

Celestial mechanics, conformal structures, and gravitational waves

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Newton's equations for the motion of N nonrelativistic point particles attracting according to the inverse square law may be cast in the form of equations for null geodesics in a $(3N + 2)$ -dimensional Lorentzian spacetime which is Ricci flat and admits a covariantly constant null vector. Such a spacetime admits a Bargmann structure and corresponds physically to a plane-fronted gravitational wave (generalized pp wave). Bargmann electromagnetism in five dimensions actually comprises the two distinct Galilean electromagnetic theories pointed out by Le Bellac and Lévy-Leblond. At the quantum level, the N -body Schrödinger equation may be cast into the form of a massless wave equation. We exploit the conformal symmetries of such spacetimes to discuss some properties of the Newtonian N -body problem, in particular, (i) homographic solutions, (ii) the virial theorem, (iii) Kepler's third law, (iv) the Lagrange-Laplace-Runge-Lenz vector arising from three conformal Killing two-tensors, and (v) the motion under time-dependent inverse-square-law forces whose strength varies inversely as time in a manner originally envisaged by Dirac in his theory of a time-dependent gravitational constant $G(t)$. It is found that the problem can be reduced to one with time-independent inverse-square-law forces for a rescaled position vector and a new time variable. This transformation (Vinti and Lynden-Bell) is shown to arise from a particular conformal transformation of spacetime which preserves the Ricci-flat condition originally pointed out by Brinkmann. We also point out (vi) a Ricci-flat metric representing a system of N nonrelativistic gravitational dyons. Our results for a general time-dependent $G(t)$ are also applicable by suitable reinterpretation to the motion of point particles in an expanding universe. Finally we extend these results to the quantum regime.

I. INTRODUCTION

Over the past few years, a new formalism has been developed for discussing the symmetries of Galilei-invariant classical and quantum-mechanical systems^{1,2} associated with the nonrelativistic spacetime picture.³⁻⁸

The key point is that everything can be best handled on an *extended spacetime*, a well-behaved Lorentz manifold devised to tackle nonrelativistic physics. The situation is reminiscent of Kaluza-Klein theory: the classical motions of a particle in a Newtonian potential correspond to *null geodesics* in a 5-dimensional—or $(3N + 2)$ -dimensional for N particles—Lorentz manifold (Q, g) carrying a covariantly constant null vector field ξ . These so-called "Bargmann structures" were introduced in Ref. 3.

The particle trajectories can also be obtained as the projections of *string* trajectories in the extended spacetime. These strings moreover satisfy the Polyakov equations of motion.

The null-geodesic-string formalism on our extended

spacetime naturally leads to studying the associated *conformal transformations* that actually provide the chronoprojective,⁹ Schrödinger,¹⁰⁻¹³ Bargmann,^{1,3} and Galilei groups with a clearcut geometrical status.

Conformally related Bargmann structures have the same null geodesics. We show that the metric associated to a time-varying gravitational constant $G(t)$ is conformally related to the G_0 case if and only if $G(t)$ changes according to the prescription of Vinti,¹⁴ whose particular case is Dirac's suggestion¹⁵

$$G(t) = G_0 \frac{t_0}{t}. \quad (1.1)$$

The theory is readily extended to N particles. The *generalized Kepler's third law* and the *virial theorem* arise due to a certain homothety of the corresponding metric. The celebrated *Lagrange-Laplace-Runge-Lenz vector* of planetary motion turns out to be associated with three conformal-Killing tensors of the extended spacetime metric.

Incorporating full electromagnetism would necessitate

an entirely relativistic framework. Electric and magnetic interactions can, however, be partially incorporated, namely in two distinct ways. In one way one gets a “magnetic” theory without the displacement current, and in the other way one gets another “electric” theory where the Faraday induction term is missing. We here recover by purely geometric means the two distinct theories of Galilean electromagnetism originally discovered by Le Bellac and Lévy-Leblond.¹⁶

The Bargmann structures are the 5-dimensional generalizations of the so called *pp-wave* solutions of Einstein’s equations in four dimensions. The latter describe *plane-fronted gravitational waves* and were discovered by Brinkmann.¹⁷ They also admit a Kaluza-Klein interpretation allowing for *magnetic mass* in dimension $D \geq 5$ and for a *Chern-Simons*-type modification of the field equations of prerelativistic electromagnetism, analogous to the situation studied recently by Jackiw.¹⁸

It is worth mentioning that Bargmann structures (in particular their *pp-wave* solutions in arbitrary dimension) provide a wide class of time-dependent classical solutions to string theory that have been extensively used¹⁹ to study string singularities.

In the case of the quantum N -body problem with the Dirac form (1.1) for $G(t)$, one is able (by using De Witt’s quantization rules in curved space) to associate to any solution of the Schrödinger equation for a time-independent Newtonian constant G_0 , a solution of the Schrödinger equation corresponding to a variable “constant” of gravitation $G(t)$. In the free case one gets the quantum operators of the Schrödinger group, as written by Niederer.¹⁰ It should be emphasized that, in our formalism, quantization of these systems (and the appearance of symmetries) relies heavily on the conformal invariance property of Schrödinger’s equation in its “Bargmannian” guise.

Our present interest in nonrelativistic conformal structures arose from a study of Dirac’s attempt to solve what is now known as the hierarchy problem. Fifty years ago Dirac¹⁵ proposed in fact that Newton’s gravitational constant G varies inversely as the age t of the Universe (1.1). Thus N celestial bodies of mass m_j and position vectors \mathbf{q}_j moving nonrelativistically would satisfy the equation

$$\frac{d^2 \mathbf{q}_j}{dt^2} = \sum_{k \neq j} G(t) m_k \frac{\mathbf{q}_k - \mathbf{q}_j}{\|\mathbf{q}_j - \mathbf{q}_k\|^3}, \quad \text{where } G(t) = G_0 \frac{t_0}{t}. \quad (1.2)$$

An identical equation would arise if one supposed that the fine-structure constant varied inversely as the age of the Universe and one considered the classical motion of nuclei and electrons in atoms and molecules. Of course in that case it is more appropriate to consider the non-relativistic many particle Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2} \sum_{j=1}^N \frac{1}{m_j} \Delta_j \Psi + V \Psi, \quad (1.3)$$

where

$$V = - \sum_{j < k} G(t) \frac{m_j m_k}{\|\mathbf{q}_j - \mathbf{q}_k\|}.$$

The classical equation (1.2) also arises if one considers the motion of particles in an expanding Universe with scale factor a in a conventional theory in which Newton’s constant really is constant $G = G_0$. If T is cosmic time then, as many have pointed out,²⁰ the relevant equation is the cosmic Newton equation:

$$a^2 \frac{d}{dT} \left(a^2 \frac{d\mathbf{q}_j}{dT} \right) = a(T) G_0 \sum_{k \neq j} \frac{\mathbf{q}_k - \mathbf{q}_j}{\|\mathbf{q}_j - \mathbf{q}_k\|^3}. \quad (1.4)$$

Equation (1.4) can easily be cast in the form of Eq. (1.2) with an apparently time-dependent gravitational constant by defining a quasi-Newtonian time coordinate t by

$$dt = \frac{dT}{a(T)^2}, \quad \text{where } G(t) = a(T) G_0. \quad (1.5)$$

In order to obtain an apparent variation inversely as the Newtonian time t the scale factor $a(T)$ must be proportional to the cosmic time T . This corresponds to an empty Milne model.

As emphasized by Lynden-Bell,²¹ the classical equations (1.2) are no more, or indeed no less difficult, to solve than the usual equations with a constant coupling constant. That is, suppose we are given a solution $\mathbf{q}_j^*(t^*)$ of the classical equation of motion (1.2) with $G = G_0$ independent of t , then

$$\mathbf{q}_j(t) = \frac{t}{t_0} \mathbf{q}_j^* \left(-\frac{t_0^2}{t} \right) \quad (1.6)$$

is a solution of the equations of motion with time-dependent Newton’s constant varying inversely as the Newtonian time.

The corresponding statement for the quantum-mechanical problem is the following: suppose that we are given a solution $\Psi_{\text{static}}(\mathbf{q}_j^*, t^*)$ of the Schrödinger equation (1.2) with a time independent potential, then

$$\begin{aligned} \Psi(\mathbf{q}_j, t) = & \left(\frac{t}{t_0} \right)^{-3N/2} \exp \left(\frac{i}{2\hbar t} \sum_{j=1}^N m_j \mathbf{q}_j^2 \right) \\ & \times \Psi_{\text{static}} \left(-\mathbf{q}_j \frac{t_0}{t}, -\frac{t_0^2}{t} \right) \end{aligned} \quad (1.7)$$

is a solution of the Schrödinger equation with a fine-structure constant varying inversely as the Newtonian time. The pre-factor $(t/t_0)^{-3N/2}$ guarantees that the wave function remains normalized.

These two results are exact and may readily be verified by elementary differentiations.

Dirac’s original hypothesis is excluded by observations, however for times short compared with t_0 any time variability of G can be modeled by a $1/t$ law, and these classical results are useful in discussing observations of the binary pulsar.^{22,23}

These results provide a fascinating example where both the classical and the quantum mechanical adiabatic theorems are in some sense exact. Their truth is suggested

by the work of Vinti¹⁴ who pointed out in the context of the two-body problem in Dirac's theory, the relevance of a result of Mestschersky²⁴ showing that the 2-body problem could be reduced to quadratures. The reader is referred to Vinti's paper for a detailed consideration of the solutions in the case $N = 2$.

In order to explain the relation between the situations with $G(t)$ and G_0 , remember that the equation of motion (1.2) may be derived from the action

$$S = \int \left(\sum_{j=1}^N \frac{m_j}{2} \left(\frac{d\mathbf{q}_j}{dt} \right)^2 - \frac{1}{2} \sum_{j \neq k} G(t) \frac{m_j m_k}{\|\mathbf{q}_j - \mathbf{q}_k\|} \right) dt.$$

Consider furthermore the transformation^{14,21}

$$D : (\mathbf{q}_j, t) \rightarrow (\mathbf{q}_j^*, t^*) = \left(\frac{t_0}{t} \mathbf{q}_j, -\frac{t_0^2}{t} \right), \quad (1.8)$$

where $j = 1, \dots, N$. It is easy to see that if $G(t)$ changes as suggested by Dirac (1.1), the action, S , changes by a mere boundary term

$$S = S^* - \int d \left(\sum_{j=1}^N \frac{m_j \mathbf{q}_j^{*2}}{2t^*} \right), \quad (1.9)$$

where S^* is the action with G_0 . This explains the result of Lynden-Bell.

