

Gravitational stability of boson stars

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We investigate the stability of general-relativistic boson stars by classifying singularities of differential mappings and compare our results with those of perturbation theory. Depending on the particle number, the star has the following regimes of behavior: stable, metastable, pulsation, and collapse.

I. INTRODUCTION

Currently there is much interest in the problem of the stability of matter confined by its self-generated gravity. This self-consistent approach dates back to the *geons* of Wheeler.¹ Recently, the work of Lee *et al.*^{2,3} stimulated further progress. They pointed out that a star, regarded as a gravitational soliton,⁴⁻⁶ can have a mass which is larger than the Chandrasekhar-type limit for gravitational collapse.⁷⁻⁹ This opens up a new avenue for studying the structure of a star under unusual matter conditions. Its stability is the most important question. So far, the dynamical stability of boson stars has been analyzed¹⁰⁻¹³ by means of perturbation theory. In this paper we will apply a method which was proposed by one of us (Ref. 14 and references therein) for nongravitational solitons. In general, the method consists of investigating the critical points of a mapping and the construction of bifurcation diagrams. In our case, a two-dimensional subspace of the dynamical variables of the boson field is mapped into the space of the integrals of motion, such as the gravitational mass M and the total particle number N . Using Arnold's classification¹⁵ of singularities of differential maps (catastrophe theory), we are able to derive general criteria for the stability of the star.

II. COUPLED EINSTEIN-SCALAR FIELD EQUATION

As a general-relativistic model of a boson star, we consider a self-interacting scalar field Φ describing a state with zero temperature. This field is self-consistently coupled to its own gravitational field via the Lagrangian

$$\mathcal{L} = \frac{1}{2\kappa} \sqrt{|g|} R + \frac{1}{2} \sqrt{|g|} [g^{\mu\nu} (\partial_\mu \Phi^*) (\partial_\nu \Phi) - U(|\Phi|^2)], \quad (2.1)$$

where $\kappa = 8\pi G$ is the gravitational constant in natural units, g the determinant of the metric $g_{\mu\nu}$, $\mu, \nu = (0, 1, 2, 3)$, R the curvature scalar, and $U(|\Phi|^2)$ the self-interaction potential. We will investigate to what extent the form of U influences the stability of the star.

From the principle of least action we obtain the coupled Einstein-Klein-Gordon equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}(\Phi), \quad (2.2)$$

$$\left[\square + \frac{dU}{d|\Phi|^2} \right] \Phi = 0, \quad (2.3)$$

where $T_{\mu\nu}(\Phi) = (\partial_\mu \Phi^*) (\partial_\nu \Phi) - (g_{\mu\nu} / \sqrt{|g|}) \mathcal{L}(\Phi)$ is the energy-momentum tensor and $\square = (1/\sqrt{|g|}) \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu)$ the generally covariant d'Alembertian.

In this paper we restrict ourselves to the static, spherical-symmetric metric

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (2.4)$$

in which the functions $\nu = \nu(r)$ and $\lambda = \lambda(r)$ depend on the Schwarzschild-type radial coordinate r . For the boson field we make the stationarity ansatz

$$\Phi(r, t) = P(r) e^{-i\omega t}, \quad (2.5)$$

which describes a spherically symmetric bound state with frequency ω . The resulting coupled system reads

$$\nu' + \lambda' = \kappa(\rho + p_r) r e^\lambda, \quad (2.6)$$

$$\lambda' = \kappa p_r r e^\lambda - \frac{1}{r} e^\lambda + \frac{1}{r}, \quad (2.7)$$

$$P'' + \left[\frac{1}{2}(\nu' - \lambda') + \frac{2}{r} \right] P' = e^\lambda \frac{dU}{dP^2} P - e^{\lambda-\nu} \omega^2 P. \quad (2.8)$$

The energy-momentum tensor becomes diagonal; i.e., $T_\mu^\nu = \text{diag}(\rho, -p_r, -p_\perp, -p_\perp)$ with

$$\rho = \frac{1}{2}(\omega^2 P^2 e^{-\nu} + P'^2 e^{-\lambda} + U), \quad (2.9)$$

$$p_r = \rho - U, \quad p_\perp = p_r - P'^2 e^{-\lambda}. \quad (2.10)$$

The form of T_μ^ν is familiar from an ideal fluid, except that the radial and tangential pressure generated by the scalar field are in general different, i.e., $p_r \neq p_\perp$. This *fractional anisotropy* $a_f := (p_r - p_\perp)/p_r$ has already been noted by Ruffini and Bonazzola.⁵ Moreover, Gleiser¹³ found that

all boson stars have the same amount of anisotropy at the radius of the star.

