PHYSICAL REVIEW D VOLUME 43, NUMBER 12 15 JUNE 1991

Vacuum polarization in a locally static multiply connected spacetime and a time-machine problem

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(Received 26 November 1990)

The quantum field theory in a multiply connected locally static spacetime is considered. It is shown that the vacuum-polarization effect in a spacetime with a nonpotential gravitational field results in divergences of the renormalized stress-energy tensor for a quantum field near the Cauchy horizon. Possible applications of this result are discussed in connection with a "time machine" problem.

I. INTRODUCTION

One of the most intriguing features of general relativity is the possible existence of spacetimes with a nontrivial topological and causal global structure. The spacetime containing wormholes described by Wheeler[1,2] is one important example. One can imagine a wormhole as a three-dimensional space with two spherical holes (mouths) in it. These holes are connected one with another by means of a handle. The length of this handle l does not depend on the distance L between the mouths in the external space and in principle this length might be much smaller then L.[3] Recently the interest in this problem increased because it was shown that a stable wormhole (if only it exists) can be transformed into a time machine (i.e., into a spacetime with closed timelike curves[4–5]). Moreover, it appeared that this relation between a wormhole and a time machine is of a quite general nature. Namely, closed timelike curves arise as a result of the generic motion of a wormhole's mouths[4] and/or the action of the generic external gravitational field.[6] The classical gravitational interaction of a wormhole with surrounding matter also transforms it into a time machine.[6]

Wormholes with handle geometry that is stationary or slowly changing in time are of particular interest. It is possible to enter into such a wormhole, to pass through the tunnel, to exit into external space again (traversable wormholes). This property distinguishes wormholes from black holes. In Refs. 3 and 4 it was shown that, for the existence of traversable wormholes, the stress-energy tensor must violate the average weak energy condition. This condition is assumed to be valid in classical physics. Until now it is not completely clear whether fundamental laws of physics allow the required violation of the energy condition (see, e.g., Refs. 7 and 8). But in any case it seems that the quantum effects must be important in wormhole physics and for a time-machine problem.

An infinite blueshift effect near the Cauchy horizon is another general property of a spacetime with a time machine. This effect is connected with the existence of a special class of trajectories which may pass through a wormhole an infinite number of times before the Cauchy horizon is formed. For those trajectories the blueshift which arises after passing a wormhole due to the Doppler effect for moving and/or due to a nonpotential gravitational field may be accumulated and become infinite. In classical physics this effect generally does not mean an instability of a time machine because of a zero measure of "dangerous" trajectories and because of the defocusing property of a gravitational field of a traversable wormhole.[4,9] In particular, a wave packet of a classical massless field due to defocusing effect may return and pass again through the wormhole only a finite number of times. One can use such a mode to extract finite energy from a wormhole, but its amplitude at the Cauchy horizon remains finite.[9]

For quantum fields due to the existence of null fluctuations which cannot be isolated or suppressed the situation is quite different. There are infinitely many modes of a null fluctuation which may be amplified after passing a wormhole in a proper direction just before the Cauchy horizon formation. Though the amplification is finite for each mode, the contribution of all modes to the renormalized stress-energy tensor of a quantum field may be divergent at the Cauchy horizon.

One can also arrive at the same conclusion by considering an amplitude of a quantum particle propagation in a multiply connected spacetime which describes a time-machine formation. This amplitude $G(X,X')$ for close points X and X' can be written as Feynman integral over all trajectories connecting these points. In addition to the trivial paths that do not go through the wormhole, there are homotopically nonequivalent classes of paths numerated by an integer winding number indicating how many times a path passes through the wormhole in a given direction. In the geometric optics approximation the main contribution to the Feynman propagator is given by geodesics connecting X and X'. The existence of closed null geodesics with $n\neq 0$ (which are possible in a spacetime with a time machine) may cause divergence of the renormalized stress-energy tensor $\langle T_{\mu\nu}\rangle^{\mathrm{ren}}$ near the Cauchy horizon.[10] Kim and Thorne in a recent paper[11] proved that such a divergence does really take place in a model of a time machine which arises as a result of a motion of a wormhole's mouths in flat spacetime and they describe the structure of $\langle T_{\mu\nu} \rangle^{\rm ren}$ near its singular points.

