale factor) to be quantized. In the approach adopted in this paper, a Lorentzian wormhole is modeled by two spacetimes connected by a "hole." Everywhere except the hole, the classical field equations are assumed to be satisfied, leaving only the *radius* of the hole as the *one* degree of freedom subject to quantization. This quantization is carried out, leading to an expectation value for the radius of the wormhole which is of order the Planck length. This indicates (at least within the context of the minisuperspace approximation) that the wormhole is stabilized against collapse by quantum effects. Such a result, if it proves to persist beyond the minisuperspace approximation, has important implications for the process of quantum-mechanical topology change. It is in fact possible to use the results of this paper to argue that quantum effects might *suppress* topology change.[11]

The paper is organized as follows. In Sec. II we describe in detail the minisuperspace model we adopt for Lorentzian wormholes. Canonical quantization is carried out in Sec. III, while Sec. IV discusses the exact wave function appropriate to the "dust-free" wormhole, and

Sec. V discusses WKB methods as applied to "dusty" wormholes. Finally, Sec. VI consists of discussion and conclusions. Throughout the paper we adopt "geometrodynamic units" so that $G\equiv1$, $c\equiv1$, and $\hbar\equiv L_P^2\equiv M_P^2$, where L_P and M_P are the Planck length and Planck mass, respectively.

II. MINISUPERSPACE MODEL FOR LORENTZIAN WORMHOLES

To construct the class of wormholes of interest, consider two copies of the Reissner-Nordström geometry. Both geometries are taken to be of mass M, while one geometry has charge $+Q$ and the other has charge $-Q$. One may temporarily wish to assume that $|Q|>M$ so that no event horizons exist and the Schwarzschild coordinate patch covers the complete geometry. This helps in visualizing the construction, but our results are not restricted to the $|Q|>M$ case. The metric of the Reissner-Nordström geometry is

$$ds^2=-\left[1-\frac{2M}{r}+\frac{Q^2}{r^2}\right]dt^2+\frac{dr^2}{1-2M/r+Q^2/r^2}+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\ , \tag{2.1}$$

while the electromagnetic field is simply $E=Q/r^2$.

From each copy of the Reissner-Nordström geometry, one removes identical four-dimensional regions of the form $\{(t,r,\theta,\phi)|r<a(\tau)\}$. One is then left with two geodesically incomplete manifolds whose boundaries are given by the timelike hypersurfaces $\{(t,r,\theta,\phi)|r=a(\tau)\}$. One now proceeds to identify these two hypersurfaces (by "sewing" them together). The resulting spacetime is geodesically complete. It possesses two asymptotically flat regions connected by a wormhole. The throat of this wormhole is at $r=a(\tau)$. Note that this spacetime is completely singularity-free because the region surrounding $r=0$ has been explicitly excluded from the manifold. In particular, for $|Q|>M$ the *naked* singularity is excluded and need not further concern us. Because this manifold is piecewise Reissner-Nordström, the Ricci scalar is everywhere zero, except at the throat itself. Observe that the electric flux lines thread the throat of the wormhole and do not terminate. Thus there is no electric charge present anywhere in the model wormhole. This geometry is "tailor made" for an application of the "thin-shell" formalism (the "junction condition" formalism).[15] At the throat of the wormhole the Riemann curvature tensor is proportional to a δ function. The Ricci tensor at the junction can be calculated in terms of the extrinsic curvature[7,15,16] (second fundamental form)

$$\mathcal{K}^i{}_j=\frac{1}{2}g^{ik}\frac{\partial g_{kj}}{\partial\eta}\bigg|_{\eta=0}\ . \tag{2.2}$$

Here η denotes the normal coordinate to the throat, while τ denotes proper time along the throat. The Ricci tensor is almost everywhere (except at the throat) that of a Reissner-Nordström geometry:[7]

$$R^\mu{}_\nu(x)=R^{(\mathrm{RN})\mu}{}_\nu(x)-2\begin{bmatrix}\mathcal{K}^i{}_j(x) & 0\\ 0 & \mathcal{K}(x)\end{bmatrix}\delta(\eta)\ , \tag{2.3}$$

so that the Einstein-Hilbert action reduces to

$$S_g=\frac{1}{16\pi}\int_{\mathcal{M}}\sqrt{g_4}R=\frac{-1}{4\pi}\int_{\partial\Omega}\sqrt{g_3}\mathcal{K}\ . \tag{2.4}$$

