Bosonization of odd-spin-structure amplitudes

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In string theory the odd-spin-structure loop amplitudes have previously been studied only in the fermionic construction for cases with exclusively bosonic external states and only in the bosonic construction for amplitudes with at least two external fermions. We establish and discuss the Bose-Fermi equivalence for odd-spin-structure amplitudes.

I. INTRODUCTION

Bosonization of string-theory vertices[1] and the construction of amplitudes[2] corresponding to these vertices has proven to be an extremely useful technique in treating graphs with external fermions. At the one-loop level the equivalence of the bosonized amplitudes to those derived from the Neveu-Schwarz and Ramond fields is based, in the case of even-spin structures, on the addition theorem for θ functions;[3]

$$\theta_3\left[\sum_{i=1}^{N/2} x_i-\sum_{i=1}^{N/2} y_i-e\right]\theta_3(e)^{N/2-1}\frac{\prod_{i<j}E(x_i,x_j)E(y_j,y_i)}{\prod_{i,j}E(x_iy_j)}$$
$$=\det\frac{\theta_3(x_i-y_j-e)}{E(x_i,y_j)}\ . \qquad (1.1)$$

The "prime form" $E(x,y)$ is proportional to the θ_1 function, the proportionality constant canceling between the two sides of Eq. (1.1):

$$E(x_i,y_j)\sim\theta_1(y_j-x_i)\ . \qquad (1.2)$$

By a suitable choice of the arbitrary parameter e, an equivalent addition theorem can be written with the θ_3's replaced by θ_1, θ_2, or θ_4. In the case of odd-spin structures, the Bose-Fermi equivalence has not been demonstrated. The odd-spin-structure (OSS) amplitudes are those which contain an odd number of Γ^{11}'s on internal fermion lines. These include all parity-odd graphs with external bosons only. Because of the chiral nature of string theory, OSS amplitudes with external fermions do not differ in their behavior under parity from even-spin-structure amplitudes. In Ref. 2 the OSS amplitudes with external fermions were considered together with even-spin-structure amplitudes in the bosonized formalism. The treatment relied heavily on analytic function theory on a closed Riemann surface so that applications to open-string theory and the connection to the Ramond-Neveu-Schwarz (fermionized) formalism remained unclear for the OSS piece. Furthermore, Lorentz invariance even for the even-spin structures was not manifest in intermediate steps, but was imposed at the end through symmetrization techniques. In the bosonized formulation the extension to multiloop graphs has also been discussed by many authors.[4]

With external bosons only, the first appearance of odd-spin structures occurs in the one-loop six-point function which contains the potential gauge anomaly. The anomaly cancellation in the case of an SO(32) gauge group was established without calculating the full amplitude by taking the gauge projection from the beginning.[5] In this calculation the OSS fermionic correlation function does not appear. In the calculation of the full six-point function, this correlation was found to be[6]

$$\langle 0|\Gamma_\mu(\rho_1)\Gamma_\mu(\rho_2)|0\rangle=g_{\mu\nu}\chi_0^{11}(\rho_2/\rho_1)$$
$$=\frac{g_{\mu\nu}}{2\pi i}\frac{\partial}{\partial\nu_2}\ln\theta_1(\nu_2-\nu_1|\tau)\ , \qquad (1.3)$$

where

$$\nu_i\equiv(\ln\rho_i)/2\pi i\ , \qquad (1.4)$$

$$\tau\equiv(\ln w)/2\pi i\ . \qquad (1.5)$$

Relative to earlier normalizations,[6] we have absorbed a factor of $1/(i\sqrt{2})$ into the definition of Γ_μ to facilitate unification with the Neveu-Schwarz sectors. That is we have put, on the genus-1 surface,

$$\Gamma_\mu(\rho)\equiv\gamma_\mu/(i\sqrt{2})+\gamma_{11}\sum_{n=1}^{\infty}(\bar d_\mu^n\rho^n+d_\mu^n\rho^{-n})\ , \qquad (1.6)$$

with

$$d_\mu^n\equiv(b_\mu^n+b_\mu'^{n\dagger}w^{n/2})/(1-w^n)^{1/2}\ , \qquad (1.7)$$

$$\bar d_\mu^n\equiv(b_\mu^{n\dagger}-b_\mu'^n w^{n/2})/(1-w^n)^{1/2}\ . \qquad (1.8)$$

the b and b' sets of oscillations as well as the d and $\bar d$ set satisfy canonical anticommutation relations. The vacuum expectation value in (1.3) is defined to include a normalized trace over the Dirac matrices. The correlation (1.3) was independently suggested from an analyticity point of view by Verlinde and Verlinde and also derived by Miki.[7]