Because of the contracted Bianchi identity $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \equiv 0$, a further equation involving $T_\phi^\phi = T_\phi^\phi = -p_\perp$ is identically satisfied. Equation (2.7) possesses a Schwarzschild-type solution

$$e^{-\lambda(r)} = 1 - \frac{\kappa\alpha(r)}{r}, \quad \alpha(r) := \int_0^r \rho x^2 dx, \quad (2.11)$$

where $\alpha(r)$ is the mass function. For the polynomial self-interaction

$$U := m^2|\Phi|^2 + \frac{1}{2}\alpha|\Phi|^4 + \frac{1}{3}\beta|\Phi|^6, \quad (2.12)$$

these equations have been solved numerically for non-singular, finite-mass, and zero-node solutions.^{3-5,16,17} Two- and higher-node solutions occurred already in Ref. 6. For a massless scalar field with $U=0$, an exact solution is known, cf. Ref. 18. For a massless real scalar field, Christodoulou¹⁹ could show that a spherically symmetric time-dependent field configuration must either disperse to infinity or, for nonvanishing Bondi mass, form a black hole.

III. INTEGRALS OF MOTION

The concept of an energy-momentum four-vector for a field configuration is notoriously subtle²⁰ in general relativity. However, the exponential decrease of the radial function $P(r) \simeq \exp(-\sqrt{m^2 - \omega^2}r)$ for $|\omega| < m$ yields an *isolated*, static system for which the *Tolman mass formula*

$$M := \int (2T_0^0 - T_\mu^\mu) \sqrt{|g|} d^3x \\ = 4\pi \int_0^\infty (2\omega^2 P^2 e^{-\nu} - U) e^{(\nu+\lambda)/2} r^2 dr. \quad (3.1)$$

applies (cf. Ref. 6). It can be derived from the *local* conservation law $\partial_\nu(T_\mu^\nu + \tau_\mu^\nu) = 0$, where $T_\mu^\nu = \sqrt{g} T_\mu^\nu$ and τ_μ^ν is the gravitational energy-momentum complex. For a boson star, the explicit expression (3.1) does not involve derivatives, in contrast with the Schwarzschild mass $M_{\text{Schwarzschild}} := 4\pi\alpha(\infty)$ that is commonly studied¹¹ in this context. Friedberg *et al.* implicitly rederived the equivalence of the Tolman and the Schwarzschild mass [see (2.27) of Ref. 3(a); cf. also Ref. 7].

A second “integral of motion” arises from the fact that the Lagrangian (2.1) is invariant under the global phase transformation $\Phi \rightarrow \Phi e^{-i\vartheta}$. Therefore the Noether current density

$$j^\mu = \frac{i}{2} \sqrt{|g|} g^{\mu\nu} (\Phi^* \partial_\nu \Phi - \Phi \partial_\nu \Phi^*) \quad (3.2)$$

is *locally* conserved, i.e., $\partial_\mu j^\mu = 0$. The time component j^0 integrated over space yields the *particle number* N or the *charge* Q :

$$N = \frac{Q}{e} = 4\pi\omega \int_0^\infty e^{(\lambda-\nu)/2} r^2 P^2 dr. \quad (3.3)$$

Since the current density (3.2) is a “measure” for the radial distribution of the “particles” in the boson star, its *effective radius* can be defined by

$$R := \frac{1}{N} \int r j^\mu d\Sigma_\mu \\ = \frac{4\pi\omega}{N} \int_0^\infty e^{(\lambda-\nu)/2} r^3 P^2 dr. \quad (3.4)$$

On account of the fractional anisotropy a_f another interesting radius R_0 could be obtained from the node $p_\perp(R_0) = 0$ in the tangential pressure p_\perp . This radius R_0 separates the interior part of the boson star from a *marginal layer* in which p_\perp becomes negative before it decreases exponentially.¹⁷

IV. SMOOTH MAPPING (WHITNEY SURFACE)

In order to investigate the stability of soliton-type solutions against radial perturbations, we consider the two-dimensional mapping

$$F: (k, \omega) \mapsto (M, N), \quad (4.1)$$

where k is a variational parameter which dilates the radius R of the star and ω the frequency eigenvalue. The parameter k induces a scaling of the metric, the frequency, and the scalar field in accordance with their normal physical dimensions

$$ds^2 \rightarrow k^2 ds^2, \quad \omega \rightarrow \omega/k, \quad P(r) \rightarrow kP(kr), \quad (4.2)$$

such that the particle number N is kept fixed.