Observe that the above total derivative actually comes from the kinetic term. The "Vinti-Lynden-Bell transformation," D in (1.8), is therefore a symmetry for a free particle. But the symmetries of a free particle form a two-parameter extension of the Galilei group, the so-called "Schrödinger group".^{10-13,25} The new transformations are

$$\delta_d : (\mathbf{q}, t) \rightarrow (d\mathbf{q}, d^2t), \quad d \in \mathbf{R} \setminus \{0\} \quad (\text{dilations}), \quad (1.10)$$

$$\kappa_k : (\mathbf{q}, t) \rightarrow \left(\frac{\mathbf{q}}{1-kt}, \frac{t}{1-kt} \right), \quad k \in \mathbf{R} \quad (\text{expansions}). \quad (1.11)$$

They form, together with

$$\epsilon_e : (\mathbf{q}, t) \rightarrow (\mathbf{q}, t + e), \quad e \in \mathbf{R} \quad (\text{time translations}), \quad (1.12)$$

a group isomorphic to $\text{SL}(2, \mathbf{R})$. This group was later rediscovered²⁶ as a symmetry of the Dirac monopole and more recently¹⁸ as a symmetry of a magnetic vortex.

Being a symmetry of a free particle, the Vinti-Lynden-Bell transformation (1.8) is expected to belong to the Schrödinger group. It is easy to see that this is indeed the case since, for $t_0 \neq 0$, we have

$$D = \epsilon_{t_0} \circ \kappa_{1/t_0} \circ \epsilon_{t_0} \circ \delta_{-1}.$$

In this paper we deal exclusively with the case of three spatial dimensions. However, all of our considerations generalize in a straightforward fashion to an arbitrary

number d of spatial dimensions. In this case, the analog of Dirac's law for the time dependence of G is

$$G(t) \propto \frac{1}{t^{d-2}}.$$

The organization of the paper is as follows.

In Sec. II we provide a review of the geometrical framework we use, that is, of Bargmann structures leading to a covariant formulation of classical mechanics and pre-relativistic electromagnetism.

In Sec. III we discuss the single-particle case with varying "constant" of gravitation.

In Sec. IV we review the conformal symmetries of a Bargmann structure and relate these to the so-called Vinti-Lynden-Bell transformations.

Sec. V extends this work to the N -body problem by passing to a $(3N + 2)$ -dimensional spacetime and deals with the homographic solutions and the virial theorem.

In Sec. VI we show how the Lagrange-Laplace-Runge-Lenz vector is associated to a three-vector's worth of conformal Killing two-tensors.

In Sec. VII we relate our work to that of Brinkmann and we also point out that Newtonian theory allows for a natural generalization including gravitational "magnetic" mass monopoles.

Finally, in Sec. VIII we show how our results can be extended to the quantum regime, in particular how mass gets quantized in the presence of gravitational dyons in this nonrelativistic context.

II. BARGMANN STRUCTURES AND COVARIANT MECHANICS

In this section we give a short outline of a geometrical framework adapted to the description of nonrelativistic classical and quantum physics.

A. Bargmann structures

Let us first recall that a Bargmann manifold³ is a 5-dimensional Lorentz manifold (Q, g) with signature $(++++-)$ and a fibration by a free action of the additive group of real numbers, $(\mathbf{R}, +)$, whose infinitesimal generator ξ is null and covariant constant.

A $\text{SO}(2)$ action would lead to a topologically nontrivial bundle suitable to generalize Newtonian theory as described in Secs. VII and VIII.

As an example, let us consider the extended spacetime $(\mathbf{R}^3 \times \mathbf{R}) \times \mathbf{R} \ni (\mathbf{q}, t, s)$, where the fifth coordinate s has dimension [action]/[mass]; start off with the Lorentz metric

$$g_U \equiv \langle d\mathbf{q} \otimes d\mathbf{q} \rangle + dt \otimes ds + ds \otimes dt - 2U(\mathbf{q}, t) dt \otimes dt, \quad (2.1)$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product on \mathbf{R}^3 and put

$$\xi \equiv \frac{\partial}{\partial s}. \quad (2.2)$$

It is a simple task to check that (\mathbf{R}^5, g_U, ξ) is actually a Bargmann manifold.

The base manifold $B = Q/(\mathbf{R}, +)$ is readily interpreted as spacetime. If we denote by $\pi : Q \rightarrow B$ the corresponding projection, we see that B is canonically endowed with a Galilei structure, i.e., a symmetric 2-contravariant tensor $h = \pi_* g^{-1}$ with signature $(+++0)$ and a one-form θ defined by $g(\xi) = \pi^* \theta$ which generates $\ker(h)$. Note that the “clock” θ is closed and the integrable spacelike distribution $\Sigma = \ker(\theta)$ then defines the *absolute time axis* B/Σ .

In our example [(2.1),(2.2)], we easily find that spacetime $B \cong \mathbf{R}^3 \times \mathbf{R}$ has the following Galilei structure: $h = \langle \partial/\partial \mathbf{q} \otimes \partial/\partial \mathbf{q} \rangle, \theta = dt$.

The Levi-Civita connection $\nabla^{(Q)}$ on Q can be shown to descend to B as a preferred symmetric “Newton-Cartan” connection^{27,28} $\nabla^{(B)}$ (which, in particular, parallel-transport h and θ) interpreted as the Newtonian gravitational field. In the particular case (2.1),(2.2), one finds, using obvious notations, $\Gamma_{tt}^A = \partial_A U = -\Gamma_{At}^s$ ($A = 1, 2, 3$) and $\Gamma_{tt}^t = -\partial_t U$ for the nonzero Christoffel symbols of $\nabla^{(Q)}$. A straightforward computation shows furthermore that the geodesics of (Q, g_U) project upon spacetime $(B, h, \theta, \nabla^{(B)})$ as the world lines of test particles in the gravitational potential U .

Consider now such a geodesic $[\tau \rightarrow q(\tau)]$ and set $p = dq/d\tau$. The two quantities $p^2 \equiv g_{ab} p^a p^b$ and $m \equiv g_{ab} p^a \xi^b$ ($a, b = 1, \dots, 5$) are conserved along the motion. Comparing with the free case, the three-(co)vector $\mathbf{p} = (p_1 p_2 p_3)$ may be interpreted as the linear momentum, $-p_t = E$ as the energy and $p_s = m$ as the mass. An easy calculation yields

$$p^2 = 2m \left(\frac{\mathbf{P}^2}{2m} - E + V \right) = -2mE_0,$$

where $V \equiv mU$ and E_0 is thus the *internal energy* of the particle under consideration. But, in the nonrelativistic theory, the internal energy is required to vanish, $E_0 = 0$. The motion of our test particle of mass m in the potential V can be described therefore by a *null geodesic* in the extended spacetime. We defer to Sec. V the general case of N bodies.

Let us now assume that two Bargmann structures (Q, g, ξ) and (Q, g^*, ξ^*) on the same spacetime extension are related by

$$g^* = \Omega^2 g \text{ and } \xi^* = \xi. \tag{2.3}$$

The nowhere vanishing function Ω is necessarily a function of time only since the new clock $\theta^* = \Omega^2 \theta$ must be closed, implying $d\Omega \wedge dt = 0$. But conformally related metrics have the same null geodesics, yielding the same world lines in spacetime. Because of (2.3), the mass is also preserved.

B. Symplectic mechanics and strings

Classical mechanics can be most elegantly formulated in a symplectic formalism.² We thus wish to present, in

this geometric setup, the equations of motion of a point particle dwelling in spacetime and subject to the action of an external gravitational field described by a Bargmann structure.

We start with the cotangent bundle T^*Q endowed with its canonical one-form $\vartheta \equiv p_a dq^a$. The equations of motion we are interested in can be obtained as follows. Consider the (closed) 8-dimensional submanifold $C \hookrightarrow T^*Q$ defined by the two previously introduced constraints ($a, b = 1, \dots, 5$):

$$E_0 = g^{ab} p_a p_b = 0 \tag{2.4}$$

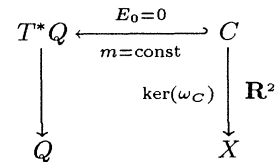
and

$$m \equiv p_a \xi^a = \text{const}. \tag{2.5}$$

We will confine considerations to massive systems only. The restriction $\omega_C = d\vartheta|_C$ to C of the symplectic two-form of T^*Q turns C into a presymplectic manifold with a 2-dimensional null foliation $\ker(\omega_C)$, whose leaves projects onto Q as two-surfaces, physically the world sheets of *strings*. These project, in turn, onto spacetime B as curves, i.e., as the world lines of *particles*. The equations of the foliation are indeed

$$\begin{aligned} \dot{p}_a &= 0, \\ \dot{q}^a &= \alpha p^a + \beta \xi^a \quad (\alpha, \beta \in \mathbf{R}), \end{aligned} \tag{2.6}$$

where the dot in the first equation stands for covariant derivative with respect to $\nabla^{(Q)}$. We thus have the diagram



where $X = C/\ker(\omega_C)$ is a 6-dimensional manifold canonically endowed with the symplectic two-form ω_X , the projection of ω_C . In Souriau’s terminology,² (X, ω_X) is the *space of motions* of our test particle.

Hence the geodesics of $(B, \nabla^{(B)})$ arise from strings in the Bargmann manifold. Interestingly enough, these strings satisfy the *Polyakov equations of motion* in (Q, g) . (An analogous situation appears for “winders” in 5-dimensional Kaluza-Klein theory,²⁹ where the strings satisfy the Goto-Nambu equations.) To see this, note that the Polyakov action for a string, i.e., for a map $q^a : (M_2, (\gamma_{ij})) \rightarrow (Q, (g_{ab})) : (\sigma^1, \sigma^2) \rightarrow q^a(\sigma^1, \sigma^2)$, is given by³⁰

$$\int g_{ab} \frac{\partial q^a}{\partial \sigma^i} \frac{\partial q^b}{\partial \sigma^j} \gamma^{ij} \sqrt{\gamma} d\sigma^1 d\sigma^2. \tag{2.7}$$

Varying with respect to the q ’s gives

$$\Delta_\gamma q^a + \Gamma_{bc}^a \partial_i q^b \partial_j q^c \gamma^{ij} = 0, \tag{2.8}$$

and varying with respect to the γ ’s gives

$$g_{ab} (\partial_i q^a \partial_j q^b - \frac{1}{2} \gamma_{ij} \gamma^{mn} \partial_m q^a \partial_n q^b) = 0. \tag{2.9}$$

In the present case, we set $\gamma = \gamma_{ij} d\sigma^i \otimes d\sigma^j = m(du \otimes dv + dv \otimes du)$ for the two-metric and these equations become

$$\frac{\partial^2 q^a}{\partial u \partial v} + \Gamma_{bc}^a \frac{\partial q^b}{\partial u} \frac{\partial q^c}{\partial v} = 0 \quad (2.10)$$

and

$$g_{ab} \partial_u q^a \partial_u q^b = g_{ab} \partial_v q^a \partial_v q^b = 0. \quad (2.11)$$

The previously introduced coordinate system (u, v) on M_2 is actually given by

$$p^a \equiv \frac{\partial q^a}{\partial u} \quad \text{and} \quad \xi^a \equiv \frac{\partial q^a}{\partial v}, \quad (2.12)$$

and Eq. (2.10) is thus equivalent to $\nabla_\xi p^a = \nabla_p \xi^a = 0$, a system which is clearly satisfied by our string equations of motion (2.6)—the second equation being automatic since $\nabla \xi \equiv 0$ on a Bargmann manifold. On the other hand, the constraint (2.4) and the isotropy condition $g(\xi, \xi) = 0$ become just the two equations in (2.11).