The elimination of spurious states from the string-theory amplitudes is most easily effected by including a ghost term in L_0. The superconformal ghosts that eliminate the negative-norm states due to the Neveu-Schwarz and Ramond fields contribute to L_0 through[8]

$$L_0^{\text{gh}}=\sum_r r\beta_{-r}\gamma_r, \quad [\gamma_n,\beta_m]=\delta_{n+m,0} . \tag{1.9}$$

The index r takes integer values on fermion lines and half-odd-integer values on boson lines. Traces over L_0^{gh} are most transparent if one notes that (1.9) can be written in terms of two independent sets of canonical oscillators:

$$L_0^{\text{gh}}=\sum_{r>0} r(\bar{d}^r d^r+\bar{e}^r e^r), \quad N^{\text{gh}}=\sum_{r>0}(\bar{d}^r d^r+\bar{e}^r e^r) , \tag{1.10}$$

where

$$d^r=(\beta_r+\gamma_r), \quad \bar{d}^r=(\beta_{-r}+\gamma_{-r}), \quad [d_r,\bar{d}_s]=\delta_{rs} , \tag{1.11a}$$

$$e^r=(\beta_r-\gamma_r), \quad \bar{e}^r=(\beta_{-r}-\gamma_{-r}), \quad [e_r,\bar{e}_s]=\delta_{rs} . \tag{1.11b}$$

The ghost contributions to one-loop amplitudes are then, for half-odd-integer r,

$$\mathrm{Tr}(-1)^{N^{\text{gh}}}w^{L_0^{\text{gh}}}=\left[\frac{\theta_3}{\eta}\right]^{-1} , \tag{1.12a}$$

$$\mathrm{Tr}(-1)^{N^{\text{gh}}}w^{L_0^{\text{gh}}}(-1)^{N^{\text{gh}}}=\left[\frac{\theta_4}{\eta}\right]^{-1} , \tag{1.12b}$$

and, for integer r,

$$\mathrm{Tr}(-1)^{N^{\text{gh}}}w^{L_0^{\text{gh}}}=\left[\frac{\theta_2}{2\eta w^{1/8}}\right]^{-1} , \tag{1.12c}$$

$$\mathrm{Tr}(-1)^{N^{\text{gh}}}w^{L_0^{\text{gh}}}(-1)^{N^{\text{gh}}}=\eta^{-2} . \tag{1.12d}$$

These factors occur multiplicatively in the correlation functions in the four one-loop spin structures.

In Sec. II we review the Bose-Fermi equivalence in the case of the even-spin-structure amplitudes. We introduce and make use of a Pfaffian[9] expression for the fermionized correlation functions which leads to a generalization of Fay's formula clarifying the Lorentz invariance of the full amplitudes in bosonized form. In Sec. III we treat the odd-spin-structure amplitudes, beginning for the sake of clarity with the dimension $D=2$ case. The final section contains a summary and discussion of our results.

II. RESULTS FOR EVEN-SPIN STRUCTURES

In the fermionic construction of even-spin structures, one encounters correlation functions of N anticommuting Neveu-Schwarz or Ramond fields:

$$C_N(z_1,z_2,\ldots,z_N)\equiv\left\langle 0\left|\prod_{i=1}^{N} r_i\cdot H(z_i)\right|0\right\rangle , \tag{2.1}$$

with various D-vectors r_i ($D=$ space-time dimension). C_N has a useful expression in terms of Pfaffians:

$$C_N(z_1,z_2,\ldots,z_N)=P(\langle 0|r_i\cdot H(z_i)r_j\cdot H(z_j)|0\rangle) . \tag{2.2}$$

The Pfaffian is defined as the square root of the determinant of an antisymmetric matrix:[9]

$$P(A_{ij})\equiv[\det(A_{ij})]^{1/2}, \quad A_{ij}=-A_{ji} . \tag{2.3}$$

The Pfaffian is trivally zero if N is odd. Using the properties of Pfaffians, the equivalence of (2.1) and (2.2) can be established by induction, beginning from the trivial equivalence of the $N=2$ case, and noting that the right-hand sides of (2.1) and (2.2) both satisfy the recursion relation

$$C_N(z_1,\ldots,z_N)=\sum_{j=2}^{N}(-1)^j C_2(z_1,z_j)C_{N-2}(z_2,\ldots,z_N{:}j\text{ omitted}) . \tag{2.4}$$

A recursion relation of this form provides an alternate definition of the Pfaffian and resolves the sign ambiguity in (2.3).[9] A useful theorem relates the Pfaffian of order N (even) to a sum over determinants of one-half the order:

$$2^{+N/2}P(A)=P(A-A^T)=\sum_{\substack{\text{eq. part's}\\ \{i\}\{j\}}}(-1)^{\sum j}\det A_{ij} . \tag{2.5}$$