The main aim of the present paper is to prove that the divergences of the renormalized stress-energy tensor is a general property of a quantum field in a locally static spacetime with a nonpotential gravitational field describing time-machine formation.

We begin by considering properties of a locally static multiply connected spacetime which possesses one or several wormholes (Sec. II A). The main feature of these spaces is that there is no global Killing vector field while such a field exists in each simply connected part of the space.[6] If we consider a continuous Killing vector along a closed curve with a nonvanishing winding number, we find out that the Killing vector at the final point coincides with the Killing vector at the initial point in direction but has a different norm and hence a global Killing vector field exists only as a multiply valued one.

The quantum field theory of a massless scalar field in a locally static multiply connected spacetime is discussed in Sec. II B. The main idea is the use of the covering-space method developed for quantum mechanics and quantum field theory in a multiply connected space by Schulman,[12] Dowker,[13] and Banach and Dowker.[14–16] This method allows one to reduce the quantum field theory in a multiply connected space to the theory in its universal covering space. It should be stressed that in a locally static multiply connected spacetime there is neither global time nor a global Killing vector field, so that the standard approach is not directly applicable and it requires some modifications. Nevertheless the covering space method is ideally suited for our case because the universal covering space is not only simply connected but also allows a global Killing vector field, which can be used for a natural definition of positive and negative frequencies and for a vacuum state choice.

Quantum effects in two-dimensional models of spacetime with a time-machine formation are considered in Sec. III. The spacetimes in such models are locally conformally flat, but the nontriviality of their topologies and causal structures makes a quantum theory of massless fields nontrivial on this background. A locally static two-dimensional spacetime is connected by a regular conformal transformation with some "standard model" where the spacetime not only has the same global properties but also is of constant curvature (Sec. III A). In Sec. III B the general covering space method is illustrated by its application to the theory of a conformal massless scalar field in a standard model. By using this method it is shown that the renormalized stress-energy tensor for the scalar field is divergent near the Cauchy horizon in any locally static spacetime with a nonpotential gravitational field. In Sec. III C it is shown that this result is generic: it does not depend on the particular choice of a quantum state and it is also valid when the gravitational field is time dependent, provided the time machine is formed.

In Sec. IV a four-dimensional locally static multiply connected spacetime is considered. We prove that for an arbitrary spatial point there exists a closed null geodesic which begins and ends at this point and has a given winding number n (Sec. IV A). This result is used in Sec. IV B to prove that the renormalized stress-energy tensor in such a spacetime is divergent at a system of hypersurfaces lying above the Cauchy horizon and to describe the structure of its divergence. Section V contains a discussion of the obtained results in connection with a general problem of quantum stability of a time machine.

In this paper we use the natural units $c=G=\hbar=1$ and the sign convention of Ref. 17.

II. QUANTUM FIELD THEORY IN A LOCALLY STATIC WORMHOLE GEOMETRY

A. Locally static spacetime with wormholes

Consider a spacetime, i.e., a four-dimensional manifold M with a metric $g_{\mu\nu}$. We call it locally static if in any simply connected region $U \subset M$ there exists a uniquely defined (up to normalization) nonvanishing timelike Killing vector field ξ, obeying the relations

$$\xi_{(\mu;\nu)}=0 \ , \tag{2.1a}$$

$$\xi_{[\mu;\nu}\xi_{\lambda]}=0 \ . \tag{2.1b}$$

The first equation is a definition of a Killing vector, while the second equation means that ξ is "surface orthogonal" in U. Equations (2.1) allow one to show that in a simply connected locally static spacetime the Killing vector exists as a global vector field and hence the spacetime is static. In a more general case of a locally static multiply connected spacetime (e.g., when wormholes are present) such a global Killing vector field may not exist. In this section we consider some general properties of locally static spacetimes (for more details, see Ref. 6).