By spherical symmetry, the extrinsic curvature contains only two nontrivial components: $\mathcal{K}^\theta{}_\theta\equiv\mathcal{K}^\phi{}_\phi$ and $\mathcal{K}^\tau{}_\tau$. These components may conveniently be extracted from Ref. 7 ($\dot a\equiv da/d\tau$—an overdot denotes a proper-time derivative):

$$\mathcal{K}^\theta{}_\theta\equiv\mathcal{K}^\phi{}_\phi=\frac{1}{a}\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}+\dot a^2\right]^{1/2},$$

$$\begin{aligned}\mathcal{K}^\tau{}_\tau&=\frac{\ddot a+M/a^2-Q^2/a^3}{(1-2M/a+Q^2/a^2+\dot a^2)^{1/2}}\\ &=\frac{d}{d\tau}\operatorname{arcsinh}\left[\dot a\Big/\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}\right]^{1/2}\right]+\frac{1}{a^2}\left[M-\frac{Q^2}{a}\right]\frac{dt}{d\tau}\ .\end{aligned} \tag{2.5}$$

Note carefully the distinction between Schwarzschild time t and proper time τ. They are related by

$$\frac{dt}{d\tau}=\frac{(1-2M/a+Q^2/a^2+\dot a^2)^{1/2}}{1-2M/a+Q^2/a^2}\ . \tag{2.6}$$

For completeness we mention that the four-velocity of a point on the throat is

$$\begin{aligned}U^\mu&=\left[\frac{dt}{d\tau},\frac{da}{d\tau},0,0\right]\\ &=\left[\frac{(1-2M/a+Q^2/a^2+\dot a^2)^{1/2}}{1-2M/a+Q^2/a^2},\dot a,0,0\right]\ ,\end{aligned} \tag{2.7}$$

and that the unit normal to the throat is

$$\xi^\mu=\left[\frac{\dot a}{1-2M/a+Q^2/a^2},\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}+\dot a^2\right]^{1/2},0,0\right]\ . \tag{2.8}$$

Since $\sqrt{g_3}d^3x\mapsto4\pi a^2d\tau$, an integration by parts leads to

$$S_g=2\int\left\{a\dot a\operatorname{arcsinh}\left[\dot a\Big/\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}\right]^{1/2}\right]-a\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}+\dot a^2\right]^{1/2}\right\}d\tau-\int\left[M-\frac{Q^2}{a}\right]dt\ . \tag{2.9}$$

It should be noted that this calculation is a direct analog of the minisuperspace techniques more commonly used in quantum cosmology.[14] The gravitational action has been written in this way with malice aforethought. We shall see that the second integral (the $\int dt$) is effectively absent from the total action. This integral is partly canceled by the electromagnetic contribution to the action, and the remaining portion $\int M\,dt$ is merely a reflection of the fact that the mass of the system imposes an "imprint at infinity" on the metric.[17] In each asymptotically flat region, the electromagnetic contribution to the action is

$$S_{\rm em}^{(1)}=\frac{-1}{16\pi}\int F^2=\frac{-1}{16\pi}\int 4\pi r^2\frac{2Q^2}{r^4}dr\,dt=\frac{Q^2}{2}\int_a^\infty\frac{dr}{r^2}dt=-\frac{Q^2}{2}\int\frac{dt}{a(t)}=S_{\rm em}^{(2)}\ . \tag{2.10}$$

A convenient technical trick at this stage is to cut off the Reissner-Nordström geometry at some large fixed radius ρ. Then attach this truncated Reissner-Nordström geometry to a piece of Minkowski space. This procedure contributes to the Einstein-Hilbert action an amount

$$S_\rho^{(1)}=\tfrac{1}{2}\int\int[\mathcal{K}]\rho^2dt=\tfrac{1}{2}\int M\,dt+O(1/\rho)=S_\rho^{(2)}\ . \tag{2.11}$$

Here $[\mathcal{K}]$ denotes the discontinuity in the second fundamental form at $r=\rho$. Now let $\rho\to\infty$ and combine terms to obtain

$$\begin{aligned}S_{\rm eff}&=S_g+S_{\rm em}^{(1)}+S_{\rm em}^{(2)}+S_\infty^{(1)}+S_\infty^{(2)}\\&=2\int\left\{a\dot a\,\mathrm{arcsinh}\left[\dot a\Big/\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}\right]^{1/2}\right]-a\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}+\dot a^2\right]^{1/2}\right\}d\tau\ .\end{aligned} \tag{2.12}$$