In Eq. (2.5) the N indices of A are partitioned into two sets $\{i\}$ and $\{j\}$ of equal order $N/2$ and the determinant is taken over the elements of A that couple the two sets. One then sums over all such equal partitions. This relation is derived inductively by noting the trivial equivalence for the $N=2$ case and verifying that both sides satisfy the recursion relation (2.4). The sign of each term depends on whether the sum of all the indices in set $\{j\}$ is even or odd. The sum over equal partitions can also be written by assigning to each index a parameter $\alpha=\pm1$ and summing over all choices of the α_i consistent with a zero sum:

$$P(A-A^T)=\sum_{\alpha_k=\pm 1}\delta_{\sum\alpha_k,0}(-1)^{\sum_j(\alpha_j+1)/2}\det A_{ij}\Big|_{\alpha_i=-\alpha_j=-1} . \tag{2.6}$$

At one-loop order in string theory, the amplitudes contain factors from orbital oscillators multiplied by traces over anticommuting fields. These traces can be reduced to vacuum expectation values of fields with rotated oscillators through the technique of Ref. 10. The resulting correlations can then be related to Pfaffians through Eqs. (2.1) and (2.2). For example,

$$\begin{aligned}M_N^H &\equiv \mathrm{Tr}(-1)^{N^{\mathrm{gh}}}w^{L_0-1/2}\prod_{i=1}^{N} r_i\cdot H(\rho_i,b,b^\dagger)\\ &= w^{-1/2}\left[\frac{\theta_3(0|\tau)}{\eta}\right]^{(D-2)/2}\left\langle 0\left|\prod_{i=1}^{N} r_i\cdot H(\rho_i,d,\bar d)\right|0\right\rangle\\ &= w^{-1/2}\left[\frac{\theta_3(0|\tau)}{\eta}\right]^{(D-2)/2} P[r_i\cdot r_j\chi(\rho_j/\rho_i,w)] .\end{aligned} \tag{2.7}$$

Here L_0 includes the term from the superconformal ghosts

$$L_0^{\mathrm{gh}}=\sum_{r\in\mathbb{Z}+1/2} r\beta_{-r}\gamma_r,\quad [\gamma_n,\beta_m]=\delta_{n+m,0} .$$

The Neveu-Schwarz field is

$$H_\mu(\rho,b,b^\dagger)=\sum_{n=0}^{\infty}(b_\mu^n\rho^{-n-1/2}+b_\mu^{n\dagger}\rho^{n+1/2}) .$$

The d oscillators in (2.7) given by the trace theorem of Ref. 10 are analogous to those in (1.7) and (1.8). η is the Dedekind η function:

$$\eta=\prod_{n=1}^{\infty}(1-w^n) . \tag{2.8}$$

The χ function is defined by

$$\langle 0|H_\mu(\rho_1,d,\bar d)H_\nu(\rho_2,d,\bar d)|0\rangle=g_{\mu\nu}\chi(\rho_2/\rho_1,w)=g_{\mu\nu}\frac{i\theta_1'(0)}{2\pi\theta_3(0)}\frac{\theta_3(v_{12}|\tau)}{\theta_1(v_{12}|\tau)} . \tag{2.9}$$

Although $\chi(\rho_j/\rho_i)$ is singular at $\rho_j=\rho_i$, the Pfaffian in (2.7) is, of course, defined as the square root of the determinant of the corresponding matrix with diagonal elements put to zero.

The Pfaffian in (2.7) can be written as a product of $D/2$ Pfaffians corresponding to a Cartan-Weyl decomposition of the Lorentz vectors r_i:

$$r_i^a\equiv(r_{i,2a-1}+ir_{i,2a})/\sqrt{2},\quad \bar r_i^a\equiv(r_{i,2a-1}-ir_{i,2a})/\sqrt{2} . \tag{2.10}$$

The superscript in (2.10) represents the Lorentz indices in the Cartan-Weyl basis, while the subscripts after the comma represent the Lorentz indices in the Minkowski basis. To see the product decomposition of the Pfaffian, note that the vacuum expectation value (VEV) in (2.7) can be written