Let u^μ be a four-velocity of a Killing observer, i.e., a unit timelike vector defined by the relation

$$u^\mu=\xi^\mu/|\xi^2|^{1/2} \ . \tag{2.2}$$

The vector u^μ does not depend on the norm of ξ^μ and it is well defined globally in a locally static spacetime. The integral lines $X^\mu(\tau)$ of u^μ,

$$dX^\mu/d\tau=u^\mu \ , \tag{2.3}$$

are known as Killing trajectories or trajectories of Killing observers. It is easy to show that Eqs. (2.1) are locally equivalent to the following system of equations:

$$u_\mu u^\mu=-1 \ , \tag{2.4a}$$

$$u_{\mu;\nu}=-w_\mu u_\nu \ , \tag{2.4b}$$

$$w_{[\mu;\nu]}=0 \ . \tag{2.4c}$$

Equation (2.4b) shows that w^μ is a four-acceleration of a Killing observer ($w^\mu=u^\nu u^\mu{}_{;\nu}$). In other words one can define a locally static spacetime as a spacetime which admits two global vector fields u^μ and w^μ obeying the conditions (2.4).

It is convenient to consider a locally static spacetime M as a collection W of Killing trajectories.[18] That is, an element of W is a curve in M which is everywhere tangent to u^μ. For each point of M one can find the trajectory of u^μ

which passes through this point so that there exists the natural mapping $\Psi: M \to W$. We assume that the topology of Killing trajectories T is R^1 and that $M = T \times W$.

Before considering further geometrical properties of a locally static spacetime we make some comments concerning its topological structure. T is a topologically trivial space; that is why the homotopic groups $\pi_k(M)$ and $\pi_k(W)$ are identical (isomorphic): $\pi_k(M) = \pi_k(W)$. In what follows we assume that W is a three-dimensional asymptotically flat space with a wormhole. Such a space can be obtained from R^3 by cutting two balls B_1 and B_2 from it and by further identification of the two-dimensional boundaries ∂B_1 and ∂B_2. We denote by W_1 the space obtained as a result of these operations. A three-dimensional space W_N with N wormholes can be obtained in a similar manner by cutting N pairs of balls and gluing the boundaries of each pair. Sometimes it is more convenient to deal not with an asymptotically flat space R^3 but with its compactified version $S^3 = R^3 \cup \infty$. We shall use the notation W_n^c for the space S^3 with N wormholes. In particular one has $W_1^c = S^2 \times S^1$. (For more details, see Ref. 2.)

By using Seifert–Van Kampen theorem[19,20] one can show that

$$\pi_1(W_N) = \pi_1(W_N^c) \tag{2.5}$$

and that the fundamental group $\pi_1(W_N^c)$ is a free group with N generators. In the simplest case of a space with one wormhole W_1, one has

$$\pi_1(W_1) = \mathbb{Z} . \tag{2.6}$$

The generator of this fundamental group is the class of homotopic paths which begin at a given point x_0, pass once through the handle in a chosen direction, and return to x_0. The integer winding number $n \in \mathbb{Z}$ numerating different homotopic classes indicates how many times a chosen closed path passes through the wormhole in the chosen direction. (For the opposite direction n is negative.)

The free group $\pi_1(W_N)$ has N generators. $\gamma^1, \gamma^2, \ldots, \gamma^N$, where γ^i is a homotopic class of a closed path passing once through the i handle in a chosen direction. For $N > 1$ the group $\pi_1(W_N)$ is non-Abelian. The first homology group $H_1(W_N)$ is connected with the fundamental group $\pi_1(W_N)$ by the relation

$$H_1(W_N) = \pi_1(W_N) / [\pi_1(W_N), \pi_1(W_N)] , \tag{2.7}$$

where $[\pi_1(W_N), \pi_1(W_N)]$ is a commutant of the group $\pi_1(W_N)$. It is easy to see that

