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Constraints on negative-energy fluxes

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Locally negative energy due to quantum coherence effects in quantum field theory is discussed. In a previous work, it was argued that a beam carrying negative energy must satisfy an uncertainty-principle-type inequality of the form $|\Delta E|\Delta t \leq 1$, where $|\Delta E|$ is the magnitude of the negative energy that may be transmitted in a time Δt. This conclusion applied only to two-dimensional spacetime, and was based on an examination of particular classes of quantum states. In the present work, we give more precise formulations of this type of inequality for a free massless scalar field in both two- and four-dimensional flat spacetime. These inequalities are proven to hold for all quantum states. The physical interpretation of these inequalities is also discussed, and it is argued that they are likely to prevent negative energy from producing such large-scale effects as violations of the second law of thermodynamics or of cosmic censorship.

I. INTRODUCTION

Quantum field theory has the remarkable property that the local energy density can be negative, even though the energy density is a positive-definite quantity in classical physics. It is a general feature of both free and interacting theories that there exist states in which the energy density at a particular point can be arbitrarily negative.[1] Nonetheless, the total energy, integrated over all space, is required to be non-negative. Negative energy densities may be thought of as a quantum coherence effect, arising in quantum states which are superpositions of different particle number eigenstates. One class of such states describe the quantum radiation by a moving mirror,[2] which can carry negative energy. They are also examples of negative-energy fluxes, where all particles are moving in one direction, but the energy flow is instantaneously in the opposite direction.

Unrestricted negative-energy fluxes would have alarming consequences, including violation of the second law of thermodynamics.[3] By shining negative energy on a hot object, it would be possible to decrease its entropy without an apparent compensating entropy increase in the quantum field. Fortunately, there are strict limitations on the magnitude and duration of a pulse of negative energy. These limitations were first noted in a previous publication,[3] henceforth referred to as I. In I it was noted that negative-energy fluxes in two-dimensional spacetime seem to obey an inequality of the form

$$|F| < t^{-2} \,, \tag{1}$$

where $|F|$ is the magnitude of the negative flux and t is its duration. This inequality implies that $|F|t$, the amount of negative energy which passes by a fixed location in time t, is less than the quantum energy uncertainty, t^{-1}, on that time scale. Such an inequality prevents the negative energy from having gross, macroscopic effects, such as a violation of the second law of thermodynamics. The possibility of using negative energy to violate the cosmic-censorship hypothesis has been recently discussed in the context of two-dimensional models.[4] It was argued that this type of inequality prevents a classically observable violation of cosmic censorship.

The arguments given in I in favor of an inequality on negative-energy fluxes were incomplete in that they were applicable only to certain restricted classes of quantum states. In particular, the inequality was proven for states in which a single mode is excited and for the states generated by slowly moving mirrors. However, this left open the possibility that other quantum states exist in which the inequality would be dramatically violated. The main purpose of the present paper is to prove that this is not the case. The first task is to formulate a precise statement of an inequality, and the second is to prove that it is true for all quantum states.

In Sec. II an inequality constraining negative-energy fluxes is proven for a massless, free scalar field in flat two-dimensional spacetime. In Sec. III an analogous inequality is proven for such a field in four-dimensional spacetime. The results are discussed in Sec. IV.

II. TWO-DIMENSIONAL FLUX INEQUALITY

Let us consider a massless scalar field in flat two-dimensional spacetime. The stress tensor is

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha} \,, \tag{2}$$

and the field operator may be expanded in terms of creation and annihilation operators as

$$\phi = \sum_k (a_k f_k + a_k^\dagger f_k^*) \,. \tag{3}$$

Here the mode functions are taken to be

$$f_k = \frac{i}{\sqrt{2\omega L}} e^{i(kx - \omega t)} \,, \tag{4}$$

where $\omega = |k|$ and periodicity of length L has been imposed in the spatial direction, so that k takes on discrete values.