In order to understand better the classical behavior of this wormhole, we shall add some matter to the model. We choose pressureless dust that is confined to lie on the throat of the wormhole. This dust should be thought of as a "regulator," used to improve the classical dynamics of the wormhole. Then $m=4\pi\sigma a^2$, the mass of the dust shell, is a constant of the motion. The matter Lagrangian simply reduces to $L_m=-m$, and the total Lagrangian describing the matter plus gravity system is

$$L_{\rm tot}=2\left\{a\dot a\ \mathrm{arcsinh}\left[\dot a\Big/\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}\right]^{1/2}\right]-a\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}+\dot a^2\right]^{1/2}\right\}-m\ . \tag{2.13}$$

The classical Wheeler-DeWitt Hamiltonian is now easily extracted; the conjugate momentum to a is

$$p\equiv\frac{\partial L}{\partial\dot a}=2a\ \mathrm{arcsinh}\left[\dot a\Big/\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}\right]^{1/2}\right]\ . \tag{2.14}$$

This relation may be easily inverted to yield $\dot a=(1-2M/a+Q^2/a^2)^{1/2}\sinh(p/2a)$ so that the Wheeler-DeWitt Hamiltonian is

$$\begin{aligned}H_{\rm tot}(p,a)&\equiv p\dot a-L_{\rm tot}\\&=2a\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}\right]^{1/2}\cosh\left[\frac{p}{2a}\right]+m\\&=2a\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}+\dot a^2\right]^{1/2}+m\ .\end{aligned} \tag{2.15}$$

The classical dynamics of the wormhole is now obtained by setting $H_{\rm tot}=0$. The fact that the Hamiltonian is zero is a standard consequence of the reparametrization invariance of the theory. To check the correctness of this calculation, observe that the constraint equation $H_{\rm tot}=0$ reproduces the classical Einstein field equations for the motion of the wormhole throat:[7]

$$\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}+\dot a^2\right]^{1/2}=-\frac{m}{2a}\ . \tag{2.16}$$

Classically, it is easy to see from the Einstein equations of motion that m must be negative. We shall soon see that quantum effects permit well-behaved wave functions for both $m=0$ and m positive. This calculation of $H_{\rm tot}$ is a nontrivial check of correctness in that it shows that classically we can either (1) consider arbitrary metric variations in the action to obtain the full Einstein field equations and then use symmetry to simplify these field equations,[7] or (2) use symmetry to simplify the action *ab initio* and then use restricted variations of the metric to obtain the same physics.

In fact, if a classical analysis is all that is required, then proceeding from the Einstein field equations is both more direct and less subject to subtle interpretation disputes. However, when it comes to quantizing the system, the Hamiltonian approach just exhibited will be much more useful.

Finally, we add some extra comments concerning the truncation procedure. The truncation procedure forces the geometry at large radius to be exactly Minkowski (rather than asymptotically Minkowski), thus allowing use of the simple version of the Hamiltonian constraint $H_{\rm eff}=0$. An alternative procedure is available. If one writes the Lagrangian and Hamiltonian in terms of the

Schwarzschild time coordinate and does not truncate the geometry at large radius, then it is a standard result of the Arnowitt-Deser-Misner (ADM) formalism that $H_{\rm Sch}=2M$, where each asymptotically flat region contributes an amount M to the ADM mass (this is the "imprint at infinity"). By defining a new "effective" Lagrangian

$$S_{\rm eff}=\int (L_{\rm Sch}-2M)dt \ , \qquad (2.17)$$

one finds that the "effective" Hamiltonian corresponding to this "effective" action satisfies $H_{\rm eff}=0$. It is this $S_{\rm eff}$ which, after it is rewritten in terms of the proper time coordinate, is equivalent to the truncation procedure previously outlined. This makes explicit our previous comment that the truncation procedure is related to the "imprint at infinity" of the wormhole's mass.

With the Hamiltonian of the wormhole model in hand, we now turn to the dynamics of the model wormhole.

III. DYNAMICS OF THE WORMHOLE: CANONICAL QUANTIZATION

From the classical equations of motion, we see that

$$\dot a=\pm\left[-1+\frac{2M}{a}-\frac{Q^2-m^2/4}{a^2}\right]^{1/2} . \qquad (3.1)$$

Thus large values of a are classically forbidden ($\dot a$ is imaginary), while for small a the behavior depends on the relative magnitudes of $Q^2-m^2/4$ and M^2. The classical turning points occur at

$$a_\pm=M\pm(M^2-Q^2+m^2/4)^{1/2} . \qquad (3.2)$$

If $(Q^2-m^2/4)<M^2$, there are two real classical turning points and the system oscillates between these turning points. For large enough values of $|m|$, only one of these turning points is physical, at $a_{\rm max}\approx 2M+|m|/2$. The second "turning point" is then $a_{\rm min}=0$, so that our picture of the motion is simple: The wormhole "emerges" from $a=0$ with infinite velocity, expands to a maximum radius of order $|m|$, and recollapses to $a=0$ in finite proper time (also of order $|m|$). Note that even if $Q>M$, we can with large enough $|m|$ ensure that $(Q^2-m^2/4)<M^2$, so that the comments of this paragraph apply.