$$\begin{aligned}\left\langle 0\left|\prod_{i=1}^{N} r_i\cdot H(\rho_i)\right|0\right\rangle &= \left\langle 0\left|\prod_{i=1}^{N}\sum_{a_i=1}^{D/2}[r_i^{a_i}\bar H^{a_i}(\rho_i)+\bar r_i^{a_i}H^{a_i}(\rho_i)]\right|0\right\rangle\\ &= \sum_{a_i=1}^{D/2}(-1)^P\left\langle 0\left|\prod_{i=1}^{N}[r_i^{a_i}\bar H^{a_i}(\rho_i)+\bar r_i^{a_i}H^{a_i}(\rho_i)]_{a\text{ ordered}}\right|0\right\rangle\\ &= \sum_{a_i=1}^{D/2}(-1)^P\prod_{b=1}^{D/2}\left\langle 0\left|\prod_{\substack{i=1\\ a_i=b}}^{N}[r_i^b\bar H^b(\rho_i)+\bar r_i^bH^b(\rho_i)]\right|0\right\rangle\\ &= \sum_{a_i=1}^{D/2}(-1)^P\prod_{b=1}^{D/2}P[r_i^b\bar r_j^b\chi(\rho_j/\rho_i)-r_j^b\bar r_i^b\chi(\rho_i/\rho_j)]\Big|_{a_i,a_j=b} .\end{aligned} \tag{2.11}$$

The parity $(-1)^P$ is $+1$ if $a_i\le a_j$ for all $i<j$ and is -1 if the indices are an odd permutation thereof. In the last expression of Eq. (2.11), we have used the fact that $\chi(\rho)$ is odd under inversion of its argument. The decomposition of the

Pfaffian into a sum over products of $D/2$ Pfaffians can also be seen by a suitable manipulation into a block-diagonal form using the well-known properties of determinants. We can now use (2.5) and (2.9) to write

$$M_N^H = w^{-1/2}\left[\frac{i\theta_1'(0)}{2\pi\theta_3}\right]^{N/2}\left[\frac{\theta_3}{\eta}\right]^{(D-2)/2}\sum_{a_i=1}^{D/2}(-1)^P\prod_{b=1}^{D/2}\sum_{\alpha_i=\pm 1}\delta_{\sum\alpha_i,0} \times\left[\prod_{\substack{i\\ \alpha_i=-1}} r_i^b\right]\left[\prod_{\substack{j\\ \alpha_j=+1}}(-1)^j\bar{r}_j^b\right]\det\frac{\theta_3(\nu_j-\nu_i|\tau)}{\theta_1(\nu_j-\nu_i|\tau)}\Bigg|_{a_i=a_j=b,\ \alpha_i=-\alpha_j=-1} . \tag{2.12}$$

To understand the essential features of (2.12), the reader might find it useful to examine first the $D=2$ case. For $D=2$, in fact, one can write a simple generalization of Fay's formula, [Eq. (1.1)] that makes clear the Lorentz invariance of the bosonized correlation function:

$$P[r_i\cdot r_j\chi(\rho_j/\rho_i)] = \left[\frac{i\theta_1'(0)}{2\pi\theta_3(0)}\right]^{N/2}\sum_{\alpha_i=\pm 1}\delta_{\sum\alpha_i,0}\left[\prod_{\substack{i=1\\ \alpha_i=-1}}^{N} r_i\right]\left[\prod_{\substack{j=1\\ \alpha_j=+1}}^{N}\bar{r}_j\right]\theta_3\left[\sum\nu_i\alpha_i\,\Big|\,\tau\right]\theta_3(0)^{N/2-1}\prod_{i<j}\theta_1(\nu_j-\nu_i|\tau)^{\alpha_i\alpha_j} . \tag{2.13}$$

The Lorentz invariance of (2.12) is established by its equivalence to the manifestly Lorentz-invariant form (2.7). The addition theorem (1.1) tells us that

$$\det\frac{\theta_3(\nu_j-\nu_i|\tau)}{\theta_1(\nu_j-\nu_i|\tau)}\Bigg|_{\alpha_i=-\alpha_j=-1} = \theta_3(0)^{-1+N/2}\theta_3\left[-\sum\alpha_i\nu_i\,\Big|\,\tau\right]\frac{\prod_{\substack{i<j\\ \alpha=-}}\theta_1(\nu_j-\nu_i)\prod_{\substack{i<j\\ \alpha=+}}\theta_1(\nu_i-\nu_j)}{\prod_{\substack{i,j\\ \alpha_i=-\alpha_j=-1}}\theta_1(\nu_j-\nu_i)}$$

$$=\theta_3(0)^{-1+N/2}\theta_3\left[\sum\alpha_i\nu_i\,\Big|\,\tau\right]\prod_{i<j}\theta_1(\nu_j-\nu_i)^{\alpha_i\alpha_j}\prod_{\substack{i=1\\ \alpha_i=+}}^{N}(-1)^i . \tag{2.14}$$