We are interested in the most general quantum state in which only particles moving in the $+x$ direction are present. A negative-energy flux then arises if the energy flow is in the $-x$ direction. Consider the energy flux at an arbitrary spatial point, which we take to be $x=0$, as a function of time:

$$F(t)=\langle T^{xt}\rangle = \frac{1}{L}\mathrm{Re}\sum_{k,k'>0}\sqrt{kk'}(\langle a_k^\dagger a_{k'}\rangle e^{i(k-k')t} + \langle a_k a_{k'}\rangle e^{-i(k+k')t}) . \quad (5)$$

Note that although the expectation values of the diagonal components of $T_{\mu\nu}$ are divergent, and hence require renormalization, the off-diagonal components have finite expectation values. Hence F is finite. The integral of $F(t)$ over all time is non-negative, but $F(t)$ may be instantaneously negative. Some explicit examples of states displaying a negative-energy flux were discussed in I.

We wish to formulate and prove an inequality involving a time integral of F. This is most conveniently done by multiplying F by a peaked function of time whose time integral is unity and whose characteristic width is t_0. A suitable choice of such a function is $t_0/[\pi(t^2+t_0^2)]$. Define the integrated flux $\hat{F}$ by

$$\hat{F}=\frac{t_0}{\pi}\int_{-\infty}^{\infty}\frac{F(t)dt}{t^2+t_0^2} . \quad (6)$$

From Eq. (5), $\hat{F}$ may be expressed as

$$\hat{F}=\frac{1}{L}\mathrm{Re}\sum_{k,k'>0}\sqrt{kk'}(\langle a_k^\dagger a_{k'}\rangle e^{-|k-k'|t_0} + \langle a_k a_{k'}\rangle e^{-(k+k')t_0}) . \quad (7)$$

In Appendix B it is shown that

$$\mathrm{Re}\sum_{k,k'>0}\sqrt{kk'}\langle a_k^\dagger a_{k'}\rangle e^{-|k-k'|t_0} \geq \sum_{k,k'>0}\sqrt{kk'}\langle a_k^\dagger a_{k'}\rangle e^{-(k+k')t_0} , \quad (8)$$

and that the right-hand side of Eq. (8) is real. Thus

$$\hat{F}\geq\frac{1}{L}\mathrm{Re}\sum_{k,k'>0}\sqrt{kk'}e^{-(k+k')t_0} \times(\langle a_k^\dagger a_{k'}\rangle+\langle a_k a_{k'}\rangle) . \quad (9)$$

However, the right-hand side of the above inequality is of a form to which the lemma proven in Appendix A may be applied. Take $h(k)=\sqrt{k}e^{-t_0 k}$. Then the lemma [Eq. (A11)] tells us that

$$\hat{F}\geq-\frac{1}{2L}\sum_{k>0}h^2(k) . \quad (10)$$

At this point we may let the periodicity length L become infinitely large, in which limit $\sum_{k>0}\to(L/2\pi)\int_0^\infty dk$. Performing the resulting integration on k yields our final result:

$$\hat{F}\geq-\frac{1}{16\pi t_0^2} . \quad (11)$$

This integrated inequality is a rigorous version of Eq. (1), the type of inequality conjectured in I and discussed in the Introduction. Note that t_0 is the characteristic time over which the flux is sampled. The inequality tells us that the magnitude of the flux cannot be more negative than t_0^{-2} times a dimensionless number which is small compared to unity. The sampling time t_0 can be chosen arbitrarily and need not be a physical time scale associated with a given energy flux. However, it is most natural to take it to be a time scale associated with the duration of the negative-energy pulse. Because the total energy integrated over all time must be non-negative, there must be a compensating positive-energy pulse either preceding or following the negative energy. The most efficient separation of positive and negative energy is obtained by δ-function pulses. Consider the flux[5]

$$F(t)=|\Delta E|[-\delta(t)+\delta(t-T)] . \quad (12)$$

This represents a pulse of negative energy followed a time T later by an exactly compensating pulse of positive energy. The inequality [Eq. (11)] yields

$$|\Delta E|\leq\frac{T^2+t_0^2}{16t_0T^2} . \quad (13)$$

This relation is true for all t_0, but the best constraint on $|\Delta E|$ is obtained by setting $t_0=T$. Then we find