In order to classify the singularities of this mapping F , let us consider the Jacobi matrix

$$J = \begin{pmatrix} \partial M / \partial k & \partial M / \partial \omega \\ \partial N / \partial k & \partial N / \partial \omega \end{pmatrix}. \quad (4.3)$$

According to Whitney’s theorem,^{14,15} the singularities of the mapping F can be one of three types, depending on the rank $R_J = 2, 1$, and 0 , respectively. Since we require the soliton solution to be an extremal point of the Lagrange manifold, we have

$$\frac{\partial M}{\partial k} = 0 \quad \text{and} \quad \frac{\partial N}{\partial k} = 0. \quad (4.4)$$

For the soliton the rank of J is $R_J < 2$ and, consequently, the singularities of the mapping F may have either $R_J = 1$ or $R_J = 0$. In that case, our soliton solution corresponds to the extremal or critical points of the Whitney surfaces which has a very definite form (see Refs. 14 and 15). In numerical examples, the dependence

$$M = M(\omega), \quad N = N(\omega) \quad \text{with} \quad \omega = \omega(\sigma(0)), \quad (4.5)$$

on the frequency ω can be smoothly converted into a function of the central density $\sigma(0) = \sqrt{\kappa/2} P(0)$ such that the critical points coincide (Fig. 1). If the rank of J is zero, the critical points are degenerate. The maxima and minima of $M = M(\sigma(0))$ and $N = N(\sigma(0))$, see Fig. 1, correspond to the A_2 singularity, in the notation of Arnold. Other points of the curves in Fig. 1 correspond to the critical points A_1 .

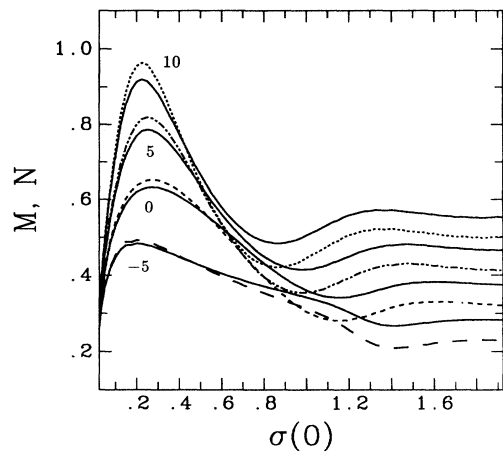


FIG. 1. The Tolman mass M (—) in units of $(1/mG)$ and particle number N (---) in dimensionless units of $(1/m^2G)$ as a function of the central density $\sigma(0) = \sqrt{\kappa/2}|\Phi(0)|$ for various $\bar{\alpha} := (2\alpha/\kappa m^2)$ and $\beta=0$ in the potential U . For a linear scalar field ($\bar{\alpha}=0$) the Kaup limit $M_{\text{Kaup}}=0.633$ is recovered (Ref. 17). The maxima and minima correspond to the A_2 singularities.

V. BIFURCATION DIAGRAM

In order to classify the nondegenerate A_1 we need to consider the bifurcation diagram $M=M(N)$.³ Equivalently we may consider the *binding energy*

$$B = M - mN = B(N) \quad (5.1)$$

as a function of the particle number (Fig. 2). A further “magnification” of $B(N)$ is achieved in Fig. 3. According to the Whitney theorem, the cuspidal points of these diagrams classify the A_2 singularity, whereas the other points of the diagram correspond to the A_1 singularity.

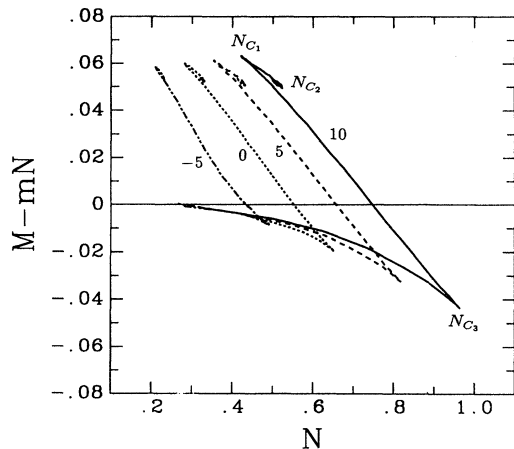


FIG. 2. The binding energy $M - mN$ as a function of N at different values $\bar{\alpha} = -5, 0, 5, 10$ (Ref. 17). In this bifurcation diagram, the lower branch of each cusp corresponds to a stable star configuration. See, for comparison, Fig. 1.