$$H_1(W_N) = \mathbb{Z}^N , \tag{2.8a}$$

$$H_1(M) \equiv H_1(T \times W_N) = \mathbb{Z}^N . \tag{2.8b}$$

For $N = 1$

$$H_1(W_1) = \pi_1(W_1) = \mathbb{Z} . \tag{2.9}$$

The paths with a given winding number n which are homotopic by definition are also homologic one to another. The homology class of a cycle (a closed path passing through the wormhole) with a winding number $n = 1$ is the generator of $H_1(W_1)$. For $N > 1$ any cycle can be specified by its winding number $n = \{n_1, n_2, \ldots, n_N\}$, where n_i indicates how many times the cycle passes through the i handle. Denote by C^i a cycle which passes once through the i wormhole. The set of these cycles forms the generators of $H_1(W_N)$.

After these quite general topological remarks we return to discussion of the geometrical properties of locally static spacetimes. Equation (2.4c) shows that the one-form $w = w_\mu dX^\mu$ is closed ($dw = 0$) and hence according to the Stokes theorem the integral of this form over any closed path C_n in $M = T \times W_N$ passing through the wormhole depends only on its winding number $n = \{n_1, n_2, \ldots, n_N\}$. Denote by $I^i[w]$ the i period of w, i.e., the integrel of the form w over a generator C^i,

$$I^i[w] = \oint_{C^i} w_\mu dX^\mu , \tag{2.10}$$

and by $I_n[w]$ the analogous integral of w over a closed path with a winding number n; then one has

$$I_n[w] = n_1 I^1[w] + n_2 I^2[w] + \cdots + n_N I^N[w] . \tag{2.11}$$

For our purpose it is convenient to fix the directions of each of the generators C^i in such a way that $I^i[w] \le 0$.

The physical meaning of a period $I^i[w]$ in a locally static spacetime is quite simple. Choose a point X_0^μ in M and denote by $X_0^\mu(\tau)$ a Killing trajectory passing through this point $[X_0^\mu = X_0^\mu(\tau_0)]$. Consider now a path $C^i: Y^\mu(\lambda)$, $\lambda \in (0,1)$ which begins at X_0^μ, passes once through the i wormhole and ends at the same Killing trajectory and which is everywhere orthogonal to u^μ: $u_\mu dX^\mu / d\lambda = 0$. The last condition means that this path is formed by the events that are simultaneous with the initial event X_0^μ. Denote by τ_1 the proper time of the endpoint of the path at the Killing trajectory $X_0^\mu(\tau)$. In the general case in a multiply connected spacetime, τ_1 does not coincide with τ_0 and the gap $\Delta_1(\tau_0) \equiv \tau_0 - \tau_1$ obeys the equation[6]

$$\frac{d\Delta_1(\tau_0)}{d\tau_0} = 1 - \exp(I^i[w]) . \tag{2.12}$$

This relation means that the time gap for the clock synchronization along any closed path passing through the i wormhole is growing with time and the period $I^i[w]$ may be considered as a measure of this growing. When this gap becomes greater than the time needed for a light ray to propagate along the closed path passing through the wormhole and to return back, closed timelike curves become possible and the so-called "time machine" arises. The existence of nonpotential gravitational fields with $I^i[w] \neq 0$ is a quite general property of nonsimply connected spacetimes. That is why locally static wormholes are generally unstable with respect to their transformation into a time machine. (For more details, see Ref. 6.)

In any simple connected region U of spacetime, the closed form w is exact,

$$w = d\varphi , \tag{2.13}$$

where φ allows the representation

$$\varphi(X)=\int_{X_0}^{X} w_\mu dX^\mu+\varphi_0 . \qquad (2.14)$$

Here the integral is taken over any path connecting X_0^μ and X^μ and lying in U. The quantity φ may be considered as a gravitational potential. One can easily see that