If $Q^2-m^2/4=M^2$, the two turning points coalesce at $a=M$.

If $Q^2-m^2/4>M^2$, both turning points are unphysical (complex). The entire range $a\in(0,\infty)$ is classically forbidden, and careful attention to suitable limiting procedures indicates that the classical solution is $a\equiv 0$.

These comments should be compared with those in an earlier paper[9] where the case $Q=M=0$ was considered. The turning points are then $a_{\rm min}=0$ and $a_{\rm max}=|m|/2$, while at small times and small distances $a(\tau)\approx\sqrt{|m|\tau/2}$. Then $a_{\rm max}\to 0$ as $|m|\to 0$. It follows that for $m=0$ the classical wormhole always remains at $a=0$. Adding quantum effects serves to "smear out" this classically pointlike object.

With the classical dynamics of the model now understood and the Wheeler-DeWitt Hamiltonian in hand, quantization is straightforward. The only remarkable aspect of the analysis is that in some cases closed-form exact expressions are obtained. Canonical quantization proceeds via the usual replacement $p\mapsto -i\hbar\,\partial/\partial a$. Naturally, the resulting quantum Hamiltonian has a factor-ordering ambiguity. This factor-ordering ambiguity may be removed in a natural (though not unique) way by demanding that the quantum Hamiltonian be Hermitian:

$$\hat H_{\rm tot}=\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}\right]^{1/4} 2a\cos\left[\frac{L_P^2}{2}\frac{1}{a}\frac{\partial}{\partial a}\right] \times\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}\right]^{1/4}+m \ . \qquad (3.3)$$

That this Hamiltonian is Hermitian may formally be seen by Taylor-series expansion of the cosine. A more careful statement, taking into account appropriate boundary conditions, is that this Hamiltonian acts on $L^2[0,\infty)$, the space of square-integrable functions on $[0,\infty)$. However, when we discuss the $m=0$ case, we shall see that in this instance the boundary conditions may be deduced from the Wheeler-DeWitt equation rather than being put in by hand. The wave function of the wormhole is determined in the usual fashion by the Wheeler-DeWitt equation $\hat H_{\rm tot}\psi(a)=0$, which may be rewritten as $\hat H_{\rm eff}\psi=-m\psi$. Thus m may be interpreted as an eigenvalue of the effective Hamiltonian associated with $L_{\rm eff}$. The mass of the dust shell is therefore quantized in this formalism. This behavior is similar to that seen by DeWitt,[18] where a minisuperspace quantization of a Friedmann-Robertson-Walker universe led to a quantization condition on the mass of the dust which that universe contained.

IV. "DUST-FREE" WORMHOLE: EXACT SOLUTION

For nonzero values of m, exact solutions of the Wheeler-DeWitt equation have proved elusive, and one must resort to WKB techniques. For the special eigenvalue of $m=0$, exact solutions of the Wheeler-DeWitt equation may be written down by inspection:

$$\psi_n(a)=\frac{\exp[-(n+\frac{1}{2})\pi(a/L_P)^2]}{(1-2M/a+Q^2/a^2)^{1/4}} . \qquad (4.1)$$

Here n is an integer-valued quantum number describing the internal state of the wormhole. Negative values of n, not being normalizable, are discarded in the usual fashion. Note that the dynamics thus implies that $\psi(0)=0$. The expectation value of the wormhole radius is $\langle\psi_n|a|\psi_n\rangle\approx L_P$. The apparent occurrence of singularities at the classical event horizons is not at all a problem in that the wave function is square integrable over these "poles." Physical quantities do not pick up infinites from the event horizons, and the presence of these event horizons is not a matter of concern.