In the general D-dimensional case, we have, therefore,

$$M_N^H = w^{-1/2}\left[\frac{\theta_3}{\eta}\right]^{-1}\sum_{a_i=1}^{D/2}(-1)^P\prod_{b=1}^{D/2}\sum_{\substack{\alpha_i=\pm 1\\ a_i=a_j=b}}\delta_{\sum\alpha_i^b,0}\frac{\theta_3\left[\sum\alpha_i^b\nu_i|\tau\right]}{\eta}\left[\prod_{\substack{i\\ \alpha_i=-}} r_i^b\right]\left[\prod_{\substack{j\\ \alpha_j=+}}\bar{r}_j^b\right]\prod_{i<j}\left[2\pi\frac{\theta_1(\nu_j-\nu_i)}{i\theta_1'(0)}\right]^{\alpha_i^b\alpha_j^b} , \tag{2.15}$$

where we have put, for clarity,

$$\alpha_i^b = \alpha_i\delta_{a_i b} .$$

The right-hand side is the bosonized expression for M_N^H. It can be obtained by a trace over bosonized H fields H_μ^B:

$$M_N^H = \sum_{n\in\mathbb{Z}}\mathrm{Tr}(-1)^{N^{\mathrm{gh}}}\left\langle 0\left| e^{-inq_0}w^{L_0-1/2}\prod_{i=1}^{N} r_i\cdot H^B(\rho_i)e^{inq_0}\right|0\right\rangle , \tag{2.16}$$

where

$$r_i\cdot H^B(\rho) = \sum_{a_i=1}^{D/2} c_{a_i} :(r_i^{a_i}e^{-i\Phi^{a_i}(\rho)}+\bar{r}_i^{a_i}e^{i\Phi^{a_i}(\rho)}): . \tag{2.17}$$

The fields Φ^a are constructed from bosonic oscillators in a way that is totally parallel to the string coordinate field except that the momentum operator p_0^b has quantized eigenvalues:

$$\Phi^b(\rho) = q_0^b - ip_0^b\ln\rho + \sum_{m=1}^{\infty}(a^{m,b}\rho^{-m}+a^{m,b\dagger}\rho^m)/\sqrt{m} . \tag{2.18}$$

The sum over n in (2.16) takes the place of the loop momentum integral. The $\theta_3(\sum\alpha_i^b\nu_i|\tau)$ in (2.15) results from this zeroth-mode summation. The cocycle's c_a are required to provide the $(-1)^P$ factor in (2.15) or equivalently to ensure that H_μ^B with different Lorentz indices μ anticommute. One representation[11] is

$$\begin{aligned} c_1 &= 1 , \\ c_{a+1} &= e^{\pm i\pi p_0^a} c_a . \end{aligned} \tag{2.19}$$

The validity of (2.16) does not depend on the choice of $\pm$ signs in 2.19. The insertion of the G-parity operator $e^{2\pi i L_0}$ into the trace (2.7) has the effect of interchanging θ_3 and θ_4 in all equations of this section. The G-parity operator to insert into the bosonized trace [Eq. (2.16)] is

$$G = \exp\left[i\pi \sum_a p_0^a \right] (-1)^{N^{\mathrm{gh}}} . \tag{2.20}$$

Similarly, the correlation of Ramond fields with even-spin structure is given by replacing all θ_3's by θ_2's. This is accomplished in the bosonized trace [Eq. (2.16)] by replacing the sum over integers $\mathbb{Z}$ by a sum over half odd integers $\mathbb{Z}+\frac{1}{2}$. That is,

$$\begin{aligned} M_N^\Gamma &\equiv -\mathrm{Tr}(-1)^{N^{\mathrm{gh}}} w^{L_0} \prod_{i=1}^N r_i \cdot \Gamma(\rho_i, b, b^\dagger) \\ &= -\left[\frac{\theta_2(0|\tau)}{\eta w^{1/8}} \right]^{(D-2)/2} \left\langle 0 \left| \prod_{i=1}^N r_i \cdot \Gamma(\rho_i, d, \bar{d}) \right| 0 \right\rangle \\ &= -w^{-1/2} \left[\frac{\theta_2(0|\tau)}{\eta} \right]^{(D-2)/2} P[r_i \cdot r_j \chi_0(\rho_j/\rho_i, w)] \\ &= -w^{-1/2} \left[\frac{\theta_2}{\eta} \right]^{-1} \sum_{a_i=1}^{D/2} (-1)^P \prod_{b=1}^{D/2} \sum_{\substack{\alpha_i = \pm 1 \\ a_i = a_j = b}} \delta_{\sum \alpha_i^b, 0} \frac{\theta_2 \left[\sum \alpha_i^b \nu_i \middle| \tau \right]}{\eta} \left[\prod_{\substack{i \\ \alpha_i = -}} r_i^b \right] \left[\prod_{\substack{j \\ \alpha_j = +}} \bar{r}_j^b \right] \prod_{i<j} \left[2\pi \frac{\theta_1(\nu_j - \nu_i)}{i\theta_1'(0)} \right]^{\alpha_i^b \alpha_j^b} . \end{aligned} \tag{2.21}$$