C. Galilean electromagnetisms

We conclude this section with a novel viewpoint on the two distinct theories of Galilean electromagnetism.

1. The magneticlike theory

Consider firstly a 2-form $\mathcal{F} = \frac{1}{2} \mathcal{F}_{ab} dq^a \wedge dq^b$ on the extended (Bargmann) spacetime which satisfies the 5-dimensional Maxwell equations $(a, b, c = 1, \dots, 5)$

$$\partial_{[a} \mathcal{F}_{bc]} = 0, \quad (2.13)$$

$$\nabla^a \mathcal{F}_{ab} = \mathcal{J}_b. \quad (2.14)$$

(Square brackets denote skew symmetrization.)

In order to get a well behaved 4-dimensional theory, we require furthermore that

$$\mathcal{F}_{ab} \xi^a = 0. \quad (2.15)$$

Now, \mathcal{F} being closed, this last condition entails that $L_\xi \mathcal{F} = 0$ and, hence, that \mathcal{F} is the pullback of a two-form $F = \frac{1}{2} F_{\alpha\beta} dq^\alpha \wedge dq^\beta$ on spacetime B . We clearly still have $dF = 0$, i.e., the first group of Maxwell's equations,

$$\partial_{[\alpha} F_{\beta\gamma]} = 0, \quad (2.16)$$

where $\alpha, \beta, \gamma = 1, \dots, 4$.

We then reduce the second group (2.14). Since ξ is covariantly constant, Eq. (2.15) readily implies that the five-current \mathcal{J} has no “fifth component” (i.e., $\mathcal{J}_a \xi^a = 0$), or, componentwise $(\mathcal{J}_a) = ((J_a), 0)$. The only non-vanishing Christoffel symbols of $\nabla^{(Q)}$ are $\Gamma_{\beta\gamma}^\alpha$ and $\Gamma_{\alpha\beta}^5$ in an adapted coordinate system. The explicit form of $\Gamma_{\alpha\beta}^5$ is given by Eq. (3.25) in Ref. 3 and will not be needed here. The $\Gamma_{\beta\gamma}^\alpha$'s turn out to be nothing but the components of the Newtonian connection $\nabla^{(B)}$ on spacetime. The 5D equation (2.14) reduces therefore to

$$h^{\alpha\beta} \nabla_\alpha F_{\beta\gamma} = J_\gamma, \quad (2.17)$$

where $(h^{\alpha\beta})$ is the 4-dimensional Galilei “metric.” Had we worked in a truly relativistic spacetime $(B, (g^{\alpha\beta}))$ by considering, from the outset, a spacelike fibration^{3,5} $g(\xi, \xi) = c^{-2}$, Eq. (2.17) would have yielded the second group of the Maxwell equations. However, our “metric” h is degenerate; as a result *the displacement current is lost* here. For example, in flat spacetime $(\mathbf{R}^4, (h^{\alpha\beta}) = \text{diag}(1, 1, 1, 0), (\theta_\alpha) = (0001))$, Eqs. (2.16) and (2.17) reduce to

$$\text{div } \mathbf{B} = 0, \quad \text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \quad (2.18)$$

$$\text{div } \mathbf{E} = \rho, \quad \text{curl } \mathbf{B} = \mathbf{j}. \quad (2.19)$$

We have put $E^A \equiv F_{A4}$, $B^C \equiv \frac{1}{2} \epsilon^{ABC} F_{AB}$ (with $A, B, C = 1, 2, 3$) to define the Galilean electromagnetic field (\mathbf{E}, \mathbf{B}) and $j^A \equiv J_A$, $\rho \equiv J_4$ to define the three-current density \mathbf{j} and the charge density ρ .

In the curved case, however, Eqs. (2.16) and (2.17) retain the same form as in the flat case, except for a possible modification of Gauss's law,

$$\text{div } \mathbf{E} + \langle \mathbf{H}, \mathbf{B} \rangle = \rho, \quad (2.20)$$

that couples the electromagnetic field to the “Coriolis” components $H^C \equiv \epsilon_A^{BC} \Gamma_{B4}^A$ [see (7.16)] of the gravitational field corresponding to certain solutions of Newton-Cartan field equations, such as the nonrelativistic Taub-Newman-Unti-Tamburino- (NUT-) type solutions given by the metric (7.4). This modification is very reminiscent of the Chern-Simons modification of Maxwell's equations.¹⁸ This point will not be pursued here.

This is the so-called “magnetic limit” of Maxwell's equations.

2. The electriclike theory

Interestingly enough, the 5-dimensional Maxwell equations admit another subtle “electric limit.”^{16,31} We could have started with the *contravariant* form of the 5-dimensional Maxwell equations, i.e., with

$$\partial^{[a} \mathcal{F}^{bc]} = 0 \quad \text{and} \quad \nabla_a \mathcal{F}^{ab} = \mathcal{J}^b,$$

which is obviously equivalent to the covariant form (2.13) and (2.14). By requiring now the mere invariance condition $L_\xi \mathcal{F} = 0$, one can push down the two-vector $\mathcal{F}^{ab} \partial_a \otimes \partial_b$ we still denote \mathcal{F} and get a two-vector $\tilde{F} = \pi_* \mathcal{F}$ on spacetime B that satisfies $(\alpha, \beta, \gamma, \dots = 1, \dots, 4)$

$$\nabla_\alpha \tilde{F}^{\alpha\beta} = \tilde{J}^\beta, \quad (2.21)$$

where $\tilde{J} = \pi_* \mathcal{J}$ and

$$h^{\lambda[\alpha} \partial_\lambda \tilde{F}^{\beta\gamma]} = 0, \quad (2.22)$$

that is, the *full second group* of Maxwell's equations with the displacement current included, which retains the familiar form in flat spacetime,

$$\operatorname{div} \tilde{\mathbf{E}} = \tilde{\rho}, \quad \operatorname{curl} \mathbf{B} - \frac{\partial \tilde{\mathbf{E}}}{\partial t} = \mathbf{j}, \quad (2.23)$$

together with a *truncated first group*, i.e., with the induction term missing

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \tilde{\mathbf{E}} = 0. \quad (2.24)$$

Notice that we have $\tilde{\mathbf{B}} = \mathbf{B}$ if we again define the “electriclike” electromagnetic field $(\tilde{\mathbf{E}}, \tilde{\mathbf{B}})$ by $\tilde{E}^A \equiv \tilde{F}^{A4}$, $\tilde{B}^C \equiv \frac{1}{2} \epsilon_{ABC} \tilde{F}^{AB}$ and the current-charge density by $\tilde{j}^A \equiv \tilde{J}^A = j^A$, $\tilde{\rho} \equiv \tilde{J}^4$.

In addition, there are extra fields $E^A \equiv \mathcal{F}^{A5}$, $S \equiv \mathcal{F}^{45}$ that push forward to zero. One easily finds that they satisfy in the flat case $\operatorname{grad} S + \partial \tilde{\mathbf{E}} / \partial t = \mathbf{0}$, $\operatorname{curl} \mathbf{E} + \partial \mathbf{B} / \partial t = \mathbf{0}$ and $\operatorname{div} \mathbf{E} + \partial S / \partial t = \rho$ where $\rho \equiv \mathcal{J}^5$. In contrast with the more familiar Kaluza-Klein case when the fibration is spacelike, one cannot, in our case where the fibration is null, associate in an intrinsic fashion these extra components to fields on spacetime.

III. A TIME-VARYING “CONSTANT” OF GRAVITATION

We now apply our general framework to cope with Dirac’s theory in which the gravitational “constant” varies with the time.

On a Bargmann manifold (Q, g, ξ) Newton’s field equations for gravity can be cast into the simple geometrical form³

$$R_{ab} = 4\pi G \varrho \xi_a \xi_b, \quad (3.1)$$

where ϱ is the mass density of the sources. The spacetime projection of these equations yields the so-called Newton-Cartan field equations.²⁸ Those reduce, in the particular case of a Bargmann manifold (Q, g_U, ξ) given by (2.1) and (2.2) with $Q \subset \mathbf{R}^5$, to the *Poisson equation*

$$\Delta U = 4\pi G \varrho.$$

Since the gravitational “constant” G can be consis-

tently assumed to depend on time only,

$$U(\mathbf{q}, t) \equiv -G(t) \frac{m_0}{\|\mathbf{q}\|} \quad \text{with } G_0 \equiv G(t_0) \quad (3.2)$$

plainly defines an $\text{SO}(3)$ -invariant vacuum solution of (3.1) on $Q = (\mathbf{R}^3 \setminus \{0\} \times \mathbf{R}) \times \mathbf{R}$.

In contrast with Einstein’s theory in which the constant of gravitation must be independent of time, unless one augments the theory by adding an extra scalar field,^{32,33} a time-dependent gravitational “constant” is quite natural in the Newtonian context. In fact, even if one is interested in the usual case where G is constant with time, our results on time-dependent $G(t)$ are directly applicable to the motion of particles in an expanding universe [Eq. (1.4) using the identification (1.5)].

Suppose now that we can find a (local) diffeomorphism \mathcal{D} of Q with the metric g_U given by (2.1) and (3.2) and the vertical vector ξ given by (2.2) such that

$$\mathcal{D}^* g_U = \Omega^2 g_{U_0} \quad (3.3)$$

for some strictly positive function Ω , and

$$\mathcal{D}_* \xi = \xi, \quad (3.4)$$

where $U_0(\mathbf{q}) = U(\mathbf{q}, t_0)$. The metrics $\mathcal{D}^* g_U$ and g_{U_0} clearly have the same null geodesics and, hence, the same world lines in spacetime. Accordingly, up to a change of coordinates, the equations of motion governed by $G(t)$ are the same as those corresponding to G_0 . Let us show that this happens for very special functions $G(t)$.