While not a cause for concern, the presence of event horizons does complicate the global geometry of the wormhole. Recall that if $|Q|>M$, then horizons do not occur and the global geometry is correspondingly simple. On the other hand, if $|Q|\le M$, the Schwarzschild coordinate patch does not completely cover the Reissner-

Nordström geometry, and the global geometry of the wormhole is more complex. For instance, if $M>0$, $Q=0$, the global geometry may be described as follows: (1) Take two Kruskal diagrams (appropriate to the description of the maximally extended Schwarzschild solution); (2) trim a small fringe off the future singularity of one diagram (the black hole); (3) trim a small fringe off the past singularity of the other diagram (the white hole); (4) then sew the two diagrams together along the trimmed fringes. The result is a model whereby one can make physical sense of the oft-repeated hope that matter which falls down a black hole will reappear in a white hole somewhere else in the "multiverse." Unfortunately, were one to undertake such a trip, one would reappear in a future incarnation of the Universe, rather than in a distant part of our own Universe, in the meantime having been squeezed down to sizes of the order of the Planck length. This is not a useful method of interstellar travel.

While it is clear from the exponential decay of the wave function that the average radius of the wormhole will be of the order of the Planck length, it is possible in some cases to make more precise statements. The general $M\neq 0$, $Q\neq 0$ case is intractable. However, for $M=0$, $Q\neq 0$, exact results may be obtained:

$$\begin{aligned}\langle a\rangle &= \frac{\int a|\psi_n|^2 da}{\int |\psi_n|^2 da} \\ &= \frac{\int [a^2/(a^2+Q^2)^{1/2}]e^{-[(2n+1)\pi a^2/L_P^2]}da}{\int [a/(a^2+Q^2)^{1/2}]e^{-[(2n+1)\pi a^2/L_P^2]}da} \\ &= Z^2\alpha L_P \frac{\sqrt{2n+1}}{2} e^{-[Z^2\alpha(n+1/2)]} \\ &\quad \times \frac{K_1(Z^2\alpha\pi(n+\frac{1}{2}))-K_0(Z^2\alpha\pi(n+\frac{1}{2}))}{\operatorname{erfc}\{[Z^2\alpha\pi(2n+1)]^{1/2}\}}\end{aligned} \tag{4.2}$$

Here K_0 and K_1 are modified Bessel functions, erfc is the complementary error function, and $Q\equiv Ze\equiv Z\sqrt{\alpha}L_P$. (It is one of the more outré features of geometrodynamic units that the charge on the electron is related to the Planck length by the fine-structure constant: $e=\sqrt{\alpha}L_P$.) This result, while exact, is in its present form unenlightening. The situation may be somewhat improved by using asymptotic expansions to show that for large n:

$$\langle a\rangle \approx \frac{1}{4}\frac{L_P}{\sqrt{2n+1}} \ . \tag{4.3}$$

Alternatively, for the $n=0$ mode, one may use the smallness of the fine-structure constant ($\alpha\approx\frac{1}{137}$) to obtain

$$\langle a\rangle = \frac{L_P}{\pi}+O(Z^2\alpha) \ . \tag{4.4}$$

To see the physical regime in which these calculations may be of interest, recall that for elementary particles, such as the electron, $Q/M\approx 10^{+22}$. It is only in the realm of charged elementary particles that the $M=0$, $Q\neq 0$ case is likely to be a good approximation to physics.

One may, on the other hand, consider the "astrophysical case." Naturally occurring black holes are expected to have $Q\ll M$ and $M\gg M_P$. Let us approximate by setting $Q=0$ and estimate $\langle a\rangle$ by Taylor series expanding the square root occurring in the integral for $\langle a\rangle$. It is this "astrophysical case" whose global geometry was considered previously. A brief calculation yields

$$\langle a\rangle = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})\sqrt{\pi}}\frac{L_P}{\sqrt{2n+1}}+O\left[\frac{L_P^2}{[(2n+1)M]}\right] . \tag{4.5}$$

Again, though the detailed calculations are tedious, they support the assertion that $\langle a\rangle\approx L_P$. It might be argued that with hindsight this result is not surprising on grounds of dimensional analysis. To see that this is not quite true, observe that the model wormhole possesses *three* independent length scales: (1) the Planck length L_P, (2) the Schwarzschild radius $2M$, and (3) the "charge radius" $Q\equiv Z\sqrt{\alpha}L_P$.

It is worth pointing out the exact sense in which I am claiming the wormhole to be stable—it is *a priori* quite possible that the minisuperspace calculation could have led to a Wheeler-DeWitt equation whose solution was a non-normalizable wave function that blew up as $a\to 0$. With such a wave function, one could at best define $\langle a\rangle=0$, indicating that the wormhole would be overwhelmingly likely to have collapsed to a point. In fact, of course, the situation is very much better than that unpleasant possibility, the true wave function behaving as $\psi\to\sqrt{a/Q}$ as $a\to 0$. This boundary condition, coming directly from solving the Wheeler-DeWitt equation, does not have to be put in "by hand."