$$|\Delta E|\leq\frac{1}{8T} . \quad (14)$$

This inequality tells us that there is a maximum separation in time between the two pulses which is within the limits allowed by the uncertainty principle. An inequality similar to Eq. (14) was derived in Ref. 4 for the case of δ-function pulses produced by moving mirrors. The present argument shows that, at least for two-dimensional flat spacetime, this conclusion is independent of the mechanism for generating the pulses. In Ref. 4 the inequality was used to argue against the possibility of using negative energy to violate cosmic censorship. More generally, we can see that the effects of a negative-energy pulse of magnitude $|\Delta E|$ cannot last for a time longer than $1/|\Delta E|$ before the positive energy arrives. Thus the magnitude of the negative-energy pulse is within the scale of the natural quantum fluctuations on a time scale T.

III. FOUR-DIMENSIONAL FLUX INEQUALITY

We now wish to turn to the problem of formulating and proving an inequality on negative-energy fluxes in four-dimensional spacetime. Again, we consider a free, massless, minimally coupled scalar field for which the energy-momentum tensor is of the form of Eq. (2). The mode functions may be taken to be

$$F_{\mathbf{k}}=\frac{i}{\sqrt{2\omega V}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} , \quad (15)$$

where $\omega=|\mathbf{k}|$ and V is the normalization volume. The energy flux in the $+x$ direction is given by the expectation value of T^{xt}:

$$F_x = \langle T^{xt} \rangle \ . \tag{16}$$

In order to discuss meaningfully whether F_x is negative as a result of quantum coherence effects, we must restrict the quantum state to have excitations only of modes with $k_x \geq 0$. Apart from this restriction, the state is completely arbitrary. As in the two-dimensional case, we may take the spatial point at which the flux is evaluated to have spatial coordinate $\mathbf{x}=0$. Then F_x may be expressed as

$$F_x(t) = \frac{1}{2V} \mathrm{Re} \sum_{\mathbf{k},\mathbf{k}'} \frac{k_x\omega' + k_x'\omega}{\sqrt{\omega\omega'}} \times (\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}'} \rangle e^{i(\omega-\omega')t} + \langle a_{\mathbf{k}} a_{\mathbf{k}'} \rangle e^{i-(\omega+\omega')t}) \ , \tag{17}$$

where it is understood that only terms with $k_x \geq 0$ and $k_k' \geq 0$ contribute to the sum. We define the integrated flux as before:

$$\hat{F}_x = \frac{t_0}{\pi} \int_{-\infty}^{\infty} \frac{F_x(t)dt}{t^2+t_0^2} = \frac{1}{2V} \mathrm{Re} \sum_{\mathbf{k},\mathbf{k}'} \frac{k_x\omega' + k_x'\omega}{\sqrt{\omega\omega'}} \times (\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}'} \rangle e^{-|\omega-\omega'|t_0} + \langle a_{\mathbf{k}} a_{\mathbf{k}'} \rangle e^{-(\omega+\omega')t_0}) \ . \tag{18}$$

Let us first consider the case of a state where all the excited modes have wave vectors along the x direction. In this case, $k_x = \omega$ and $k_x' = \omega'$, and so we can write $\hat{F}_x$ for such a state, which we will denote by $\hat{G}$, as

$$\hat{G} = \frac{1}{V} \mathrm{Re} \sum_{\mathbf{k},\mathbf{k}'} \sqrt{\omega\omega'} (\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}'} \rangle e^{-|\omega-\omega'|t_0} + \langle a_{\mathbf{k}} a_{\mathbf{k}'} \rangle e^{-(\omega+\omega')t_0}) \ . \tag{19}$$

However, the inequality proven in Appendix B [Eq. (8)] may be applied to this expression to show that

$$\hat{G} \geq \frac{1}{V} \mathrm{Re} \sum_{\mathbf{k},\mathbf{k}'} \sqrt{\omega\omega'} e^{-(\omega+\omega')t_0} \times (\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}'} \rangle + \langle a_{\mathbf{k}} a_{\mathbf{k}'} \rangle) \ . \tag{20}$$

This is now in a form to which the lemma of Appendix A may be applied.