III. RESULTS FOR ODD-SPIN STRUCTURES

Although the results for even-spin structures have been relatively well understood for several years, the relation between the fermionized and bosonized expressions for odd-spin structures has not been up to now well established. For amplitudes with external fermions, the fermionized construction presents severe complications[12] and has not been well studied in the case of multiple external fermion pairs. The bosonized expressions, on the other hand, seem relatively simple.[1,2,13] Conversely, the OSS amplitudes with external bosons have been clearly derived only in the fermionized version,[6] although it is tempting to infer the corresponding bosonized expressions from the amplitudes with external fermions. In this section, beginning from the well-understood parity-odd amplitudes with external bosons, we derive the corresponding bosonized expressions. The way in which the bosonized vertices provide the required D-dimensional ϵ tensor is made clear together with the Lorentz invariance of the bosonized formalism.

From the fermionized point of view, the OSS amplitudes with external bosons have a structure apparently quite different from that of the even-spin structures. In place of (2.7) one encounters correlation functions of the form

$$\begin{aligned} M_N^{\mathrm{OSS}} &= -\mathrm{Tr}(-1)^{N^{\mathrm{gh}}} \Gamma^{D+1} w^{L_0} \prod_{i=1}^N r_i \cdot \Gamma(\rho_i, b, b^\dagger) \\ &= -\eta^{D-2} \left\langle 0 \left| \gamma^{D+1} \prod_{i=1}^N r_i \cdot \Gamma(\rho_i, d, \bar{d}) \right| 0 \right\rangle . \end{aligned} \tag{3.1}$$

The trace in the center expression of (3.1) and the VEV in the final expression are defined to include a normalized trace over the Dirac matrices. Using the results of Ref. 6, Eq. (3.1) may be written

$$M_N^{\mathrm{OSS}} = -\eta^{D-2} \epsilon_{\mu_1 \mu_2 \cdots \mu_D} \sum_{i_j=1}^N r_{i_1,\mu_1} r_{i_2,\mu_2} \cdots r_{i_D,\mu_D} (-1)^P P'[r_i \cdot r_j \chi_0^{11}(\rho_j/\rho_i)] , \tag{3.2}$$

with χ_0^{11} given in Eq. (1.3). The validity of (3.2) is established inductively starting from the trivial $N=D$ case and making use of the recursion relation of the Pfaffians. In (3.2), D-vectors are chosen from among the r_i in all possible ways such that $i_j < i_k$ for $j<k$. The parity $(-1)^P$ is $+1$ if the ordering $\{r\} = \{r_{i_1} \cdots r_{i_D};$ other r's$\}$ is an even permutation of the index order and -1 otherwise. The prime on the Pfaffian indicates that the D r_i contracted with the ϵ tensor are

omitted from the Pfaffian. The ϵ tensor can be incorporated into a larger Pfaffian writing, for D even,

$$M_N^{\mathrm{OSS}}=-(-1)^{D/2}\eta^{D-2}P(r_i\cdot r_j\chi_0^{11}(\rho_j/\rho_i),\mathbf{r}_i)\ . \tag{3.3}$$

By the Pfaffian in (3.3) we mean the square root of the determinant of the antisymmetric matrix A_{ij}, in which

$$\begin{aligned}&A_{ij}=0,\quad i=j,\\&A_{ij}=r_i\cdot r_j\chi_0^{11}(\rho_j/\rho_i),\quad i,j\le N,\ i\neq j\ ,\\&A_{ij}=r_{i,j-N},\quad i\le N<j\le N+D\ ,\\&A_{ij}=0,\quad N<i,j\le N+D\ .\end{aligned} \tag{3.4}$$

$r_{i,j-N}$ is the $(j-N)$th component of the D-vector r_i. For clarity we study first the $D=2$ case suppressing the factor η^{-2} coming from the ghosts. Equation (3.3) becomes

$$M_N^{\mathrm{OSS}}=-\eta^2\left[\frac{d}{dt}P[t\epsilon_{r_ir_j}+r_i\cdot r_j\chi_0^{11}(\rho_j/\rho_i)]\right]_{t=0}\ . \tag{3.5}$$