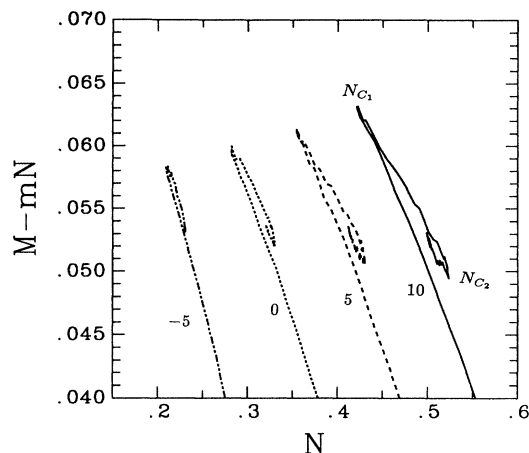


FIG. 3. “Magnified” view of the binding energy for the same parameters as in Fig. 2.

Each cusp represents some *Whitney surface*, which is a part of the mass-energy surface. As shown in Ref. 14, the minimum on this surface corresponds to the stable soliton, the maximum corresponds to the unstable soliton. At the cuspidal point, the minimum coalesces with the maximum, and the soliton loses its stability. Thus, the lower branch of the lowest cusp corresponds to the *absolutely stable soliton*. The upper branch of the first cusp, which is, at the same time, the lower branch of the second cusp, corresponds to the unstable soliton. The upper branch of the second cusp also corresponds to unstable solitons, which, however, suffer from a different kind of instability than the soliton of the lower branch of the second cusp.

For the boson star, the degrees of freedom of the configuration space are very large. The fact that the lower branch of the lower cusp corresponds to an absolutely stable soliton means that there we have minima for all directions in configuration space.

The higher branch of the first lower cusp corresponds to a maximum. This maximum occurs in that section of the mass-energy surface which depends on the radius R of the star. For the second cusp the appearance of a new instability depends on the mutual branching to other cusps. There are the two possibilities that the next (third) branch goes higher or lower than the second one. In the first case, according to Whitney’s theorem, the minimum transforms into a maximum at the transition from the second branch to the third one, and a new instability appears. Vice versa, in the second case, one maximum transforms into a minimum after the transition through the cuspidal point and one instability disappears. The numerical data (Fig. 3) show that for the third cuspidal point one instability disappears, in accordance with the general picture developed here. This disappearance of one instability has not been pointed out in previous works on the stability of boson stars.

In the literature^{3,21} the corresponding diagrams are obtained by applying the method of small perturbations to the problem of stability. In our approach, the bifurcation diagram is a key point of the analysis and is gained by

analyzing the topology of the Whitney surface. In the first step of the analysis we calculate the bifurcation diagram for the mapping which relates the integrals of motion M and N to some degrees of freedom of the boson star. Following Ref. 14 we connect this bifurcation diagram to the mass-energy surface $M = M(R, N)$.

The mass-energy surface described by the bifurcation diagram corresponds to some complicated manifold of catastrophe, which probably has never been seen before in the theory of singularities. The order of this grand catastrophe depends on the number of cusps in the bifurcation diagram. The *static description* of the bifurcation diagram is that at every cusp there is a transition from minimum to maximum or vice versa. For every cusp there occurs one Whitney surface. As an application to our case we find that at the first cusp a minimum transforms to a maximum. This maximum will not be affected at the following cusps, if at the second cusp the following branch goes to higher mass values. If this branch reached lower mass values, we will get a more complicated “Whitney surface” in analogy to the case of the swallow’s tail.²²

For neutron stars or white dwarfs, we get a complicated bifurcation diagram consisting of the many cusps (see Fig. 8 of Ref. 8). Again, each cusp corresponds to a Whitney surface. The first four lower branches of this bifurcation diagram (in the direction of increasing density) describe a section of manifold of catastrophe which is usually called a butterfly. Consequently, for these four branches this two-dimensional manifold of catastrophe corresponds to the two-dimensional Whitney surface in the bosonic case.