Before doing this explicitly, let us show that the result will be a lower bound on the flux for any state for which the excited modes have $k_x \geq 0$, not just states with all wave vectors in the x direction. For a general state we could perform the sum on $\mathbf{k}'$ in Eq. (18) and define $g(\mathbf{k})$ so that

$$\hat{F}_x = \sum_{\mathbf{k}} g(\mathbf{k}) \ . \tag{21}$$

Now classify modes by the sign of $g(\mathbf{k})$, and define a new quantum state $|\tilde{\psi}\rangle$, from the original state $|\psi\rangle$, by a relabeling of the modes $\mathbf{k} \to \tilde{\mathbf{k}}$. This relabeling is defined as follows: If $g(\mathbf{k}) > 0$, then $\mathbf{k}$ is obtained by rotating $\mathbf{k}$ so that it lies in the yz plane. However, if $g(\mathbf{k}) \leq 0$, the $\mathbf{k}$ is obtained by rotating $\mathbf{k}$ so that it lies along the x axis. The net effect of this relabeling is to remove positive contributions from $\hat{F}_x$ and to enhance the magnitude of negative contributions. Thus the flux in the new state is less than or equal to that in the original state. But the new state $|\tilde{\psi}\rangle$ is a state with only modes along the x axis excited. Consequently, the right-hand side of Eq. (20) is a lower bound not just for such states, but for all states. Thus, if we apply the lemma for Appendix A to Eq. (20), we find

$$\hat{F}_x \geq -\frac{1}{2V} \sum_{\mathbf{k}} |h(\mathbf{k})|^2 \ , \tag{22}$$

where $h = \sqrt{\omega} e^{-\omega t_0}$. In the limit that $V \to \infty$, $\sum_{\mathbf{k}} \to (V/4\pi^3) \int_0^\infty d\omega\, \omega^2$. (Recall that $k_x \geq 0$.) Performing the integration on ω yields our final result:

$$\hat{F}_x \geq -\frac{3}{32\pi^2 t_0^4} \ . \tag{23}$$

This inequality is the four-dimensional version of Eq. (11). In effect, it states that if a negative-energy flux lasts for a time τ, then its magnitude will be less than about $\tau - 4$. Thus the magnitude of the negative energy which passes through an area A in this time is less than $A\tau^{-3}$. If the collecting area A were to be made arbitrarily large, then this energy would be unbounded. However, if the dimensions of the absorbing system are greater than τ in any direction, then the different parts of the system are not in casual contact on this time scale. In this case the system does not act as a coherent whole, but rather as a collection of disjoint subsystems, each having linear dimensions of order τ, and these subsystems become the objects of interest, rather than the whole system. Thus we should require that $A \leq \tau^2$, in which case the magnitude of the negative energy absorbed is less than $1/\tau$, again within the limits of quantum fluctuations on this time scale. Let us return to the case of δ-function pulses; the four-dimensional analog of Eq. (12) is

$$F_x(t) = \frac{|\Delta E|}{A} [-\delta(t) + \delta(t-T)] \ . \tag{24}$$

This represents a plane δ-function pulse of negative energy which has a magnitude $|\Delta E|$ over a collecting area A and which is followed a time T later by compensating positive energy. Here we may regard T as being the time scale τ for the duration of the negative energy. If we insert this into Eq. (23), set $t_0 = T$, and require that $A \leq T^2$, then we find that

$$|\Delta E| \leq \frac{3}{16\pi T} \ . \tag{25}$$

Again, there is a constraint which requires the positive energy to arrive within a time $1/|\Delta E|$.