We seek the conformal equivalence (3.3) by putting $(\mathbf{q}^*, t^*, s^*) \equiv \mathcal{D}(\mathbf{q}, t, s)$ as shorthand. Being a bundle automorphism (3.4), \mathcal{D} defines a spacetime local diffeomorphism $(\mathbf{q}, t) \rightarrow (\mathbf{q}^*, t^*)$ which, in turn, defines a local diffeomorphism $t \rightarrow t^*$ of the time axis. Owing to the fundamental $\text{SO}(3)$ symmetry of the problem, we confine our considerations to $\text{SO}(3)$ -equivariant diffeomorphisms, i.e., $\mathbf{q}^* = \Omega(r, t) \mathbf{q}$ where $r \equiv \|\mathbf{q}\|$. But, as previously noticed (2.3), Ω cannot depend on r ; after some calculation we get $t^* = \int \Omega(t)^2 dt$ and

$$\mathcal{D}^* g_U = \Omega(t)^2 \left(\langle d\mathbf{q} \otimes d\mathbf{q} \rangle + dt \otimes ds + ds \otimes dt + 2 \frac{G_0 m_0}{r} dt \otimes dt \right) + \Omega(t)^2 (dt \otimes \psi + \psi \otimes dt),$$

with

$$\psi = ds^* - ds + \alpha(t) r dr + \frac{1}{2} \alpha(t)^2 r^2 dt + 2 \frac{\beta(t) m_0}{r} dt,$$

where $\alpha(t) \equiv \dot{\Omega}(t) / \Omega(t)$ and $\beta(t) \equiv G(t^*) \Omega(t) - G_0$. Now Eq. (3.3) will be satisfied provided ψ vanishes. But this requires $d\psi = (\dot{\alpha} - \alpha^2 + 2\beta/r^3) dt \wedge dr = 0$, that is $\dot{\alpha} = \alpha^2$ and $\beta = 0$. Hence,

$$G(t^*) = \frac{G_0}{\Omega(t)} \quad \text{and} \quad \Omega(t) = \frac{a}{t+b},$$

with $a, b = \text{const.}$ We thus obtain $s^* = s - \frac{1}{2} \alpha(t) r^2 + d$

with $d = \text{const.}$ Condition (3.4) then holds automatically because

$$\frac{\partial}{\partial s^*} = \frac{\partial}{\partial s}.$$

Thus $t^* = -a^2/(t+b) + c$ with $c = \text{const.}$ Let us now summarize.

The solution g_U of Newton’s field equation associated with

$$U(\mathbf{q}, t) = -G(t) \frac{m_0}{\|\mathbf{q}\|}$$

can be $SO(3)$ -equivariantly brought into the conformal class of g_{U_0} where

$$U_0(\mathbf{q}, t) = -G_0 \frac{m_0}{\|\mathbf{q}\|},$$

provided

$$G(t) = G_0 \frac{a}{t-c}. \quad (3.5)$$

The local diffeomorphisms \mathcal{D} which commute with the rotations and such that $\mathcal{D}^*g_U = \Omega^2 g_{U_0}$ and $\mathcal{D}_*\xi = \xi$ will be called “Vinti-Lynden-Bell transformations.” They are given by

$$\begin{aligned} \mathbf{q}^* &= \frac{a}{t+b} \mathbf{q}, \\ t^* &= -\frac{a^2}{t+b} + c, \\ s^* &= s + \frac{\mathbf{q}^2}{2(t+b)} + d, \end{aligned} \quad (3.6)$$

$a \neq 0, b, c$, and d being arbitrary real constants. The conformal factor is finally

$$\Omega(t) = \frac{a}{t+b}.$$

This result is consistent with that of Ref. 14. Dirac’s original suggestion (1.1) for $G(t)$ corresponds to choosing $a = t_0, b = c = 0$ in (3.6). Vinti actually proposed the slight modification (3.5) to Dirac’s prescription in order “to avoid infinities in the resulting exact classical solutions for the orbit in the two-body problem.” Here, we get Vinti’s formula as the general solution of the problem of conformal equivalence for Newtonian gravity between a constant and a time-varying “constant” of gravitation.

Notice that the change (3.6) in the extra coordinate s is precisely the change of the action as given in (1.9) and that the spacetime part of (3.7) can actually be obtained by having the 5×5 matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c/a & bc/a - a \\ 0 & 1/a & b/a \end{pmatrix} \quad (3.7)$$

act projectively on the affine space of the five-vectors

$$\begin{pmatrix} \mathbf{q} \\ t \\ 1 \end{pmatrix}$$

representing spacetime events. We record for further purposes that the matrices (3.7) form an open subset of $SL(2, \mathbf{R})$.

IV. BARGMANN CONFORMAL SYMMETRIES

In this section we discuss the general notion of conformal symmetries of the 5-dimensional Bargmann spacetime. It will be shown in Sec. VIII that these symmetries are actually specific to the Schrödinger equation.

It was recognized by Niederer¹⁰ and Hagen¹¹ in the

early 1970’s that the maximal kinematical symmetry group of the free Schrödinger equation is larger than the mere Bargmann group, i.e., the 11-dimensional extended Galilei group. This group is 13-dimensional and has been called the “extended Schrödinger group.” In our formalism, it simply consists of those conformal transformations \mathcal{C} of the canonical flat Bargmann structure (\mathbf{R}^5, g_0, ξ) , the extended Galilei spacetime, that commute with the structural group, i.e., such that

$$\mathcal{C}^*g_0 = \Omega^2 g_0 \quad \text{and} \quad \mathcal{C}_*\xi = \xi, \quad (4.1)$$

with

$$g_0 = \langle d\mathbf{q} \otimes d\mathbf{q} \rangle + dt \otimes ds + ds \otimes dt \quad \text{and} \quad \xi = \frac{\partial}{\partial s}. \quad (4.2)$$

See Refs. 13 and 9 for a more detailed account. It is amusing to note that the conformal transformations were already used in five dimensions to study the parabolic diffusion equation at the beginning of the century.³⁴

Let us first determine those conformal transformations of (\mathbf{R}^5, g_0, ξ) which simply project down to the base B as spacetime transformations, i.e., which preserve the vertical *direction*. Infinitesimally, this amounts to finding all vector fields X such that

$$L_X g_0 = 2\lambda g_0 \quad \text{and} \quad L_X \xi = \mu \xi, \quad (4.3)$$

for some functions λ and μ . The solutions of this system form a 14-dimensional Lie algebra for the Lie bracket (the so-called “chronoprojective Lie algebra”) and are given by

$$\begin{aligned} \mathbf{X} &= \boldsymbol{\alpha} \times \mathbf{q} + (\chi + \kappa t)\mathbf{q} + \beta t + \boldsymbol{\gamma}, \\ X^t &= \kappa t^2 + \delta t + \epsilon, \\ X^s &= -[\frac{1}{2}\kappa \mathbf{q}^2 + \langle \boldsymbol{\beta}, \mathbf{q} \rangle + \eta + (\delta - 2\chi)s], \end{aligned} \quad (4.4)$$

with $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbf{R}^3; \chi, \kappa, \delta, \epsilon, \eta \in \mathbf{R}$. This yields $\lambda = \chi + \kappa t$ and $\mu = \delta - 2\chi$, and the subalgebra of conformal bundle automorphisms ($\mu = 0$) is thus characterized by

$$\chi = \frac{\delta}{2}. \quad (4.5)$$

Integrating this Lie algebra leaves us with a 13-dimensional Lie group, the (neutral component of the) so-called *extended Schrödinger group* “acting” on the extended spacetime according to

$$\begin{aligned} \mathbf{q}^* &= \frac{A\mathbf{q} + \mathbf{b}t + \mathbf{c}}{ft + g}, \\ t^* &= \frac{dt + e}{ft + g}, \end{aligned} \quad (4.6)$$

$$s^* = s + \frac{f(A\mathbf{q} + \mathbf{b}t + \mathbf{c})^2}{2(ft + g)} - \langle \mathbf{b}, A\mathbf{q} \rangle - \frac{t}{2}\mathbf{b}^2 + h,$$

where $A \in SO(3); \mathbf{b}, \mathbf{c} \in \mathbf{R}^3; d, e, f, g, h \in \mathbf{R}$ and $dg - ef = 1$. The corresponding conformal factor in (4.1) is therefore

$$\Omega = \frac{1}{ft + g}.$$

It is now possible to give other nonrelativistic “symmetry” groups (listed below) a neat geometrical interpretation associated with the flat Bargmann structure.

(i) The 11-dimensional subgroup defined by $d = g = 1$, $f = 0$ in (4.6) is the (neutral component of the) *Bargmann group* which consists of those ξ -preserving *isometries* of the extended spacetime.

(ii) The *Galilei group* is recovered as the 10-dimensional quotient of the Bargmann group by its center, $(\mathbf{R}, +)$, parametrized by h [see (4.6)]. The action of the restricted Galilei group on spacetime is given by the first two equations in (4.6) with $d = g = 1$, $f = 0$; it corresponds to the projection onto spacetime \mathbf{R}^4 of the Bargmann group action on \mathbf{R}^5 , the extended spacetime.

(iii) Again, factoring the extended Schrödinger group (4.6) by its center, $(\mathbf{R}, +)$, yields the 12-dimensional *Schrödinger group*, originally discovered as the “maximal invariance group of the free Schrödinger equation.”¹⁰ The Schrödinger group is thus isomorphic to $(\mathrm{SO}(3) \times \mathrm{SL}(2, \mathbf{R})) \otimes (\mathbf{R}^3 \times \mathbf{R}^3)$, i.e., to the multiplicative group of those 5×5 matrices

$$\begin{pmatrix} A & \mathbf{b} & c \\ \mathbf{0} & d & e \\ \mathbf{0} & f & g \end{pmatrix} \quad (4.7)$$

with entries as above.

(iv) The 14-dimensional group of conformal automorphisms of the flat Bargmann structure, the “chronoprojective group”⁹ that preserves the *directions* of g and ξ separately [see (4.3),(4.4)], can be thought of as a preferred Lie subgroup of $\mathrm{O}(5,2)$; its commutator subgroup turns out to be the extended Schrödinger group (4.6).