For the second cusp of the bosonic bifurcation diagram there is a Whitney surface also; i.e., at a cuspidal point of this cusp there is a transition from minimum to maximum. At the following cusps, if there is a transition to higher mass values, the maxima stay maxima. We understand such applications of catastrophe theory not only for the boson star, but also for fermion Q stars, white dwarfs, and neutron stars. The latter have been considered by Harrison *et al.*⁸ They “saw” the Whitney surface without drawing knowledge from catastrophe theory.

The simplest way to identify the instabilities, which have been qualitatively predicted by catastrophe theory, is to consider perturbation theory. Because the perturbative equations set up a Sturm-Liouville eigenvalue problem, the characteristic frequencies of the perturbation series have increasing absolute values. This holds also in the case of the boson star.^{11–13} Moreover, it was shown for the boson star¹¹ and for the neutron star⁸ that, at each cusp, one of these frequencies changes sign. Thus, each such instability can be identified with a corresponding one obtained from applying catastrophe theory.

Until now we have considered only the mapping of the two-dimensional space (k, ω) into the two-dimensional space (M, N) . Since the first space counts the number of the degrees of freedom of the star, we may extend the domain (k, ω) of the mapping by including the characteristic frequencies. In the next step, new states of the star with different dynamical behavior could be taken

into account than those we see already in the bifurcation diagram. These states correspond to other points on the mass-energy surface. For burning stars, for example, oscillations with large amplitudes occur at their final stage as red giants. The evolution of these oscillations cannot be described by perturbation theory. This kind of dynamical behavior of the star will be discussed in Sec. VI.

In flat spacetime, the dependence M on N has been investigated by Friedberg *et al.*²¹ Although they pointed out that the minimal energy branch of M versus N is stable, which is true, they have not investigated the stability of these solitons for all values M and N .

VI. THE DIFFERENT REGIMES OF THE STAR’S BEHAVIOR

In our method, like in the theory of singularities of smooth mappings, the bifurcation diagram plays an important role. It is a skeleton of the catastrophe or skeleton of the mass-energy surface. The extremal point of this surface corresponds to the soliton solution. There exist also other types of solutions, with a different dynamical behavior. Perturbation theory gives small oscillations near the soliton solution. We went beyond perturbation theory, which helped us to investigate the stability of the star.

Using the catastrophe theory we can construct the mass-energy surface. Each section of this mass-energy surface contains the different degrees of freedom of the star, which can be identified, by perturbation theory, with the characteristic frequencies. Using such sections one can predict dynamical regimes of the star which cannot be described within the framework of perturbation theory.

In order to predict the different regimes of behavior of the star we construct the section $M(1/R)$ (where R is the effective radius of the star) of the mass-energy surface $M(R, N)$ at fixed N . In other words, we construct an adiabatic potential.²³ The shape of $M(1/R)$ follows from the bifurcation diagram.

The type of regimes depends on the critical values $N_{C_1}, N_{C_2}, N_{C_3}$, of the particle number. For the coupling constants $\bar{\alpha} = (2\alpha/\kappa m^2) = 10$ and $\beta = 0$ in self-interacting potential (2.12), we find that $N_{C_1} = 0.42$, $N_{C_2} = 0.52$, $N_{C_3} = 0.96$ are the cuspidal points, see Fig. 2.

A. Stable soliton and oscillation

From the bifurcation diagram (Fig. 2) it can be inferred that, for $N < N_{C_1}$ the function $M(1/R)$ has only one minimum, which corresponds to the stable soliton solution (lower branch of the first cusp).

The dependence of M on R at some fixed value $N < N_{C_1}$ is schematically presented in Fig. 4. The smooth extremal point which corresponds to the minimum of the curve $M(1/R)$ is associated with the soliton solution. The value of M in this point we call M_{soliton} . The marginal extremal point at $R = \infty$ corresponds to the “homogeneous state,” or plane-wave solution in flat spacetime. In the self-generated gravitational field, it can be defined