Once one adds dust to the wormhole throat, relatively few exact statements can be made. It is, however, possible to show that the mass eigenvalues possess an infinite degeneracy—this can be traced back to the fact that the Hamiltonian is essentially a trigonometric function. Let us proceed by noting the identity

$$\begin{aligned}\cos\left[\frac{\partial}{\partial x}\right][f(x)g(x)] &= \cos\left[\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right][f(y)g(z)]\Big|_{x=y=z} \\ &= \left[\cos\left[\frac{\partial}{\partial y}\right]\cos\left[\frac{\partial}{\partial z}\right]-\sin\left[\frac{\partial}{\partial y}\right]\sin\left[\frac{\partial}{\partial z}\right]\right][f(y)g(z)]\Big|_{x=y=z} \\ &= \left[\cos\left[\frac{\partial}{\partial x}\right]f(x)\right]\left[\cos\left[\frac{\partial}{\partial x}\right]g(x)\right]-\left[\sin\left[\frac{\partial}{\partial x}\right]f(x)\right]\left[\sin\left[\frac{\partial}{\partial x}\right]g(x)\right] .\end{aligned} \tag{4.6}$$

With a little bit of work, one can use this identity to evaluate $\hat{H}_{\rm eff}[e^{-\beta(a/L_P)^2}\psi]$ as

$$\hat{H}_{\rm eff}[e^{-\beta(a/L_P)^2}\psi]=\cos(\beta)e^{-\beta(a/L_P)^2}\hat{H}_{\rm eff}[\psi]+\sin(\beta)e^{-\beta(a/L_P)^2}\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}\right]^{1/4} \times 2a\sin\left[\frac{L_P^2}{2a}\frac{\partial}{\partial a}\right]\left[\left[1-\frac{2M}{a}+\frac{Q^2}{a^2}\right]^{1/4}\psi(a)\right] . \tag{4.7}$$

Thus suppose we take ψ to be an eigenfunction of $\hat{H}_{\rm eff}$ with $\hat{H}_{\rm eff}\psi=m\psi$; then

$$\hat{H}_{\rm eff}(e^{-2\pi n(a/L_P)^2}\psi)=+m(e^{-2\pi n(a/L_P)^2}\psi) ,$$
$$\hat{H}_{\rm eff}(e^{-2\pi(n+1/2)(a/L_P)^2}\psi)=-m(e^{-2\pi(n+1/2)(a/L_P)^2}\psi) . \tag{4.8}$$

So one sees that the infinite degeneracy occurring in the $m=0$ case is no accident. Moreover, this informs us that quantum-mechanically positive values of m are as well behaved as negative values.

V. "DUSTY" WORMHOLE: WKB TECHNIQUES

For the case $m\neq 0$, exact solutions have proved to be elusive, and recourse has been made to WKB techniques. Since the Hamiltonian is not quadratic in momenta, a slight variant of the usual WKB technology is appropriate. Consider an *arbitrary* classical Hamiltonian $H(p,q)$; we wish to find the WKB approximations to the true energy eigenvalues and eigenfunctions $\hat{H}(\hat{p},\hat{q})\psi=E\psi$. Proceed as follows.

Step 1. Set $H(p,q)=E$ and invert to obtain $p(E,q)$, the momentum at the point q of a classical trajectory of energy E.

Step 2. Quantize the energy eigenvalues by setting

$$\oint p(E,a)da=(l+\delta)\hbar . \tag{5.1}$$

Here δ is a number that depends on both boundary conditions and the Hamiltonian H. In the old Bohr-Sommerfeld quantization, δ is just taken to be zero. For a Hamiltonian quadratic in momenta, the WKB method shows that δ is typically a simple fraction (e.g., $\frac{1}{2}$, $\frac{3}{4}$, etc.). For Hamiltonian nonquadratic in momenta, δ must be evaluated on a case-by-case basis and is often transcendental.[19] Since for the purposes of this paper a precise calculation of δ would add little to our understanding, δ will not be evaluated, but shall merely be carried along as an arbitrary constant.