IV. DISCUSSION

We have seen that there are inequalities which constrain the magnitude and duration of a flux of negative energy. These inequalities may be formulated as integrals of the flux multiplied by a sampling function. The particular sampling function used in Eqs. (6) is a convenient choice; however, it is expected that similar theorems could be proven using other sampling functions. Both the two- and the four-dimensional versions of the integrated inequality [Eqs. (11) and (23), respectively] are valid for all quantum states with particles moving only in the $+x$ direction. These inequalities ensure that on a time scale t, a system cannot absorb negative energy whose magnitude exceeds about $1/t$. Thus this negative energy is within the scale of the quantum energy fluctuations on this time scale. This prevents flat-space negative-energy fluxes from having dramatic, macroscopic effects.

The key to the proofs of these inequalities is the lemma [Eq. (A11)] proven in Appendix A. Because the sum S_N contains N^2 terms, we might have expected that S_N could vary as $-N^2$ as N increases. In fact, the lemma guarantees that it cannot become negative any faster than $-N$. It is this limit on the negativity of S_N that is crucial to the existence of limits on the negativity of the energy flux.

Davies[6] has noted that it is possible to find two-dimensional moving-mirror trajectories which produce rapidly changing negative-energy fluxes. Such fluxes can even be made to diverge in some appropriate limit. These rapidly changing fluxes appear to violate an inequality of the form of Eq. (1) because an arbitrary amount of negative energy can be radiated in a given time interval. Davies suggested that this is due to the fact that the arguments given in I for states produced by moving mirrors assume slow motion. However, the proof of Eq. (11) given in the present paper has no such restriction. We may understand why a rapidly varying flux is consistent with Eq. (11) in the following way: If we wish to accurately sample such a flux in the region where it is becoming very negative, we must choose t_0 to be very small and, hence, make the right-hand side of Eq. (11) very negative. In the case of a rapidly changing flux, the effective value of t that we should understand in Eq. (1) is not the entire duration of the negative flux, but rather the (much shorter) time during which the greater amount of negative energy is emitted. Conversely, if we choose t_0 to be large, the right-hand side of Eq. (11) is less negative because the sampling function is picking up some of the compensating positive flux that must precede or follow the negative flux.

It is of interest to note that there also seem to be constraints on the density of negative energy. First, consider the two-dimensional case. If we wish to fill a region of length l with negative energy, then we need to shine a beam of negative energy of duration $t \geq l$ into the region. Thus the energy in the region is bounded by $E \geq -t^{-1} \geq -l^{-1}$, and the average energy density is bounded by

$$\rho_{\rm av} \geq -\frac{1}{l^2} \ . \qquad (26)$$

In four dimensions we consider a region of length l and cross-sectional area A. Again, the duration of the flux which injects negative energy into the region must be $t \geq l$, so that the energy in the region is bounded by $E \geq -Al^{-3}$ and the average energy density by

$$\rho_{\rm av} \geq -\frac{1}{l^4} \ . \qquad (27)$$

The above argument is somewhat heuristic as it assumes that the negative-energy density is built up by shining negative energy into a region. One could conceive of a process which is better visualized as being the result of forcing positive energy to leave the region. In the latter case it is unclear that there would be a constraint on the negative-energy density. The task of proving more rigorous inequalities on $\rho_{\rm av}$ remains.

Both the inequalities on fluxes and those on energy densities are presumably examples of a wider class of inequalities which constrain all quantum violations of classical energy conditions. All of these inequalities prevent quantum coherence effects from producing such large-scale effects as gross violations of the second law of thermodynamics or of cosmic censorship. However, this does not mean that the effects of negative energy are never observable. Such effects can lead to a suppression of the natural quantum vacuum fluctuations. To the extent that the latter produce observable effects, then so can negative energy. The observability of negative energy has been discussed by several authors.[7–9] In particular, Grove[9] has shown how negative energy can reduce the excitations that would otherwise occur in a switched detector.

The treatment in this paper has been restricted to flat spacetime, and so it will be of interest to generalize these inequalities to curved spacetime. It is clear that there can be observers in curved spacetime for whom the flux appears to be unconstrained by an inequality of the form of Eq. (1). An example is an observer near the horizon of an evaporating black hole who sees a steady negative flux going across the horizon to compensate the thermal Hawking radiation being emitted to infinity.[10] However, this flux leads to no violation of the second law of thermodynamics. Furthermore, such observers are not inertial, and so one must be careful in the interpretation of their measurements. It is well known that accelerated detectors in the Minkowski vacuum state appear to detect particles.