The two-dimensional ϵ tensor contracted with two r_i's is

$$\epsilon_{r_ir_j}\equiv r_{i,1}r_{j,2}-r_{i,2}r_{j,1}=i(r_i\bar r_j-r_j\bar r_i)\ , \tag{3.6}$$

whereas, using the antisymmetry of χ_0^{11},

$$r_i\cdot r_j\chi_0^{11}(\rho_j/\rho_i)=r_i\bar r_j\chi_0^{11}(\rho_j/\rho_i)-r_j\bar r_i\chi_0^{11}(\rho_i/\rho_j)\ . \tag{3.7}$$

We can therefore use the theorem (2.6) to write the $D=2$ case as

$$M_N^{\mathrm{OSS}}=-i\eta^2\sum_{\alpha_i=\pm1}\delta_{\sum\alpha_i,0}\left[\prod_{\substack{i\\ \alpha_i=-1}}r_i\right]\left[\prod_{\substack{j\\ \alpha_j=+1}}(-1)^j\bar r_j\right]\left[\frac{d}{dt}\det[t+\chi_0^{11}(\rho_j/\rho_i)]\right]_{t=0}\ . \tag{3.8}$$

The last set of large parentheses in (3.8) appeared also in the work of Verlinde and Verlinde.[7] The structure of the OSS amplitudes seems in the fermionic construction quite different from that of the even-spin-structure amplitudes given in (2.12). In order to make connection with the bosonized amplitudes, we make use of a further identity from Fay:[3]

$$\theta_1\left[\sum_{i=1}^{N/2}x_i-\sum_{i=1}^{N/2}y_i\right]\theta_1'(0)^{N/2-1}\frac{\prod_{i<j}\theta_1(x_j-x_i)\theta_1(y_i-y_j)}{\prod_{i,j}\theta_1(y_j-x_i)}=\left[\frac{d}{dt}\det\left[t-\frac{\theta_1'(y_j-x_i)}{\theta_1(y_j-x_i)}\right]\right]_{t=0}\ . \tag{3.9}$$

This can be obtained by differentiating Eq. (1.1) $N/2-1$ times with respect to e and setting e to $-\tau/2-\frac{1}{2}$. Using the definition (1.3) of χ_0^{11}, we then have

$$\left[\frac{d}{dt}\det[t+\chi_0^{11}(\rho_j/\rho_i)]\right]_{\substack{t=0\\ \alpha_i=-\alpha_j=-1}}=\theta_1\left[\sum_{i=1}^{N}\alpha_i\nu_i\right]\left[\frac{\theta_1'(0)}{-2\pi i}\right]^{N/2-1}\prod_{i<j}\theta_1(\nu_j-\nu_i)^{\alpha_i\alpha_j}\prod_{\substack{i\\ \alpha_i=+1}}(-1)^i\ . \tag{3.10}$$

Since $\theta_1'(0)=2\pi\eta^3w^{1/8}$, we can then write the $D=2$ case as

$$M_N^{\mathrm{OSS}}=-\sum_{\alpha_i=\pm1}\delta_{\sum\alpha_i,0}\left[\prod_{\substack{i\\ \alpha_i=-1}}r_i\right]\left[\prod_{\substack{j\\ \alpha_j=+}}\bar r_j\right]\frac{\theta_1\left[\sum_{i=1}^{N}\alpha_i\nu_i\right]}{\eta w^{1/8}}\prod_{i<j}\left[\frac{\theta_1(\nu_j-\nu_i)}{i\theta_1'(0)/2\pi}\right]^{\alpha_i\alpha_j}\ . \tag{3.11}$$

It is a simple matter to generalize the $D=2$ case to that of an arbitrary even number of dimensions D. To compare with Eq. (2.15), one has, for the odd-spin structures,

$$M_N^{\mathrm{OSS}}=-\eta^{-2}\sum_{a_i=1}^{D/2}(-1)^P\prod_{b=1}^{D/2}\left[\sum_{\alpha_i=\pm1}\delta_{\sum\alpha_i^b,0}\frac{\theta_1\left[\sum\alpha_i^b\nu_i\middle|\tau\right]}{\eta w^{1/8}}\left[\prod_{\substack{i\\ \alpha_i=-}}r_i^b\right]\left[\prod_{\substack{j\\ \alpha_j=+}}\bar r_j^b\right]\prod_{i<j}\left[\frac{\theta_1(\nu_j-\nu_i)}{i\theta_1'(0)/2\pi}\right]^{\alpha_i^b\alpha_j^b}\right]\ . \tag{3.12}$$

From this point it is a simple matter to write M_N^{OSS} as a trace over bosonic fields only:

$$M_N^{\rm OSS}=-\sum_{n\in\mathbb{Z}+1/2}\mathrm{Tr}\left\langle 0\left| e^{-inq_0}e^{i\pi p_0}w^{L_0-5/8}\prod_{i=1}^{N} r_i\cdot H^B(\rho_i)e^{inq_0}\right|0\right\rangle . \tag{3.13}$$