$$\xi^\mu \equiv \exp(\varphi) u^\mu \qquad (2.15)$$

is a Killing vector field in U normalized by the condition

$$|\xi^2(X_0)|^{1/2}=\exp(\varphi_0) . \qquad (2.16)$$

It is also easy to verify that the 1-form

$$\eta=\eta_\mu dX^\mu \equiv \exp(-\varphi) u_\mu dX^\mu \qquad (2.17)$$

is closed,

$$d\eta=0 , \qquad (2.18)$$

and hence in the region U one has

$$\eta=dt . \qquad (2.19)$$

The surface $t=$const is the set of events in U that are simultaneous to one another in the reference frame of Killing observers. The spacetime metric in U can be written in the standard form $[X^\mu=(t,\mathbf{x})]$

$$ds^2=-\alpha^2(\mathbf{x})dt^2+dh^2 , \qquad (2.20)$$

where

$$dh^2=h_{ij}(\mathbf{x})dx^i dx^j \qquad (2.21)$$

is a three-dimensional metric on W_N and

$$\alpha(\mathbf{x})=|\xi^2(\mathbf{x})|^{1/2}=e^{\varphi} \qquad (2.22)$$

is a redshift factor. In order to fix the norm of ξ in U one may choose $\alpha(\mathbf{x}_0)=1$. For this choice the time t in U coincides with the proper times of the Killing observer at a point $\mathbf{x}_0$.

The gravitational potential φ cannot be defined globally on M as a single-valued function if there is a generator C^i for which the period $I^i[w]$ does not vanish. Such a gravitational field is called nonpotential. One of the important properties of a locally static spacetime with a nonpotential gravitational field is that the work done by this field over a particle which passes through the wormhole and returns to the initial point does not vanish. For a generator C^i this work is proportional to

$$A_i=\exp(-I^i[w]) . \qquad (2.23)$$

It is evident that the standard form of the line element (2.20) cannot be used globally in M unless one considers α and t as multivalued functions. There exists another very helpful possibility to use the so-called universal covering space. We shall use the following results concerning multiply connected manifolds (see, e.g., Ref. 21). For any given connected manifold M there exists a unique (up to isomorphism) universal covering manifold $\widetilde{M}$. This manifold is a principle bundle over M with a group $\pi_1(M)$ and a projection $p:\widetilde{M}\rightarrow M$. The manifold $\widetilde{M}$ is simply connected and it allows the following realization. Choose in M an initial point X_0 and some final point X and consider a path $C(X_0;X)$ connecting these two points. The class of all paths connecting X_0 with X and homotopic to $C(X_0;X)$ may be considered as a point of $\widetilde{M}$. The operator p projects this class of homotopic paths to their common end point. If M is a (pseudo-) Riemannian space with a metric $g_{\mu\nu}$ then the universal covering space $\widetilde{M}$ can be naturally supplied with a metric $\widetilde{g}_{\mu\nu}$ by assuming that the interval $d\widetilde{s}^2$ between two nearby paths connecting X_0^μ with Y^μ and with $Y^\mu+dY^\mu$ coincides with the interval ds^2 between Y^μ and $Y^\mu+dY^\mu$ in M.

Consider now a locally static spacetime M. It is evident that if $M=T\times W$ then $\widetilde{M}=T\times\widetilde{W}$, where $\widetilde{W}$ is a universal covering space for W. For a given initial point X_0, the value of $\varphi(X)$ determined by Eq. (2.14) depends only on the homology class of the paths connecting X_0 and X and hence φ (as well as ξ^μ and α) is a well-defined single-valued function on the universal covering space $\widetilde{M}$. The closed form η is exact in a simply connected space $\widetilde{M}$ and hence there exist a global time t in it.