In summary, in this paper we have provided a precise formulation and proof of inequalities which constrain negative-energy fluxes in flat spacetime. These inequalities are of the form required to prevent macroscopically observable violations of the second law of thermodynamics[3] and of cosmic censorship.[4] Further work is needed to generalize this work to curved spacetime, but it is reasonable to expect that there are inequalities that constrain all quantum violations of classical energy conditions.

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APPENDIX A

In this appendix we wish to state and prove a lemma concerning the expectation value of a bilinear product of creation and annihilation operators. Let $|\psi\rangle$ be a general quantum state for a boson field. In the Fock representation it may be expressed as

$$|\psi\rangle=\sum_{\{n_i\}} c(\{n_i\})|\{n_i\}\rangle \ , \tag{A1}$$

where $|\{n_i\}\rangle$ is a particle number eigenstate with n_i particles in mode i, and the label i runs over all modes. Here $\sum_{\{n_i\}}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots$ is a sum on all possible values of the occupation numbers n_i. The coefficients $c(\{n_i\})$ satisfy the normalization condition

$$\sum_{\{n_i\}} |c(\{n_i\})|^2=1 \ . \tag{A2}$$

Let the $\{h_i\}$ be an arbitrary set of real numbers, and define the operator P_N by

$$P_N=\sum_{i,j=1}^{N} h_i h_j(a_i^\dagger a_j+a_i a_j+\text{H.c.}) \ , \tag{A3}$$

and let

$$S_N=\langle\psi|P_N|\psi\rangle =2\,\text{Re}\sum_{i,j=1}^{N} h_i h_j(\langle a_i^\dagger a_j\rangle+\langle a_i a_j\rangle) \ . \tag{A4}$$

Here N is an arbitrary integer, and H.c. denotes the Hermitian conjugate. The quantity S_N is an expectation value of products of creation and annihilation operators of the form which appears in the expectation value of the energy-momentum tensor. We wish to establish a lower bound on this quantity.

First, let us write S_N explicitly as

$$S_N=\sum_{\{n_i\}}\left\{\sum_{i=1}^{N} h_i^2[n_i|c|^2+\sqrt{n_i(n_i-1)}c^*(n_i-2)c] + \sum_{i\neq j}^{N} h_i h_j[\sqrt{n_j(n_i+1)}c^*(n_i+1,\ n_j-1)c+\sqrt{n_i n_j}c^*(n_i-1,\ n_j-1)c]\right\}+\text{c.c.} \tag{A5}$$

Here c.c. denotes complex conjugate, and the arguments of the coefficients c are only written explicitly if they are different from n_i. Thus $c=c(\{n_i\})=c(n_1,n_2,\ldots)$, whereas $c(n_i-2)=c(n_1,n_2,\ldots,n_i-2,\ldots)$, etc. Next, write

$$S_N=\sum_{\{n_i\}}\left\{\sum_{i=1}^{N} h_i^2\{(n_i+1)|c|^2+n_i|c|^2+\sqrt{n_i(n_i-1)}[c^*(n_i-2)c+\text{c.c.}]\} + \sum_{i\neq j}^{N} h_i h_j[\sqrt{n_j(n_i+1)}c^*(n_i+1,\ n_j-1)c+\sqrt{n_i n_j}c^*(n_i-1,n_j-1)c+\text{c.c.}]\right\}-\sum_{i=1}^{N} h_i^2 \ , \tag{A6}$$

where we have used the normalization condition [Eq. (A2)] to add and subtract the term