Step 3. In the classically allowed region,

$$\psi_{\rm WKB}(q)=\frac{1}{\sqrt{(\partial H/\partial p)[p(E,q),q]}}\times\exp\left[\pm i\int^q\frac{p(E,x)dx}{\hbar}\right] , \tag{5.2}$$

while in the classically forbidden region,

$$\psi_{\rm WKB}(q)=\frac{1}{\sqrt{|(\partial H/\partial p)[p(E,q),q]|}}\times\exp\left[\pm\int^q\frac{|p(E,x)|dx}{\hbar}\right] . \tag{5.3}$$

Observe that in the allowed region,

$$|\psi_{\rm WKB}(q)|^2=\frac{1}{|(\partial H/\partial p)[p(E,q),q]|}=\frac{1}{|\dot{q}(E,q)|} . \tag{5.4}$$

As usual, the particle is most likely to be in those regions where classically it travels the slowest. It is easy to see that when $H=p^2/2m+V(q)$ this generalized prescription reduces to the usual WKB approximation. This generalized WKB approximation may be systematically derived in the usual manner from the first two terms of a formal power-series expansion in $\hbar$.

Applying this formalism to the problem at hand, in place of E we write m, the mass of the dust shell which is to be quantized. We note that $p(m,a)$ is a *multivalued* function:

$$p(m,a)=2a\,{\rm arccosh}\left[-\frac{m}{2a}\frac{1}{1-2M/a+Q^2/a^2}\right]=\pm 2a\,{\rm arccosh}\left[-\frac{m}{2a}\frac{1}{1-2M/a+Q^2/a^2}\right]+2a2\pi in . \tag{5.5}$$

Here arccosh maps $[1,\infty)$ to $[0,\infty)$, and the + (−) denotes outgoing (ingoing) directions. The quantization condition on m reads

$$\oint p(m,a)da=2\int_{a_{\rm min}}^{a_{\rm max}}2a\,{\rm arccosh}\left[-\frac{m}{2a}\frac{1}{1-2M/a+Q^2/a^2}\right]=(l+\delta)\hbar . \tag{5.6}$$

Note that the imaginary contribution to $p(m,a)$, being a total derivative, does not contribute to the quantization condition. The WKB estimate for the eigenvalue m is thus implicitly given as a function of l: $m=m(l)$. Note that each m eigenvalue has an infinite degeneracy (with respect to n). The quantum number n does, however, contribute when estimating the WKB wave function:

$$\psi_{\mathrm{WKB}}(a)=\frac{\exp\left[i\int^{a}p\,dx/\hbar\right]}{\sqrt{\dot a(m,a)}}=\frac{\exp[-2n\pi(a/L_P)^2]}{[1-2M/a+(Q^2-m^2/4)/a^2]^{1/4}}\,e^{i\Theta(a)}\,, \tag{5.7}$$

so that in the classically allowed region the wave function is indeed oscillatory, but with an envelope that for $n>0$ is exponentially damped. Note that we have in this manner recovered the degenerate modes discussed in the previous section. If we now flip $m\rightarrow -m$ and use $\operatorname{arccosh}(-x)=\operatorname{arccosh}(x)+i\pi$, we find that

$$\psi_{\mathrm{WKB}}(a)=\frac{\exp\left[i\int^{a}p\,dx/\hbar\right]}{\sqrt{\dot a(m,a)}}=\frac{\exp[-(2n+1)\pi(a/L_P)^2]}{[1-2M/a+(Q^2-m^2/4)/a^2]^{1/4}}\,e^{i\Theta(a)} \tag{5.8}$$

are WKB eigenmodes corresponding to an eigenmass $-m$. Suppose we go outside the classically allowed region, to large values of a; then

$$\operatorname{arccosh}\left[-\frac{m}{2a}\,\frac{1}{1-2M/a+Q^2/a^2}\right]\approx\operatorname{arccosh}(0)=i\pi/2\,, \tag{5.9}$$

so that as $a\rightarrow\infty$, one has

$$p(m,a)\rightarrow 2a2\pi i(n+\tfrac{1}{4})\,,\qquad \psi_{\mathrm{WKB}}(a)\rightarrow\exp[-2\pi(a/L_P)^2(n+\tfrac{1}{4})]\,, \tag{5.10}$$

independent of l and $|m|$. For the "flipped" eigenvalues $(-m)$, one sees

$$p(m,a)\rightarrow 2a2\pi i(n+\tfrac{3}{4})\,,\qquad \psi_{\mathrm{WKB}}(a)\rightarrow\exp[-2\pi(a/L_P)^2(n+\tfrac{3}{4})]\,. \tag{5.11}$$

Thus the large-radius asymptotic behavior of the WKB solution is identical to that of the exact $m=0$ solution.