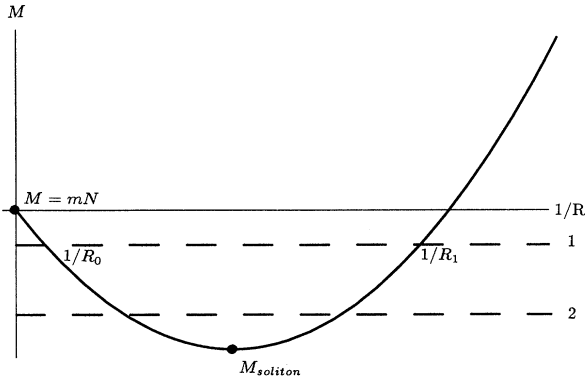


FIG. 4. “Adiabatic” potential $M = M(1/R)$ of the star for $N < N_{C_1}$ (schematic construction following Refs. 22 and 24).

as an *effectively free boson field solution* for which the binding energy $B = M - mN$ is vanishing. At an infinite value of the radius R of the star the values M and N are finite ones. This follows from the fact that in the limit $R \rightarrow \infty$ the density of the star goes to zero (see Ref. 14, p. 29). The exact construction of the corresponding solution will be deferred to a future publication. These two extremal solution characterize a static configuration of the star. As we see from this figure, the “free boson” field is unstable. This extremum corresponds to the maximum. It means that the homogeneous state of the star will collapse from the size $R = R_0 = \infty$ to the size $R = R_1$ (see Fig. 4). From R_1 , the size of the star will again increase. The value of R increases up to the initial value R_0 . Thus the star will be in an oscillating regime. For such oscillations of the star, the curve presented in Fig. 4 is a kind of “adiabatic” potential which can also be obtained from a scale transformation. There are also other oscillations of the star in this potential that correspond to other values of M . For example, the other oscillation regime of the star corresponds to the horizontal line 1. In this case the amplitude of the oscillation is lower than the oscillations in the regime associated with the unstable free boson field described above. On the other hand, the oscillating regime, corresponding to line 2, has a smaller amplitude than the one corresponding to line 1 and so on. The limiting case is a stable soliton without oscillation. In the region of $N < N_{C_1}$, Fig. 4 gives a complete picture of the star’s behavior.

The interesting point here is that any arbitrary configuration of fixed particle number N cannot have a mass smaller than M_{soliton} . The reason is that the configurations of the star are limited by the mapping F . For instance, the mapping $F: (k, \omega) \rightarrow (M, N)$ allows only a class of configurations away from the static soliton which preserve the integrals of motion M and N . Of course, the perturbed configurations of the stable soliton will evolve along the line dictated by the differential map. But there may arise other ways of choosing the parameters k and ω . Moreover, one could think of extending the domain (k, ω) of the mapping F to a higher-dimensional space $(k_1, \dots, k_N, \omega_1, \dots, \omega_N)$, where $k_1, \dots, k_N,$

$\omega_1, \dots, \omega_N$ correspond to additional degrees of freedom of the star. Such a space provides us with further lines of evolution of the perturbations, but the stability will still be determined by the bifurcation diagram $M(N)$. Such a diagram comes from a numerical solution describing the stationary points of a Lagrange manifold of Einstein’s equation. In this way one can describe virtually all perturbations which preserve the integrals of the motion M and N . If we considered configurations which cannot be categorized by this chosen map, for example, when the total mass M and the total number of particles N are not necessarily fixed, then one could presume that the star, via some oscillation process, settles down to a stationary stable configuration with some reduced mass M_{reduced} .

The oscillations arise for $M > M_{\text{soliton}}$ of the mass of the star. In this region $N < N_{C_1}$, at some value of $M > M_{\text{soliton}}$, there may occur a gravitational collapse. We expect that the configuration will oscillate rather than collapse to a black hole not up to arbitrarily large mass M . There should exist the critical value of $M_c = M_{\text{Schwarzschild}}$ which depends on the radius of the star R_c with given mass $M_c(R_c)$. At this radius R_c the star will stop oscillating and start collapsing to a black hole. However, in order to show this rigorously the analysis of Christodoulou¹⁹ has to be extended to the case of the massive or even nonlinear scalar field.

B. Collapse

On the other hand, for $N > N_{C_3}$, the section $M(1/R)$ following from the bifurcation diagram of the mass-energy surface, has only the marginal extremum corresponding to the effectively free boson field solution. As we can infer from Fig. 5, the section $M(1/R)$ is a monotonically decreasing function. The homogeneous free field solution corresponds to the marginal extremum. It is a maximum of $M(1/R)$. For the increase of the kinetic energy of the star, the radius decreases. This means that in “this region” of N there exists a collapse for any value of the mass. In the first stage, when $R_{\text{star}} \gg (\kappa/4\pi)M_{\text{Schwarzschild}}$, it is a wave collapse (see Ref. 24 for details and references therein) which later induces the gravitational collapse.