$$\sum_{\{n_i\}}\sum_{i=1}^{N} h_i^2|c|^2=\sum_{i=1}^{N} h_i^2 \ . \tag{A7}$$

Relabeling the sums on the occupation numbers n_i enables us to write

$$S_N=\sum_{\{n_i\}}\left\{\sum_{i=1}^{N} h_i^2\{n_i|c(n_i-1)|^2+(n_i+1)|c(n_i+1)|^2+\sqrt{n_i(n_i+1)}[c^*(n_i-1)c(n_i+1)+\text{c.c.}]\} + \sum_{i\neq j}^{N} h_i h_j\{\sqrt{n_j n_i}c^*(n_j-1)c(n_i-1)+\sqrt{(n_j+1)(n_i+1)}c^*(n_j+1)c(n_i+1) + [\sqrt{(n_i+1)n_j}c^*(n_j-1)c(n_i+1)+\text{c.c.}]\}\right\}-\sum_{i=1}^{N} h_i^2 \ . \tag{A8}$$

Here examples of the relabelings which we have used include

$$\sum_{\{n_i\}}\sqrt{n_j(n_i+1)}c^*(n_i+1,\ n_j-1)c=\sum_{\{n_i\}}\sqrt{n_j n_i}c^*(n_j-1)c(n_i-1) \ ,$$
$$\left[\sum_{\{n_i\}}\sqrt{n_j(n_i+1)}c^*(n_i+1,\ n_j-1)c\right]^*=\sum_{\{n_i\}}\sqrt{(n_j+1)(n_i+1)}c^*(n_j+1)c(n_i+1) \ . \tag{A9}$$

Finally, we may factor the terms which are quadratic in the coefficients c, and write

$$S_N=\sum_{\{n_i\}}\left|\sum_{i=1}^{N} h_i[\sqrt{n_i}c(n_i-1)+\sqrt{(n_i+1)}c(n_i+1)]\right|^2-\sum_{i=1}^{N} h_i^2 \; . \tag{A10}$$

From this form it is apparent that

$$S_N \geq -\sum_{i=1}^{N} h_i^2 \; . \tag{A11}$$

This is the lower bound on S_N which we wish to establish.

APPENDIX B

In this appendix we wish to establish the inequality [Eq. (8)]

$$\mathrm{Re}\sum_{k,k'>0}\sqrt{kk'}\langle a_k^\dagger a_{k'}\rangle e^{-|k-k'|t_0} \geq \sum_{k,k'>0}\sqrt{kk'}\langle a_k^\dagger a_{k'}\rangle e^{-(k+k')t_0} \; . \tag{B1}$$

Let

$$B_{mn}=\sqrt{mn}\,e^{-\alpha(m+n)}\mathrm{Re}\langle a_m^\dagger a_n\rangle \tag{B2}$$

and

$$A_{mn}=(e^{-\alpha|m-n|}-e^{-\alpha(m+n)})\sqrt{mn}\;\mathrm{Re}\langle a_m^\dagger a_n\rangle=(e^{-\alpha[|m-n|-(m+n)]}-1)B_{mn} \; . \tag{B3}$$

With $k=2\pi mL^{-1}$, $k'=2\pi nL^{-1}$, and $\alpha=2\pi t_0L^{-1}$, we see that Eq. (B1) is equivalent to

$$\sum_{m,n=1}^{\infty} A_{mn} \geq 0 \; . \tag{B4}$$

First, we note that $B_{nn}\geq 0$ because $\langle a_n^\dagger a_n\rangle$ is just the mean number of particles in mode n. Furthermore, the sum of the B_{mn} over any square block centered on the diagonal is the norm of a state vector and, hence, non-negative:

$$\sum_{m,n=J}^{J+M} B_{mn}=\left|\left|\sum_{n=J}^{J+M}\sqrt{n}\,e^{-\alpha n}a_n|\psi\rangle\right|\right| \geq 0 \; . \tag{B5}$$

Here J and M are any non-negative integers. Hence the right-hand side of Eq. (B1) is both real and non-negative. We wish to demonstrate Eq. (B4) by considering first a finite sum:

$$\sum_{m,n=1}^{N} A_{mn}=\sum_{l=1}^{N}\left[A_{ll}+\sum_{j=0}^{N-1-l}(A_{l,N-j}+A_{N-j,l})\right] \; . \tag{B6}$$

In Eq. (B6) we are summing over all of the elements of the matrix A in the following way: For each diagonal element sum up the elements in the same row lying to the right of and the same column lying below the diagonal element. The index j in Eq. (B6) runs up the column and across the row from right to left.