The standard theory says that the fundamental group $\pi_1(M)$ (which from now on we call Γ) acts as a discrete group of isometries on $\widetilde{M}=T\times\widetilde{W}$. The action of this group on T is trivial (it leaves the points of T unchanged) while it produces nontrivial transformations of $\widetilde{W}$. An arbitrary element γ of the fundamental group $\pi_1(W)$ can be written in the form $\gamma=\gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_k}$ where γ^i $(i=1,2,\ldots,N)$ are the generators of this group. If $n=\{n_1,n_2,\ldots,n_N\}$ is a winding number corresponding to γ, then one has

$$\alpha(\gamma X)=A_1^{-n_1}A_2^{-n_2}\cdots A_N^{-n_N}\alpha(X) , \qquad (2.23a)$$

$$t(\gamma X)=A_1^{n_1}A_2^{n_2}\cdots A_N^{n_N}t(X) , \qquad (2.23b)$$

$$h_{i'j'}(\gamma X)=\frac{\partial X^i}{\partial X'^{i'}}\frac{\partial X^j}{\partial X'^{j'}}h_{ij}(X) . \qquad (2.23c)$$

Here $X'=\gamma X$ is the result of the action of γ on a point X of $\widetilde{M}$.

B. Quantum field theory in a multiply connected locally static spacetime

The quantum field theory in spacetimes carrying nontrivial topology has been investigated for many reasons. A quite general approach to these theories was proposed by Schulman,[12] Dowker,[13] and by Dowker and Banach[14–16] in which the nontrivial fundamental group of spacetime is used to pull back the field theory onto the universal covering spacetime manifold. In this section we combine some results of this approach and use them to study quantum effects in locally static spacetimes with wormholes.

For simplicity we restrict ourselves by considering the simplest possible case of scalar massless field Φ described by the Lagrangian

$$L=-\tfrac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi-\tfrac{1}{2}\xi R\Phi^2 , \qquad (2.24)$$

where R is a scalar curvature and a parameter ξ for a conformal invariant case is equal to $\frac{1}{6}$. There are two equivalent ways to quantize the field Φ in M. The most usual one is to work directly with the solutions of the field equation

$$\Box\Phi-\xi\Phi=0 \tag{2.25}$$

on M. The second possibility is to use the universal covering space $\tilde{M}$. The discrete group Γ acts on $\tilde{M}$ from the left and the quotient $\Gamma\backslash\tilde{M}$ coincides with the original spacetime M. In our particular case of a locally static spacetime there is no global Killing vector field in M, while such a field exists in $\tilde{M}$. The existence of a global Killing vector field in the universal covering space $\tilde{M}$ makes possible a natural definition of positive- and negative-frequency solutions. That is why we prefer here the covering space approach.

The basic idea of this approach is to identify M with a fundamental domain $\Gamma\backslash\tilde{M}$ and regard a field theory on M as a field theory on $\tilde{M}$ obeying certain conditions. Because the changing of the fundamental domain is a symmetry transformation and the metric $d\tilde{s}^2$ on $\tilde{M}$ is invariant under these transformations, the invariance of the field action implies the following symmetry property of the field Lagrangian:

$$\gamma L[\Phi(X)]=L[\Phi(\gamma X)]=L[\Phi(X)] \, . \tag{2.26}$$

The Lagrangian is quadratic in Φ so that the field transformations are of the form

$$\Phi(\gamma X)=a(\gamma)\Phi(X) \, , \tag{2.27}$$

where $a^2(\gamma)=1$. The group property of Γ gives

$$a(\gamma_1\gamma_2)=a(\gamma_1)a(\gamma_2) \, . \tag{2.28}$$

[For the sake of simplicity we assume that $a(\gamma)$ does not depend on a point X.] The fields Φ on $\tilde{M}$ obeying Eq. (2.27) are known as automorphic fields. They can be naturally obtained from a given field $\Phi(X)$ by the projection operator $p^*:\Phi\rightarrow\vec{\Phi}$, where

$$\vec{\Phi}(X)=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}a(\gamma^{-1})\Phi(\gamma X) \, . \tag{2.29}$$

It should be stressed that the group Γ is infinite and hence the procedure of averaging over Γ must be specified in a more accurate way. (For necessary details, see Refs. 15 and 16.)