Due to this *collapse*, this state is unstable. The different horizontal lines, drawn in Fig. 5, correspond to different collapse regimes of the star with different initial radii. A similar case is known for the creation of the two-dimensional plasma cavitons.^{24,25}

C. Pulsation, oscillation, and collapse

In the region $N_{C_2} < N < N_{C_3}$ the dependence of M on R has as many extremal points as there exist branches of cusps at a given value of N . In the case where there are two cusp branches at given N , the dependence $M(1/R)$ has two extremal points and a marginal extremum corresponding to the “free boson field” solution $M = mN$ (Fig. 6). The point of the minimum of $M(1/R)$ corresponds to the stable soliton with the mass M_{soliton} . The instability of the homogeneous state results in an oscillation regime similar to that which has been described in Sec.

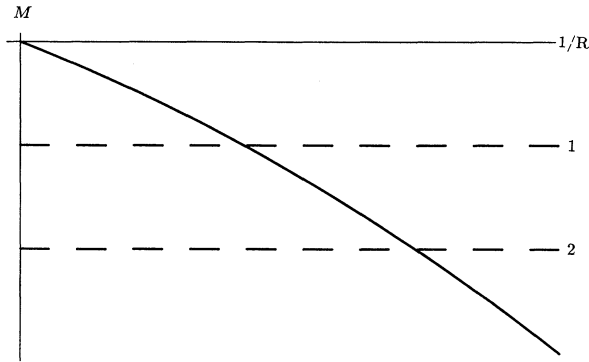


FIG. 5. "Adiabatic" potential $M = M(1/R)$ of the star for $N > N_{C_3}$, describing the collapse.

VI A. The maximum of $M(1/R)$ corresponds to the unstable soliton with mass M_u . The instability of this soliton can occur in two possible ways. One of them is collapse, decreasing the radius of the star. The *second* one is increasing the star's radius (dispersion). Such an instability gives rise to an oscillation in the same manner as the marginal free-particle extremum. There can be a lot of such oscillations (see Sec. VI A) before the boson star will collapse. That means that this oscillating regime may be also unstable and, after several periods, collapses to the state of unstable soliton. Thus the existence of this unstable oscillating regime is due to the existence of the unstable soliton.^{24,25}

Thus, in the most interesting region $N_{C_2} < N < N_{C_3}$, the function $M(1/R)$ has two extremal points, which correspond to the stable soliton (lower branch of the first cusp, see Fig. 2) and the unstable soliton (higher branch of the first cusp), respectively, and one marginal extremum (free boson field solution). The shape of such a function indicates that at each value of N there exist the following configurations: a stable soliton with a mass $M_{\text{soliton}}(N)$ and an unstable one with a mass $M_u(N)$. At $M(N) > M_u(N)$ it is very difficult to predict the evolution of the star.

In the region $N_{C_1} < N < N_{C_2}$ the dependence of

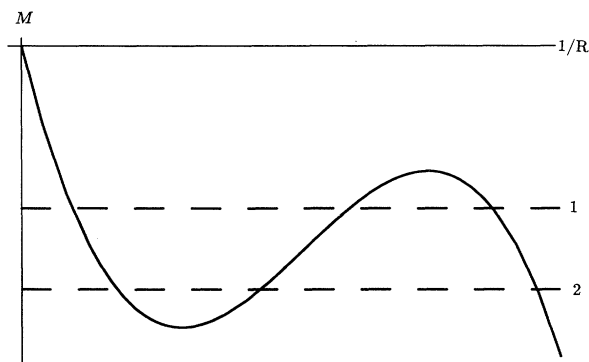


FIG. 6. "Adiabatic" potential $M = M(1/R)$ of the star for $N \in [N_{C_2}, N_{C_3}]$, describing oscillation and collapse.

$M(1/R)$ may have more than two smooth extrema which are minima and maxima. For definiteness, let us consider the case when $M(1/R)$ has three extremal points. Two of these points correspond to stable solitons with the mass M_{s_1} and M_{s_2} and one corresponds to the unstable soliton with mass M_u . There are two types of oscillations, corresponding to the minima M_{s_1} and M_{s_2} which exist of $M < M_u$. In the first case, the maximal kinetic energy of the star corresponds to the radius of the first stable soliton. It is an oscillation in the first minimum. For the second case, the maximal kinetic energy corresponds to the radius of the second stable soliton. It is an oscillation in the second minimum.