(v) Finally, the three-parameter subgroup ($A = 1$, $\mathbf{b} = \mathbf{c} = \mathbf{0}$) of the Schrödinger group (4.7) is the group of *non-relativistic conformal transformations* (1.10)–(1.12) isomorphic to $\mathrm{SL}(2, \mathbf{R})$ interpreted as the group of projective transformations of the time axis. The dilatations and expansions introduced in Sec. I form the triangular Borel subgroup ($A = 1$, $\mathbf{b} = \mathbf{c} = \mathbf{0}$, $e = 0$).

Remark 1. Comparing with (3.7) we conclude that, for each value of the parameters a, b, c , the spacetime projection of the Vinti–Lynden-Bell transformation (3.6) belongs to the $\mathrm{SL}(2, \mathbf{R})$ subgroup. This can also be understood by observing that for an element \mathcal{C} of the extended Schrödinger group, one has $\mathcal{C}^*g_U = \Omega^2g_0 - 2\mathcal{C}^*(Udt \otimes dt)$, see (2.1) and (4.1). Thus, Eq. (3.3) is satisfied as soon as

$$\mathcal{C}^*(Udt \otimes dt) = \Omega^2U_0 dt \otimes dt. \quad (4.8)$$

A closer inspection shows that this corresponds to the calculation of Sec. III.

Remark 2. In the same spirit, one can ask which potentials U are conformally related to the free case, namely

$$\mathcal{A}^*g_U = g_0 \quad \text{and} \quad \mathcal{A}_*\xi = \xi. \quad (4.9)$$

A short calculation yields¹³

$$U(\mathbf{q}, t) = \frac{1}{2}u(t)\mathbf{q}^2 + \langle \mathbf{v}(t), \mathbf{q} \rangle + w(t), \quad (4.10)$$

where u, \mathbf{v}, w are arbitrary functions of time. Physically, we can have a time-dependent spherically symmetric *harmonic oscillator* plus a *homogenous force*. This explains why the classical symmetries of these systems are related to those of the free particle. This fact has been exploited in the quantum-mechanical framework³⁵ to conformally relate the general solutions of the free Schrödinger equation to those of the Schrödinger equation in the presence of a potential of the form (4.10), e.g., a (time-dependent) harmonic oscillator.

Remark 3. One could also ask which central potentials U are invariant with respect to nonrelativistic conformal transformations given by Eqs. (1.10)–(1.12). Because of (4.8), this requires

$$U(\mathbf{q}, t) = \frac{\text{const}}{\mathbf{q}^2}. \quad (4.11)$$

A generalization of this result to the case of N bodies has been obtained in Ref. 36.

Finally, adding a Dirac magnetic monopole would only change the symplectic structure introduced in Sec. II by a term proportional to the area two-form of S^2 , which is manifestly invariant with respect to our nonrelativistic conformal transformations.

The most general conformally invariant system is thus an *inverse square potential*³⁷ plus a *Dirac monopole*.^{26,38,39}

V. THE N -BODY PROBLEM

In nonrelativistic physics, it is consistent to confine attention to a finite number N of bodies moving in Euclidean space \mathbf{R}^3 . An equivalent description is to give a curve in the configuration space \mathbf{R}^{3N} . To obtain a spacetime description, we may then add one extra absolute time variable to obtain a Newtonian spacetime of dimension $3N + 1$. The motion of the bodies corresponds to a world line in this N -body spacetime.

A. The N -body Bargmann structure

The metric of this $(3N + 2)$ -dimensional Bargmann structure is

$$g_V \equiv \sum_{j=1}^N \frac{m_j}{m} \langle d\mathbf{q}_j \otimes d\mathbf{q}_j \rangle + dt \otimes ds + ds \otimes dt - \frac{2}{m} V(\mathbf{q}_1, \dots, \mathbf{q}_N, t) dt \otimes dt, \quad (5.1)$$

where m_1, \dots, m_N are the masses of the bodies and $m = m_1 + \dots + m_N$ is the total mass of the system. As before

$$\xi \equiv \frac{\partial}{\partial s} \quad (5.2)$$

is the $(\mathbf{R}, +)$ generator that defines the principal null,

covariant-constant fibration.

For the planetary N -body problem of celestial mechanics, we take

$$V(\mathbf{q}_1, \dots, \mathbf{q}_N, t) = - \sum_{j < k} G \frac{m_j m_k}{\|\mathbf{q}_j - \mathbf{q}_k\|}, \quad (5.3)$$

and thus define the metric in $Q = (\mathbf{R}^{3N} \setminus \Delta) \times \mathbf{R} \times \mathbf{R}$ where Δ is the collision subset.

Note that the potential V in (5.3) consistently leads to a Ricci-flat metric g_V given by (5.1), i.e., a solution of the vacuum field equations (3.1).

The nonzero Christoffel symbols are

$$\Gamma_{tt}^{A_j} = (1/m_j) \partial_{A_j} V, \quad \Gamma_{A_j t}^s = -(1/m) \partial_{A_j} V$$

and $\Gamma_{tt}^s = -(1/m) \partial_t V$ where we have put $\mathbf{q}^{A_j} \equiv \mathbf{q}_j^A$ with $A = 1, 2, 3$, and $j = 1, \dots, N$. The equations of the null geodesics are readily interpreted as Newton's equations of motion, viz.,

$$\frac{d^2 \mathbf{q}^{A_j}}{dt^2} = - \frac{1}{m_j} \partial_{A_j} V, \quad (5.4)$$

together with a supplementary equation for the action

$$\frac{d^2 s}{dt^2} = \frac{1}{m} \left(2 \frac{dV}{dt} - \frac{\partial V}{\partial t} \right). \quad (5.5)$$

Note that the metric g_V is indeed conformally defined by its null geodesics, i.e., the solutions of the N -body equations of motions. In other terms, the "shape" of the extended spacetime is defined, up to a factor, by the motions of matter in the universe.

Regarding the time-variation of the "constant" of gravitation, our previous arguments for the external Newtonian field apply here just as well. We can therefore claim that the only gravitational "constant" $G(t)$ in (5.3) that can be $\text{SO}(3)$ -equivariantly associated with G_0 is again given by Vinti's formula (3.5).

We remark in passing that, had we been considering the analogous problem in electrostatics with a Coulomb's potential

$$V(\mathbf{q}_1, \dots, \mathbf{q}_N, t) = \sum_{j < k} \frac{e_j e_k}{\|\mathbf{q}_j - \mathbf{q}_k\|}$$

for a system of charges e_1, \dots, e_N , we would have found exactly the same possible time dependence for the fine structure constant

$$e^2 \propto \frac{1}{t}. \quad (5.6)$$

B. Homographic solutions

The only known general class of nonplanar exact solutions of the N -body problem are the so-called homographic solutions. Their existence is related to the conformal structure of the Bargmann manifold. These particular solutions have the form

$$\mathbf{q}_j(t) = \Omega(t) \mathbf{q}_j^0. \quad (5.7)$$

Substituting this ansatz into the equations of motion (5.4), where V is given by (5.3) with $G = \text{const}$, yields

$$\mathbf{q}_j^0 = \frac{1}{\lambda} \sum_{k \neq j} G m_k \frac{\mathbf{q}_j^0 - \mathbf{q}_k^0}{\|\mathbf{q}_j^0 - \mathbf{q}_k^0\|^3} \quad \text{with } \lambda = -\Omega^2 \ddot{\Omega}. \quad (5.8)$$

The solutions $\mathbf{q}^0 = (\mathbf{q}_1^0, \dots, \mathbf{q}_N^0)$ of these equations (with $\lambda = \text{const} > 0$) are called *central configurations* and provide by (5.7) some exact solutions of the N -body problem that, up to now, have not been completely classified (see Ref. 40 for an account of recent progress in the classification of the nonplanar central configurations with equal masses).

To gain some further insight, let us first observe that if we set $dt^0 = dt/\Omega(t)^2$ and, of course, $\mathbf{q}_j^0 = \mathbf{q}_j/\Omega(t)$, then

$$\left(\frac{d\mathbf{q}_j}{dt} \right)^2 dt = \left[\left(\frac{d\mathbf{q}_j^0}{dt^0} \right)^2 + \lambda \Omega \mathbf{q}_j^{0^2} \right] dt^0 + d(\Omega \dot{\Omega} \mathbf{q}_j^{0^2})$$

and

$$V(\mathbf{q}) dt = \Omega V(\mathbf{q}^0) dt^0.$$

Thus, up to a total derivative, we get the Lagrangian of N particles interacting with a combined repulsive oscillator and Newtonian gravitational field (with time-dependent coefficients). But this latter system admits a static equilibrium configuration, namely when the gravitational attraction is canceled by the linear repulsion.

Again, this can be rephrased geometrically in terms of conformal transformations of the N -body Bargmann manifold (Q, g_V^0, ξ^0) with $Q = (\mathbf{R}^{3N} \setminus \Delta) \times \mathbf{R} \times \mathbf{R}$ and $g_V^0 = \sum_{j=1}^N (m_j/m) (d\mathbf{q}_j^0 \otimes d\mathbf{q}_j^0) + dt^0 \otimes ds^0 + ds^0 \otimes dt^0 - (2/m)V(\mathbf{q}^0) dt^0 \otimes dt^0$ and $\xi^0 = \partial/\partial s^0$ with V given by (5.3). A simple calculation, akin to the previous remark, shows that the mapping $\mathcal{A} : (\mathbf{q}^0, t^0, s^0) \rightarrow (\mathbf{q}, t, s)$ of Q , whose inverse is given by

$$\begin{aligned} \mathbf{q}^0 &= \frac{\mathbf{q}}{\Omega(t)}, \\ t^0 &= \int \frac{dt}{\Omega(t)^2}, \\ s^0 &= s + \frac{\dot{\Omega}}{2m\Omega} \sum_{j=1}^N m_j \mathbf{q}_j^2, \end{aligned} \quad (5.9)$$

transforms the original Bargmann structure according to

$$g_V \equiv (\mathcal{A}^{-1})^* g_V^0 = \Omega^2 g_{V_{\text{eff}}}^0 \quad \text{and} \quad \xi \equiv \mathcal{A}_* \xi^0 = \xi^0, \quad (5.10)$$

where

$$V_{\text{eff}}(\mathbf{q}^0, t^0) = \Omega \left(V(\mathbf{q}^0) + \frac{1}{2} \Omega^2 \dot{\Omega} \sum_{j=1}^N m_j \mathbf{q}_j^{0^2} \right). \quad (5.11)$$

The critical points \mathbf{q}^0 of V_{eff} are the static equilibria (or the central configurations), i.e., the solutions of $\text{grad}_j V_{\text{eff}}(\mathbf{q}^0, t^0) = 0$ (implying $\lambda = -\Omega^2 \dot{\Omega} = \text{const}$). These actually define some specific null geodesics of

$g_{V_{\text{eff}}}^0$, which, according to Eq. (5.10), happens to be conformally related to g_V . We again note (5.10) that the total mass is preserved by the transformation \mathcal{A} . Central configurations are indeed associated with null geodesics of g_V , hence to some particular solutions $[t \rightarrow (\mathbf{q}_1(t), \dots, \mathbf{q}_N(t))]$ of the original set (5.4) of Newton's equations.