Note that for fixed l, each of the A_{mn} which appears on the right-hand side of this sum is related to the corresponding B_{mn} by a factor of $(e^{2\alpha l}-1)$ because $(m+n)-|m-n|=l+N-j-|N-j-l|=2l$. Thus we may write

$$\begin{aligned}
\sum_{m,n=1}^{N} A_{mn} &= \sum_{l=1}^{N}(e^{2\alpha l}-1)\left[B_{ll}+\sum_{j=0}^{N-1-l}(B_{l,N-j}+B_{N-j,l})\right]\\
&=(e^{2\alpha N}-1)B_{NN}+(e^{2\alpha(N-1)}-1)[B_{N-1,N-1}+(B_{N-1,N}+B_{N,N-1})]+\cdots\\
&\geq (e^{2\alpha(N-1)}-1)[B_{NN}+B_{N-1,N-1}+(B_{N-1,N}+B_{N,N-1})]\\
&\quad +(e^{2\alpha(N-2)}-1)\left[B_{N-2,N-2}+\sum_{j=0}^{1}(B_{N-2,N-j}+B_{N-j,N-2})\right]+\cdots\\
&\geq (e^{2\alpha(N-2)}-1)\sum_{i,j=0}^{2}B_{N-i,N-j}+(e^{2\alpha(N-3)}-1)\left[B_{N-3,N-3}+\sum_{j=0}^{2}(B_{N-3,N-j}+B_{N-j,N-3})\right]+\cdots\\
&\geq \cdots \geq (e^{2\alpha(N-k)}-1)\sum_{i,j=0}^{k}B_{N-i,N-j}+(e^{2\alpha(N-k-1)}-1)\\
&\qquad\times\left[B_{N-k-1,N-k-1}+\sum_{j=0}^{k}(B_{N-k-1,N-j}+B_{N-j,N-k-1})\right]+\cdots\\
&\geq \cdots \geq (e^{2\alpha}-1)\sum_{i,j=0}^{N-1}B_{N-i,N-j}=(e^{2\alpha}-1)\sum_{m,n=1}^{N}B_{mn}\geq 0 \; .
\end{aligned} \tag{B7}$$

Here we have repeatedly used the non-negativity of a sum on B_{mn} over a square block [Eq. (B5)] and the fact that $(e^{2\alpha j}-1)\geq(e^{2\alpha(j-1)}-1)$. From Eqs. (B5) and (B7), we can see that

$$\sum_{m,n=1}^{N} A_{mn}\geq 0 \ . \tag{B8}$$

Because this is valid for all N, we have the result which we wish to prove [Eq. (B4)] or the equivalent [Eq. (B1)].

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[1]H. Epstein, V. Glaser, and A. Jaffe, Nuovo Cimento **36**, 1016 (1965).

[2]S. A. Fulling and P. C. W. Davies, Proc. R. Soc. London **A348**, 393 (1976); P. C. W. Davies and S. A. Fulling, *ibid.* **A356**, 237 (1977).

[3]L. H. Ford, Proc. R. Soc. London **A364**, 227 (1978).

[4]L. H. Ford and T. A. Roman, Phys. Rev. D **41**, 3662 (1990).

[5]I would like to thank T. Roman for helpful discussions on this argument.

[6]P. C. W. Davies, Phys. Lett. **113B**, 215 (1982).

[7]D. Deutsch, A. C. Ottewill, and D. W. Sciama, Phys. Lett. **119B**, 72 (1982).

[8]P. G. Grove, Class. Quantum Grav. **3**, 793 (1986).

[9]P. G. Grove, Class. Quantum Grav. **5**, 1381 (1988).

[10]Other examples which involve mirrors moving in black-hole spacetimes have been discussed by A. C. Ottewill and S. Takagi, Prog. Theor. Phys. **79**, 429 (1988).