There is also a very interesting regime, for which the star has a mass $M = M_u$. In this case, there will exist a spatial type of oscillation in which it is difficult to predict in which direction (first or second minimum) the star will move from the point of maximum. It is connected with the indefiniteness of the behavior of the star in the state of the unstable soliton (dispersion or collapse). Such a regime of the star's behavior is characterized by a *pulsation*. The pulsation consists of at least two different types of oscillations.

In the same manner one can analyze the case when $M(1/R)$ has more than three extremal points.

One general conclusion which can be drawn here is that in the region $N_{C_1} < N < N_{C_2}$ there is a significant distribution of the instabilities of the star. The number of instabilities depends on the number of cusps and on the mutual branching of these cusps. If the next branch of some cusp corresponds to the higher values of the star's mass, then the number of instabilities increases by one. Vice versa if the next branch of this cusp corresponds to lower values of the star's mass, then the number of instabilities decreases by one.

VII. DISCUSSION

Our stability criteria include the results of Refs. 10–13 obtained by perturbation analysis.

Until now, there exist some more or less successful attempts to prove the stability of the boson star. These results fight with the difficulty of the mathematical problem; they tried to solve the problem of stability in a quantitative way like Harrison *et al.*⁸ or Shapiro and Teukolsky.^{7,9} Gleiser¹³ and Jetzer¹² got an upper limit for stability for the linear case and an additional $|\Phi|^4$ potential. This limit was much higher than the first maximum in the $(M, \sigma(0))$ diagram.

Later, Gleiser and Watkins¹¹ showed that in the linear case a change in stability occurs at the first extremum. At the following extrema, the higher modes (in the context of Refs. 8 and 9) are negative so that the star becomes more unstable. This result perfectly complies with those which are known from the analysis of neutron stars. Furthermore, Jetzer¹² showed that the zero-node solution of the boson star is stable until one reaches the first cusp. Lee and Pang¹⁰ did not require the particle number N to be constant, so they found that these solu-

tions are unstable.

The picture of the star's behavior, obtained here on the basis of the application of catastrophe theory to solitons,¹⁴ has a general character and has an analogous form for neutron stars (Refs. 7 and 8, see Fig. 8), fermion Q balls,²⁶ and for dilaton stars.²⁷ In fact, for neutron stars the diagram $M(N)$, obtained from numerical integration of a certain equation of state for cold, catalyzed matter, exhibit similar bifurcations as in the case of boson stars (see Fig. 8 of Ref. 8 and and Fig. 47 of Ref. 7). For the first cusp, Harrison *et al.*⁸ could even deduce correct stability criteria from the analysis of the mass-energy surface $M(\rho_0, N)$, see Fig. 9 of Ref. 8. Although this was done without knowledge of catastrophe theory or Arnold's classification of singularities, these results of 1965 are in complete agreement with the more general criteria developed here.

In comparison with Ref. 19 one should point out that in the limit $m \rightarrow 0$ all solutions, corresponding to the stable solitons with $0 < M_{\text{soliton}} < mN$ disappear and only unstable solutions remain. Such solutions disperse either to infinity or collapse. The collapse configuration can form a black hole. This agrees completely with the results of Ref. 19.

After the submission of our paper, the recent paper of Seidel and Suen²⁸ appeared, in which the numerical evolution of various configurations of boson stars using the full nonlinear Einstein equations have been studied. Moreover, perturbations which include a redistribution of scalar particles in the star and also accretion and annihilation of the bosons are considered so that the total mass M and the total number of particles N are not

necessarily fixed. Their result, that the U -branch star (unstable soliton) will either collapse to form a black hole or will disperse, agrees with our conclusion. The exception is that the unstable soliton will eventually settle down to a stable soliton. This is due to the possibility that the star is allowed to change the value of N or M . Such damping mechanisms have not been considered in our paper. The crucial role of the migration of unstable soliton (U branch) to a stable soliton (S branch) has also been pointed out in Ref. 28. According to this paper, a stable soliton (S -branch star), which is slightly perturbed, will oscillate with a fundamental frequency. This coincides with our conclusion about the oscillation regime of star near a stable soliton configuration. Thus the picture of the star's behavior obtained on the basis of theory of singularities of smooth maps (nonelementary catastrophe theory) completely coincides with conclusions obtained on the basis of extensive studies of the numerical evolution of boson stars.

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