Remark. A similar explanation can be given to the observation of Forgács and Zakrzewski,⁴¹ who found that the action $\int f(t)\dot{y}^2(t) dt$ can be brought into that of a free particle by the change of variable $t \rightarrow \int dt/f(t)$.

C. The (cosmic) virial theorem

We conclude this section with a remark about scaling and the *virial theorem*.

Consider the Newtonian N -body problem described by the metric g_V in (5.1) with V as in (5.3) and $G = \text{const}$. This Bargmann structure admits a 5-dimensional Lie algebra of fiber-preserving conformal Killing vectors. This algebra consists of four isometries (rotations and time translations) and of the homothetic-Killing vector field

$$X = \sum_{j=1}^N \mathbf{q}_j \cdot \frac{\partial}{\partial \mathbf{q}_j} + \frac{3}{2} t \frac{\partial}{\partial t} + \frac{1}{2} s \frac{\partial}{\partial s}. \quad (5.12)$$

This latter generates the *homothety group* ($\Lambda > 0$)

$$\begin{aligned} \mathbf{q}_j &\rightarrow \mathbf{q}_j^* = \Lambda \mathbf{q}_j, \\ t &\rightarrow t^* = \Lambda^{3/2} t, \\ s &\rightarrow s^* = \Lambda^{1/2} s, \end{aligned} \quad (5.13)$$

with $j = 1, \dots, N$, under which $g_V \rightarrow \Lambda^2 g_V$ and $\xi \rightarrow \Lambda^{-1/2} \xi$. Thus

$$L_X g_V = 2 g_V \quad \text{and} \quad L_X \xi = -\frac{1}{2} \xi. \quad (5.14)$$

Such homotheties lift to the cotangent bundle (T^*Q, ϑ) as canonical symplectic similitudes ($\vartheta^* = \Lambda^{1/2} \vartheta$), namely as

$$\begin{aligned} p_{A_j}^* &= \Lambda^{-1/2} p_{A_j}, \\ p_t^* &= \Lambda^{-1} p_t, \\ p_s^* &= p_s, \end{aligned} \quad (5.15)$$

where $A_j = 1, 2, 3$, and $j = 1, \dots, N$. These homotheties (5.15) preserve therefore the $(6N + 2)$ -dimensional submanifold C defined by the constraints (2.4) and (2.5) and the null foliation $\ker(d\vartheta_C)$. Hence, they permute the classical motions. This yields a *generalized Kepler third law*: if $[t \rightarrow \mathbf{q}(t)]$ is a solution of Newton's equations for N bodies then so is $[t \rightarrow \Lambda \mathbf{q}(\Lambda^{-3/2} t)]$.

As a by-product, we get the *virial theorem* used by astrophysicists to estimate the mass of clusters of galaxies.²⁰ We have seen in fact that if \bar{X} denotes the canonical lift of the vector field X to T^*Q , then

$$L_{\bar{X}} \vartheta = \frac{1}{2} \vartheta, \quad (5.16)$$

and thus if $Y \in \ker(d\vartheta_C)$ is a generator of the equations of motion (2.6), we have $Y(\vartheta_C(\bar{X})) = \frac{1}{2} \vartheta_C(Y)$. Using (2.6), $-p_t = E$ (energy) and $p^i = m$ (total mass), this can be rewritten as $Y(\sum p_{A_j} q^{A_j}) - \frac{3}{2} E m \alpha = \frac{1}{2} \alpha (\sum p_{A_j} p^{A_j} - E m)$. Introducing the kinetic energy $T = \sum p_{A_j} p^{A_j} / (2m)$ we get

$$2T - V = \frac{1}{m\alpha} Y \left(\sum p_{A_j} q^{A_j} \right). \quad (5.17)$$

If we assume that the system is in equilibrium then the average of the right-hand side of (5.17) vanishes and we are left with

$$2T = V \quad (\text{time average}).$$

All these results can be extended to include the case of a time-dependent gravitational “constant” $G(t)$. As we previously mentioned in Sec. III, this case can also be reinterpreted as giving the equations of motion for particles in an expanding universe [Eq. (1.4) if we use the interpretation (1.5)]. The virial theorem (5.17) then becomes the “cosmic virial theorem.”²⁰ If G is not constant with time, $\partial/\partial t$ will no longer be a Killing vector field of the metric g_V given in (5.1). In fact,

$$(L_{\partial/\partial t} g_V)_{ab} = -\frac{2}{m} \frac{\dot{G}(t)}{G(t)} V \xi_a \xi_b. \quad (5.18)$$

Using this equation, it is easy to obtain the “cosmic energy equation.”²⁰

VI. HIDDEN SYMMETRIES AND KILLING TENSORS

We now briefly describe the appearance of “hidden” symmetries.

In our formalism, a *manifest symmetry* belongs to the group of conformal automorphisms of the bundle (Q, g, ξ) , whose infinitesimal generators are the *conformal Killing vector fields* κ that commute with the $(\mathbf{R}, +)$ -generator, namely $(a, b = 1, \dots, 5)$,

$$\nabla_{(a} \kappa_{b)} = \lambda g_{ab} \quad \text{and} \quad [\xi, \kappa] = 0, \quad (6.1)$$

for some function λ on Q . (Parentheses denote symmetrization.)

The functions $H_\kappa = p_a \kappa^a$, linear in momentum on T^*Q , are constants of the motion of our test particle.

For example, in the case of the one-body problem with $G = G_0$, rotations, time-translations, and vertical translations act by isometries and the associated conserved quantities are respectively the *angular momentum* \mathbf{L} , the *energy* $E = -H_{\partial/\partial t}$, and the *mass* $m = H_\xi$.

Now, the so called “accidental” or “hidden” symmetries associated with the Lagrange-Laplace-Runge-Lenz vector can also be discussed in this new setup. Observe that the *quadratic quantities*

$$H_\kappa = \frac{1}{2} p_a p_b \kappa^{ab} \quad (6.2)$$

are conserved along null geodesics of (Q, g) whenever

$$\nabla_{(a}\kappa_{bc)} = \lambda_{(a}g_{bc)} \quad (6.3)$$

for a symmetric and trace-free tensor κ (see, e.g., Refs. 42 and 43). These objects are called *conformal Killing two-tensors*. Since we want to preserve the principal bundle structure on the extended spacetime (Q, ξ) , we will only deal with conformal-Killing tensors κ such that

$$L_{\xi} \kappa = 0, \quad (6.4)$$

i.e., projectable ones.

In the Kepler case, a lengthy calculation is needed to prove that the following expression is indeed a solution of the Killing equations (6.3) and (6.4):

$$\kappa^{ab} = \eta^{ab} - \frac{1}{5}\hat{\eta}g^{ab} \quad \text{with} \quad \hat{\eta} = \eta^{ab}g_{ab}, \quad (6.5)$$

where the nonvanishing contravariant components of η are given by

$$\eta^{AB} = \omega^A q^B + \omega^B q^A - \hat{\eta} \delta^{AB} \quad (6.6)$$

with $A, B, C = 1, 2, 3$ and

$$\eta^{45} = \eta^{54} = \hat{\eta} (= \omega_C q^C) \quad (6.7)$$

for some $\omega \in \mathbf{R}^3$.

It is finally easy to check, with the help of Eq. (6.2), that $H_{\kappa} = (\omega, \mathbf{A})$ where

$$\mathbf{A} = \mathbf{L} \times \mathbf{p} + m^2 m_0 G_0 \frac{\mathbf{q}}{r} \quad (6.8)$$

is the *Lagrange-Laplace-Runge-Lenz vector*. See Ref. 44 for an alternative treatment in the 4-dimensional setting.

VII. RELATION TO THE WORK OF BRINKMANN AND KALUZA-KLEIN THEORY

We now wish to point out the relation of our results to the work of Brinkmann,¹⁷ who discussed the circumstances under which two metrics g and g^* related by a conformal rescaling

$$g^* = \Omega^2 g$$

might both be Einstein type, i.e., both satisfy

$$R_{ab} = \Lambda g_{ab}.$$

He distinguished three cases:

- (A) $\Omega = \text{const}$,
- (B) $g^{ab} \partial_a \Omega \partial_b \Omega \neq 0$,
- (C) $g^{ab} \partial_a \Omega \partial_b \Omega = 0$, $\partial_a \Omega \neq 0$.

It is case (C), called “improper conformal rescalings,” that is relevant for us since our coordinate t and hence any function of it satisfies condition (C). Brinkmann included the possibility that g^* was the pull-back of g under a diffeomorphism so his results are directly applicable. He showed that in case (C), g and g^* must admit a covariantly constant null vector field. Thus, in particular, they must be Ricci flat. In other words he established that g and g^* must admit what we have referred to as a *Bargmann structure*. This implies a uniqueness property

of the Vinti-Lynden-Bell transformations.

Brinkmann then went on to determine explicitly all Einstein-Bargmann structures in 4 and 5 dimensions and to give a set of necessary and sufficient conditions in all higher dimensions.

A. Case $D = 4$

In 4 dimensions the most general Einstein-Bargmann structure is expressed as

$$g \equiv dq^1 \otimes dq^1 + dq^2 \otimes dq^2 + dt \otimes ds + ds \otimes dt - 2U(q^1, q^2, t) dt \otimes dt \quad (7.1)$$

with

$$\xi \equiv \frac{\partial}{\partial s} \quad (7.2)$$

and

$$\Delta U \equiv (\partial_1^2 + \partial_2^2) U = 0. \quad (7.3)$$

Metrics of this form are referred to as *plane-fronted gravitational waves with parallel rays* (or *pp waves*) in general relativity literature. The special cases when U is quadratic in the \mathbf{q} 's are called *exact plane gravitational waves*. They admit a five-parameter group of isometries acting on the null three-surfaces $t = \text{const}$.