Turning to the other extreme, as $a\rightarrow 0$, the "velocity factor" $1/\sqrt{\dot a}$ dominates and

$$\psi_{\mathrm{WKB}}(a)\rightarrow\frac{\sqrt{a}}{(Q^2-m^2/4)^{1/4}}\,. \tag{5.12}$$

Again, this is similar to the exact $m=0$ solution, but with a "shifted" value of the charge, $Q\mapsto(Q^2-m^2/4)^{1/2}$.

In fact, if one just blithely sets $m=0$, one can recover the exact solutions from the WKB estimates. For $m=0$, $p(a)=2a\operatorname{arccosh}(0)=2ai\pi(n+\frac{1}{2})$, while $|\dot a(a)|=(1-2M/a+Q^2/a^2)^{1/2}$, the entire real line being classically forbidden. The WKB wave function is then

$$\psi_{\mathrm{WKB}}(a)=\frac{\exp[-\pi(a/L_P)^2(n+\frac{1}{2})]}{(1-2M/a+Q^2/a^2)^{1/4}}\,, \tag{5.13}$$

in agreement with the exact calculation.

Returning to estimates of the eigenmass of the dust shell, the quantization integral in the general $Q\neq 0$, $M\neq 0$ case is undoable. Though the integral may be computed in closed form for the $Q\neq 0$, $M=0$ case, the result is not enlightening. A feel for the physics may best be obtained by considering the $Q=0$, $M=0$ case:

$$2\int_0^{m/2}2a\operatorname{arccosh}\left[-\frac{m}{2a}\right]da=(l+\delta)\hbar\,. \tag{5.14}$$

Rescaling to dimensionless variables, the integral is trivial with the result that $m(l)=-M_P\sqrt{2(l+\delta)}$. Thus the mass of the dust shell is quantized in terms of the Planck mass, as is only to be expected.

In summary, the WKB analysis indicates that the results obtained for the exact $m=0$ eigenfunctions are generic. The wave function is well behaved at the origin and exponentially damped at large radius. The average radius is of the order of the Planck length.

VI. SUMMARY AND CONCLUSIONS

In summarizing the content of this paper, one should make a very careful "reality check" as to how much of these calculations to actually believe. Perhaps the most damaging technical criticism that can be made concerning this calculation is that it is performed in minisuperspace instead of using Wheeler's full superspace. It is quite possible (maybe even likely) that the brutal truncation from an infinite number of degrees of freedom $g_{ij}(\mathbf{x},t)$ down to *one* degree of freedom $a(\tau)$ has also brutally truncated the real physics. Unfortunately, given our current lack of calculational abilities, we simply have no choice. In mitigation of this point, observe that although the application is unique, the minisuperspace technology employed is a standard quantum-gravity technique.

I should mention some other potentially serious problems. (1) Though the factor ordering choice made in $\hat H_{\mathrm{tot}}$ is in some sense "natural," it is by no means unique. Fortunately, this criticism does not apply to the WKB analysis—and the WKB analysis indicates that the qualitative features of the exact solutions continue to hold for $m\neq 0$. (2) Since the expected wormhole radius is of the order of the Planck length, it is far from clear that the Einstein-Hilbert action is an appropriate description for gravity. If R^2 terms are present (and this is expected on rather general grounds), the analysis of this paper is incomplete.[20] In particular, a naive application of the "thin-shell" formalism is no longer appropriate since this would now involve squares of δ functions.

The "bottom line" is this: This paper has marshaled a number of calculations which serve to indicate that minisuperspace models of Wheeler wormholes are quantum-mechanically stable with a natural radius of the order of the Planck length. It is this qualitative feature of the analysis that should be taken as the main thrust of this paper—rather than any of the particular model calculations. Unfortunately, it is rather difficult to judge to what extent if any these results might survive if one attempts to go beyond the minisuperspace approximation.

The implication that Wheeler wormholes are stable and amenable to some limited calculational techniques bears upon a second question—the quantum-mechanical process of topology fluctuation.[11] The occurrence of a fluctuation in topology may be viewed as equivalent to the collapse (and subsequent detachment) of a Wheeler wormhole. However, the minisuperspace calculation presented in this paper can be interpreted as indicating that the required collapse does not occur. Thus, if we are willing to believe this result beyond the minisuperspace approximation (and this is a big if), it is possible to argue that the putative quantum-mechanical stability of the Wheeler wormhole might in fact *prevent* fluctuations in topology.[11] Though the idea that quantum gravity engenders topological fluctuations has been current in the community for a rather long time, the number of calculations that can actually be carried out is distressingly small.

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