In the fermionized form the vertex is expressed in terms of Neveu-Schwarz fields for emission from a space-time bosonic line and in terms of Ramond fields for emission from a space-time fermion. The bosonized formalism provides a unification of these emission vertices since the H^B occurring in (3.13) is the same as that in (2.16). Similarly, the Gliozzi-Scherk-Olive (GSO) projection in the two cases (in the F_2 picture) takes the unified form

$$P_{\rm GSO}=\frac{1-e^{i\pi p_0}(-1)^{N^{\rm gh}}}{2} . \tag{3.14}$$

In the F_2 picture the bosonic propagator is

$$\Delta_B=\frac{1}{L_0-1/2}P_{\rm GSO} , \tag{3.15}$$

while the Fermion propagator is

$$\Delta_F=\frac{F_0}{L_0-5/8}P_{\rm GSO} . \tag{3.16}$$

In both cases,

$$L_0=p^2/2+\sum_{a=1}^{5}(p^a)^2/2+\sum_{n=1}^{\infty}\left[n(a_\mu^{n\dagger}a_\mu^n)+\sum_{a=1}^{5}n(a^{n,a\dagger}a^{n,a})\right]+L_0^{\rm gh} . \tag{3.17}$$

Note, however, that on bosonic lines p^a has integer eigenvalues, while on fermionic lines the eigenvalues are in $\mathbb{Z}+\frac{1}{2}$.

In the case $N=10$, the fact that $\theta_1(0|\tau)=0$ requires in (3.12) that the r_i be associated in pairs with opposite α_i to the Cartan-Weyl indices b. Equation (3.12) then contains five factors of the form (with $i<j$)

$$\theta_1(\alpha_j(\nu_j-\nu_i))/\theta_1(\nu_j-\nu_i)=\alpha_j , \tag{3.18}$$

$$\begin{aligned}M_{10}^{\rm OSS}&=-\eta^{-7}w^{-5/8}[i\theta_1'(0)/2\pi]^5\sum_{a_i=1}^{5}(-1)^P\prod_{b=1}^{5}\sum_{\alpha_i=\pm1}\delta_{\sum\alpha_i^b,0}\left[\prod_{\substack{i\\ \alpha_i=-}} r_i^b\right]\left[\prod_{\substack{j\\ \alpha_j=+}}\bar r_j^b\right]\\ &=-\eta^8 i^5\sum_{a_i=1}^{5}(-1)^P\prod_{b=1}^{5}\sum_{\alpha_i=\pm1}\delta_{\sum\alpha_i^b,0}\left[\prod_{\substack{i\\ \alpha_i=-}} r_i^b\right]\left[\prod_{\substack{j\\ \alpha_j=+}}\bar r_j^b\right]\left[\prod_{i<j}\alpha_j\right]\\ &=-\eta^8\epsilon_{\mu_1\mu_2\cdots\mu_{10}}r_1^{\mu_1}r_2^{\mu_2}\cdots r_{10}^{\mu_{10}} .\end{aligned} \tag{3.19}$$

In (3.19) one recovers the Lorentz-covariant ϵ tensor form which is evident in (3.1). It is instructive to consider the two-dimensional case in which the second and third forms of $M^{\rm OSS}$ in (3.19) correspond to Eq. (3.6).

IV. SUMMARY AND OUTLOOK

We have demonstrated the Fermi-Bose equivalence for the odd-spin-structure amplitudes. Although the bosonized form might not have been unexpected based on the results for the even-spin structures, the present work provides the precise relation between this bosonized form and the earlier fermionized form of the amplitude. In addition, the equivalence of Eq. (3.12) to (3.2) establishes the Lorentz invariance of the bosonized amplitudes and the correct provision of the totally antisymmetric ϵ tensor. The parity-odd nature of the amplitudes is not at all apparent in the form (3.12), which seems quite similar to the forms (2.15) and (2.21) of the parity-conserving amplitudes.

It would be interesting to determine whether the proof of anomaly freedom[5,6] of the SO(32) theory or the finiteness[14] of the parity-odd N-point functions is simpler from the bosonized point of view. It is also possible that the Pfaffian form of the fermionized amplitudes, which we have heavily relied upon, will make these properties more transparent. We leave these questions and other applications of our results to future investigations.

Note added. After submission of this article it has come to our attention that the identity due to Fay quoted in our Eq. (3.9) was also employed recently by P. DiVecchia, Nordita Report No. 90/2 P (to be published in Proceedings of the XXIII International Symposium Ahrenshoop, Ahrenshoop Germany, 1989).

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