B. Case $D = 5$

In 5 dimensions, the most general Einstein-Bargmann structure is, according to Brinkmann, of the form

$$g \equiv \langle d\mathbf{q} \otimes d\mathbf{q} \rangle + dt \otimes (ds + \mathbf{A} \cdot d\mathbf{q}) + (ds + \mathbf{A} \cdot d\mathbf{q}) \otimes dt + (-2U + \frac{1}{2}V^2) dt \otimes dt, \quad (7.4)$$

with ξ just as before; \mathbf{A}, U, V being functions of \mathbf{q} and t such that

$$\text{curl } \mathbf{A} = \text{grad } V \quad \text{and} \quad \Delta U = 0. \quad (7.5)$$

(i) If $V = 0$, we obtain the obvious generalization to five spacetime dimensions of plane-fronted waves.

(ii) If U is taken to be quadratic in the \mathbf{q} 's, the metric admits, in general, a seven-parameter group of isometries acting on the surfaces $t = \text{const}$ and, hence, preserving the fibration.

(iii) In the flat case, $U = V = 0, \mathbf{A} = \mathbf{0}$, the group of isometries that preserve the slices $t = \text{const}$ is larger; it may be viewed as the commutator subgroup of the Bargmann group called the *Carroll group*, isomorphic to the 10-dimensional Lie group of all 5×5 matrices

$$\begin{pmatrix} A & \mathbf{0} & c \\ -{}^t \mathbf{b} A & 1 & h \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix} \quad (7.6)$$

(with $A \in \text{SO}(3); \mathbf{b}, c \in \mathbf{R}^3; h \in \mathbf{R}$), acting on the “position-action” affine plane spanned by

$$\begin{pmatrix} \mathbf{q} \\ s \\ 1 \end{pmatrix}$$

as deduced from (4.6), where we have set $t = 0$ with $d = g = 1$ and $e = f = 0$. It is worth remembering that the Carroll group has been originally introduced by Lévy-Leblond as the contraction $c \rightarrow 0$ of the Poincaré group.⁴⁵

(iv) If we set $V = 0$ and U given by Eq. (3.2), we obtain the Bargmann structure associated with a single Newtonian point particle.

(v) If $V \neq 0$ we obtain something more, namely a generalization of Newtonian nonrelativistic theory which, although not envisaged in the usual elementary treatments, occurs in the formulation due to Cartan.^{27,28} In fact, Cartan’s version of the theory is not merely a reformulation but an extension since it allows a new phenomenon, the possibility of *magnetic mass*.⁴⁶ In general relativity, this possibility was recognized first in the Taub-NUT solutions. By taking a suitable limit as $c \rightarrow \infty$, Koppel⁴⁷ found a new solution of Newton-Cartan field equations on spacetime. Here, we interpret it as the “nonrelativistic Taub-NUT solution” of the vacuum field equations (3.1) corresponding to the Bargmann structure $((\mathbf{R}^3 \setminus \{0\} \times \mathbf{R}) \times \mathbf{R}, g, \xi)$ where g is given by (7.4) and

$$U(\mathbf{q}) = -G \frac{m_0}{\|\mathbf{q}\|}, \quad \mathbf{A} \cdot d\mathbf{q} = -\ell \cos \theta d\phi, \quad V(\mathbf{q}) = -\frac{\ell}{\|\mathbf{q}\|}. \tag{7.7}$$

The vector field ξ is still given by (7.2) and ℓ is a new constant homogeneous to an [action]/[mass]. To interpret V , we note that the equations of motion of a test particle read in this case

$$\frac{d^2 \mathbf{q}}{dt^2} = -\text{grad} \left(U - \frac{1}{4} V^2 \right) + \frac{d\mathbf{q}}{dt} \times \text{grad} V, \tag{7.8}$$

thus ℓ corresponds to a Newtonian magnetic mass monopole. The solution (7.7) is clearly singular at the origin $\mathbf{q} = \mathbf{0}$ for all nonzero values of the two parameters m_0 and ℓ .

It is worth mentioning that the gravitational “constant” G could actually depend on time. If G varies inversely as time while ℓ remains a constant, the Vinti-Lynden-Bell transformation (3.6) still brings the system into a time-independent form. This is so because both the monopole term coming from curl \mathbf{A} and the ℓ^2/r^2 term coming from V^2 are symmetric with respect to nonrelativistic conformal transformations. The metrics (7.4) and (7.7) fall short of having an extra “hidden” symmetry: if one had $U - V^2/2$ rather than $U - V^2/4$ in (7.8), the metric would admit a conformal Killing tensor⁴³ yielding a conserved Lagrange-Laplace-Runge-Lenz vector.

The one-form

$$\varpi \equiv ds - \ell \cos \theta d\phi \tag{7.9}$$

appearing in the metric (7.4) turns out to be directly related to the canonical connection α living on the Hopf circle bundle $S^3 \rightarrow S^2$, viz.,

$$\alpha \equiv \frac{\varpi}{2\ell},$$

provided s is taken to be periodic with period $4\pi\ell$. The associated parameters (θ, ϕ, ψ) , with $\psi \equiv s/(2\ell) \pmod{2\pi}$, are the Euler angles on S^3 whilst our Taub-NUT-like Bargmann structure (Q, g, ξ) is now globally defined by the $SO(2)$ bundle $Q \cong (S^3 \times \mathbf{R}^+) \times \mathbf{R} \rightarrow B \cong (S^2 \times \mathbf{R}^+) \times \mathbf{R}$ endowed with the metric

$$g = dr \otimes dr + r^2 g_{S^2} + 2\ell(\alpha \otimes dt + dt \otimes \alpha) + \left(2G \frac{m_0}{r} + \frac{\ell^2}{2r^2} \right) dt \otimes dt, \tag{7.10}$$

and the circle generator

$$\xi = \frac{1}{2\ell} \frac{\partial}{\partial \psi}. \tag{7.11}$$

However, in contrast with the relativistic Taub-NUT solution in which the relativistic time must be periodic, the Newtonian time variable need not be periodic in our case. The periodicity has an interesting quantum-mechanical consequence, which we will discuss in Sec. VIII.

Remarks. When U , V , and \mathbf{A} are independent of t , the metrics (7.4) and (7.5) admit an additional Killing vector field $\partial/\partial t$, which, if $-2U + \frac{1}{2}V^2 > 0$ will be space-like. This allows us to give the five-metric a conventional Kaluza-Klein interpretation. If $V = 0$, and

$$-2U(\mathbf{q}) = 1 + \sum_{j=1}^N \frac{X_j}{\|\mathbf{q} - \mathbf{q}_j\|}, \tag{7.12}$$

one obtains a metric representing N -point particles in equilibrium.⁴⁸ The masses, scalar charges, and electric charges are in the ratio $1 : \sqrt{3} : 2$, which implies that the gravitational and scalar attractions are exactly cancelled by the electrostatic repulsion. These metrics are in a certain sense dual to the multi-Taub-NUT metrics which have an interpretation as Kaluza-Klein monopoles.^{29,49} The case $N = 1$ may be obtained from the 4-dimensional Schwarzschild metric by boosting it in the fifth direction up to the speed of light.⁴⁸

In the case where $V \neq 0$, we obtain metrics whose Kaluza-Klein interpretation is that of particles with both electric and magnetic charges, i.e., *Kaluza-Klein dyons*.

C. Case $D \geq 5$

Let us complete our review of Einstein-Bargmann structures by giving Brinkmann’s result for spacetime dimensions D exceeding 5.

The metric takes the form $(A, B, \dots = 1, 2, \dots, D - 2)$:

$$g \equiv \hat{g}_{AB}(\mathbf{q}, t) dq^A \otimes dq^B + dt \otimes \varpi + \varpi \otimes dt + H(\mathbf{q}, t) dt \otimes dt, \tag{7.13}$$

where

$$\varpi \equiv ds + A_K(\mathbf{q}, t) dq^K$$

and

$$\begin{aligned} R_{AB} &= \widehat{R}_{AB} = 0, \\ R_{At} &= -\frac{1}{2}[\partial_A(\widehat{g}^{KL}\partial_t\widehat{g}_{KL}) + \widehat{\nabla}^K F_{KA}] = 0, \\ R_{tt} &= -\frac{1}{2}[\widehat{\Delta}H + \partial_t(\widehat{g}^{KL}\partial_t\widehat{g}_{KL}) \\ &\quad + \frac{1}{2}F^{KL}F_{LK} - 2\widehat{\nabla}^K\partial_t A_K] = 0. \end{aligned} \quad (7.14)$$

Here F is defined to be

$$F_{KL} \equiv \partial_K A_L - \partial_L A_K - \partial_t \widehat{g}_{KL}. \quad (7.15)$$

We display, for completeness, the associated nontrivial Christoffel symbols:

$$\begin{aligned} \Gamma_{BC}^A &= \widehat{\Gamma}_{BC}^A, \\ \Gamma_{Bt}^A &= -\frac{1}{2}\widehat{g}^{AK}F_{KB}, \\ \Gamma_{tt}^A &= \widehat{g}^{AK}(\partial_t A_K - \frac{1}{2}\partial_K H), \\ \Gamma_{BC}^s &= \partial_{(B}A_{C)} - A_K\widehat{\Gamma}_{BC}^K - \frac{1}{2}\partial_t\widehat{g}_{BC}, \\ \Gamma_{Bt}^s &= \frac{1}{2}\widehat{g}^{KL}F_{KBA_L} + \frac{1}{2}\partial_B H, \\ \Gamma_{tt}^s &= \frac{1}{2}\widehat{g}^{KL}A_K(\partial_L H - \partial_t A_L) + \frac{1}{2}\partial_t H. \end{aligned} \quad (7.16)$$

The metric (7.13) is likely to have a number of applications to Kaluza-Klein supergravity and superstring theory. We could, for instance, take \widehat{g} to be the metric on a Calabi-Yau space. We defer discussion of these possibilities to another time.

We would like to finish this section by giving a Taub-NUT like exact solution for the Newtonian N -body field equations. It actually generalizes the preceding solutions (7.10) and (7.11) as well as the classical “inverse square law” (5.1)–(5.3) to a situation where the N massive bodies are allowed to carry an additional “magnetic mass,” viz., *gravitational dyons*. The corresponding Ricci-flat metric of the Bargmann manifold we are dealing with in dimension $D = 3N + 2$ is of the general form (7.13): namely,

$$g \equiv \sum_{j=1}^N \frac{m_j}{m} \langle d\mathbf{q}_j \otimes d\mathbf{q}_j \rangle + dt \otimes \varpi + \varpi \otimes dt + H(\mathbf{q}) dt \otimes dt, \quad (7.17)$$

with

$$\varpi \equiv ds + \sum_{j < k} \epsilon_{jk} \frac{\langle \mathbf{u}_{jk} \times \mathbf{q}_{jk}, d\mathbf{q}_{jk} \rangle}{r_{jk}^2 + r_{jk} \langle \mathbf{u}_{jk}, \mathbf{q}_{jk} \rangle} \quad (7.18)$$

and

$$H(\mathbf{q}) = \frac{2G}{m} \sum_{j < k} \frac{m_j m_k}{r_{jk}} + \sum_{j=1}^N \frac{m}{2m_j} \left(\sum_{k \neq j} \frac{\epsilon_{jk}}{r_{jk}} \right)^2. \quad (7.19)$$

The notations are as follows: $\mathbf{q}_{jk} \equiv \mathbf{q}_j - \mathbf{q}_k$, $r_{jk} \equiv \|\mathbf{q}_{jk}\|$ with $j, k = 1, \dots, N$, while the unit vector $\mathbf{u}_{jk} = \text{const}$

defines an otherwise arbitrary direction—the direction of the Dirac string—entering the *local* expression of the one-form (7.18). Finally, the masses m_j and magnetic masses ℓ_j are skew-symmetrically encoded into the coefficients

$$\epsilon_{jk} \equiv \frac{\ell_j m_k - \ell_k m_j}{m}, \quad (7.20)$$

where $m \equiv \sum_{j=1}^N m_j$. Again, the global differential structure of this Bargmann manifold can be worked out in a similar (although more involved) way as before.

VIII. QUANTIZATION

We are now ready to derive the quantum version of the preceding classical results. We follow the general rules given by DeWitt⁵⁰ of quantization in a curved manifold. In dealing with topologically nontrivial solutions of Newtonian gravity, we will resort to prequantization² in order to establish a mass quantization formula analogous to the Dirac quantization formula for the electric charge.

A. The covariant Schrödinger equation

The kinetic energy $g^{ab}p_a p_b$ (2.4) is quantized as $-\hbar^2(\Delta_g - \frac{1}{6}R_g)$, where Δ_g is the Laplace operator on (Q, g) and R_g is the scalar curvature—we have identically $R_g = 0$ as a consequence of Newton’s field equations (3.1). This result can also be obtained by geometric quantization⁵¹ using the vertical polarization of (T^*Q, ω) . It is consistent with the quantization rule of the Killing tensors given in (8.8).

The mass Hamiltonian $p_a \xi^a$ (2.5) is linear in momentum; its quantization is therefore canonical. By quantizing the constraints (2.4) and (2.5) according to Dirac’s prescription, we obtain the following set of partial differential equations on the Bargmann manifold (Q, g, ξ) :

$$\Delta_g \psi = 0 \quad \text{and} \quad \xi \psi = \frac{im}{\hbar} \psi, \quad (8.1)$$

which was shown³ to be strictly equivalent to the Schrödinger equation on a general Newton-Cartan spacetime. (Higher spin wave equations can also be formulated in a similar fashion. This has been worked out^{5,52} for the spin- $\frac{1}{2}$ Lévy-Leblond equation¹ on a spin Bargmann manifold.)

In the N -body problem with metric g_V and the $(\mathbf{R}, +)$ generator ξ given by (5.1) and (5.2), we find that (8.1) can be cast into the form

$$\sum_{j=1}^N \frac{m}{m_j} \Delta_j \psi + 2\partial_t \partial_s \psi + \frac{2V}{m} \partial_s^2 \psi = 0.$$

The second equation in (8.1) tells that the wave function ψ is indeed of the form

$$\psi(\mathbf{q}, t, s) = e^{ims/\hbar} \Psi(\mathbf{q}, t), \quad (8.2)$$

where $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ and $m = m_1 + \dots + m_N$ denotes

the total mass. Hence the wave function Ψ finally satisfies the N -body Schrödinger equation

$$-\sum_{j=1}^N \frac{\hbar^2}{2m_j} \Delta_j \Psi + V\Psi = i\hbar \frac{\partial \Psi}{\partial t}. \quad (8.3)$$

Recall that, on any n -dimensional (pseudo) Riemannian manifold (Q, g) with scalar curvature R_g , the operator $\Delta_g - (n-2)/[4(n-1)]R_g$ is conformally invariant. For a Bargmann manifold with $R_g = 0$ (e.g., when Newton field equations hold), and, for $n = 3N + 2$,

$$g^* = \Omega^2 g \quad (8.4)$$

implies that

$$[U((A, \mathbf{b}, c, e, f, g, h)^{-1}) \Psi](\mathbf{q}, t)$$

$$= (ft + g)^{-3/2} \exp \left[\frac{im}{\hbar} \left(\frac{f(A\mathbf{q} + \mathbf{b}t + c)^2}{2ft + g} - \langle \mathbf{b}, A\mathbf{q} \rangle - \frac{t}{2} \mathbf{b}^2 + h \right) \right] \Psi \left(\frac{A\mathbf{q} + \mathbf{b}t + c}{ft + g}, \frac{dt + e}{ft + g} \right).$$

Infinitesimally, this yields the operators^{10,11,36,39}

$$\begin{aligned} \mathbf{p} &= -i\hbar \frac{\partial}{\partial \mathbf{q}} \quad (\text{linear momentum}), \\ \mathbf{L} &= \mathbf{q} \times \mathbf{p} \quad (\text{angular momentum}), \\ \mathbf{g} &= m\mathbf{q} - t\mathbf{p} \quad (\text{boost}), \\ H &= i\hbar \frac{\partial}{\partial t} \quad (\text{energy}), \\ D &= 2tH - \langle \mathbf{q}, \mathbf{p} \rangle + \frac{3}{2}i\hbar \quad (\text{dilatation}), \\ K &= t^2 H - tD - \frac{1}{2}m\mathbf{q}^2 \quad (\text{expansion}). \end{aligned}$$

(ii) In the case of a time-varying gravitational “constant,” $G(t) = G_0 t_0/t$ studied in Sec. III, let \mathcal{D} denote a (local) diffeomorphism of the extended spacetime, and set $g \equiv g_{V_0}$, $g^* \equiv \mathcal{D}^* g_V$. A short calculation shows that both $\psi \equiv \psi_{V_0}$ and $\psi^* \equiv \mathcal{D}^* \psi_V$ satisfy the Schrödinger equation (8.1) if \mathcal{D} is a Vinti–Lynden–Bell transformation [a straightforward generalization of (3.6) to the N -body case]. By (8.2) and (8.5), we find that if Ψ_{V_0} is a solution of the Schrödinger equation for the N -body problem with $G = G_0$, then

$$\begin{aligned} \Psi_V(\mathbf{q}, t) &= \left(\frac{t}{t_0} \right)^{-3N/2} \exp \left(\frac{i}{2\hbar t} \sum_{j=1}^N m_j \mathbf{q}_j^2 \right) \\ &\times \Psi_{V_0} \left(-\frac{t_0}{t} \mathbf{q}, -\frac{t_0}{t} t \right) \end{aligned} \quad (8.7)$$

is a solution of the Schrödinger equation with the time-varying gravitational “constant” as given above.

Let us mention that a second-order conserved quantity associated with a Killing tensor κ should be quantized⁴³ according to

$$\Delta_g \psi = \Omega^{2+3N/2} \Delta_{g^*} \psi^* \quad (8.5)$$

whenever

$$\psi^* = \Omega^{-3N/2} \psi.$$

(i) In particular, this fact entails that the extended Schrödinger group (the local diffeomorphisms \mathcal{C} such that $\mathcal{C}^* g_0 = \Omega^2 g_0$ and $\mathcal{C}_* \xi = \xi$) “acts” on the space of the solutions ψ of the free Schrödinger equation (e.g., for $N = 1$) according to

$$\psi \rightarrow (\mathcal{C}^{-1})^* (\Omega^{-3/2} \psi). \quad (8.6)$$

Using Eqs. (4.6) and (8.2), this “representation” takes the form

$$H_\kappa = -\frac{\hbar^2}{2} \nabla_a \kappa^{ab} \nabla_b. \quad (8.8)$$

Applied to the conformal-Killing tensor κ given by (6.5)–(6.7), this formula yields the following expression for the quantized Lagrange-Laplace-Runge-Lenz vector \mathbf{A} in (6.8):

$$\mathbf{A} = \frac{1}{2} (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}) + m^2 m_0 G_0 \frac{\mathbf{q}}{r}. \quad (8.9)$$

B. Mass prequantization

Let us finally discuss how the mass gets quantized when the Newtonian Taub-NUT solution [(7.10) and (7.11)] is considered, i.e., when the fibers of the Bargmann bundle $Q \rightarrow B$ are compact.

The basic object is the classical space of motions (X, ω_X) of our test particle of mass m already introduced in Sec. II. As a widely accepted rule, we require that this symplectic manifold be *prequantizable*.² In other words, we suppose there exists a circle bundle $Y \rightarrow X$ carrying a connection one-form ϑ_Y whose curvature descends to X as the symplectic two-form ω_X . It is a well-known fact that the prequantization (Y, ϑ_Y) exists if and only if ω_X/\hbar defines an integral cohomology class, which in our case just means

$$m\ell = n \frac{\hbar}{2}, \quad n \in \mathbf{Z}. \quad (8.10)$$

Under these circumstances, the one-form ϑ_C of the 8-dimensional constrained manifold C [(2.4) and (2.5)] descends to the discrete quotient $C_n = C/\mathbf{Z}_n$ obtained by taking $\psi \pmod{2\pi/n}$ where ψ denotes here the S^1 coordinate of the Hopf bundle $S^3 \rightarrow S^2$ [see (7.11)]. The latter integrality condition is, indeed, expressed as

$$\vartheta_C \left(\frac{\partial}{\partial \psi} \right) = 2m\ell \in \hbar\mathbf{Z}.$$

Thus ϑ_C is the pullback of a one-form ϑ_{C_n} . Moreover, the integrable distribution $F_n \equiv \ker(\vartheta_{C_n}) \cap \ker(d\vartheta_{C_n})$ turns out to be 1 dimensional and the quotient manifold $Y \equiv C_n/F_n$ is, at last, our 7-dimensional prequantum bundle carrying the connection form ϑ_Y , which is the image of ϑ_{C_n} . This construction is shown on the following diagram:

$$\begin{array}{ccccc} C & \xrightarrow{Z_n} & C_n & \xrightarrow{\mathbf{R}} & Y \\ & & & \downarrow F_n & \downarrow S^1 \\ & & & \ker(\omega_Y) & \downarrow X \end{array}$$

Note that the prequantization condition (8.10) provides us, in this topologically nontrivial situation, with a *mass quantization* in a purely nonrelativistic setting, i.e., purely as a consequence of nonrelativistic quantum gravity theory in which $c \rightarrow \infty$ but both G and \hbar are nonzero. Mass quantization has already been discovered in the relativistic context by Dowker and Roche.⁵³

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