It is convenient to introduce the following scalar product $\langle\langle u_1,u_2\rangle\rangle$ in the space of complex-valued solutions of Eq. (2.25) in the universal covering space

$$\langle\langle u_1,u_2\rangle\rangle=i\int_\Sigma B_\mu[u_1,u_2;X]d\sigma^\mu \, , \tag{2.30a}$$

$$B_\mu[u_1,u_2;X]\equiv u_1(X)\partial_\mu\bar{u}_2(X)-\bar{u}_2(X)\partial_\mu u_1(X) \, , \tag{2.30b}$$

where $d\sigma^\mu$ is a surface element of a Cauchy surface Σ in $\tilde{M}$. The value of this scalar product does not depend on the particular choice of Σ. In particular one may choose the surface Σ defined by the equation $t=0$. This choice is especially convenient for the calculations because Σ_0 is invariant under the action of Γ. We define a positive-frequency solution of Eq. (2.25) as a solution which allows the representation

$$u(X)\equiv u(t,\mathbf{x})=\int_0^\infty\frac{d\omega}{\omega}e^{-i\omega t}u_\omega(\mathbf{x}) \, , \tag{2.31}$$

where t and $\mathbf{x}$ are the coordinates on $\tilde{M}$ in which the metric has the form (2.20) and $u_\omega(\mathbf{x})$ is a solution of the equation

$$[\alpha^{-1}h^{-1/2}\partial_i(\alpha h^{1/2}h^{ij}\partial_j)+\alpha^{-2}\omega^2]u_\omega(\mathbf{x})=0 \, . \tag{2.32}$$

The set of positive-frequency solutions with the scalar product (2.30) forms a Hilbert space which we denote $H_{\tilde{M}}$.

By using Eqs. (2.23) and the definition (2.31) of a positive-frequency solution $u(X)$, one can verify that $\gamma u(X)$ is again a positive-frequency solution and hence γ is an operator in $H_{\tilde{M}}$. Moreover, it is easy to show that

$$\langle\langle u_1,u_2\rangle\rangle=\langle\langle\gamma u_1,\gamma u_2\rangle\rangle \tag{2.33}$$

and hence γ is a unitary operator in $H_{\tilde{M}}$.

Let U_j be an orthonormal basis in $H_{\tilde{M}}$ then the quantum field Φ in the universal covering space can be written in the form

$$\Phi(X)=\sum_J[U_J(X)\mathbf{b}^J+\bar{U}_J(X)\mathbf{b}^{*J}] \tag{2.34}$$

and the vacuum $|0\rangle\rangle$ in $\tilde{M}$ is defined by the relation $\mathbf{b}^i|0\rangle\rangle=0$. The Hadamard function $\tilde{G}^1(X,X')$ in the universal covering space is given by

$$\begin{aligned}\tilde{G}^1(X,X')&\equiv\langle\langle0|\Phi(X)\Phi(X')+\Phi(X')\Phi(X)|0\rangle\rangle\\&=\sum_J[U_J(X)\bar{U}_J(X')+U_J(X')\bar{U}_J(X)] \, .\end{aligned} \tag{2.35}$$

For automorphic solutions (2.27) it is more convenient to use another scalar product

$$\langle u_1,u_2\rangle\equiv i\int_{\Sigma_M}B_\mu[u_1,u_2;X]d\sigma^\mu \, . \tag{2.36}$$

Here u_1 and u_2 are automorphic solutions and Σ_M is the intersection $\Sigma\cap(\Gamma\backslash\tilde{M})$ of a Cauchy surface Σ invariant under the action of Γ with the fundamental domain $\Gamma\backslash\tilde{M}$. We denote by H_M the Hilbert space of automorphic positive-frequency solutions with the scalar product (2.36). If u_i is an orthonormal basis in H_M then the quantum field Φ in M allows the decomposition

$$\Phi(X)=\sum_i[u_i(X)\mathbf{a}^i+\bar{u}_i(X)\mathbf{a}^{*i}] \, . \tag{2.37}$$

One can choose a state $|0\rangle$ defined by the relations $\mathbf{a}^i|0\rangle=0$ as a natural vacuum state in M. The corresponding Hadamard function $G^1(X,X')$ is

$$\begin{aligned}G^1(X,X')&\equiv\langle0|\Phi(X)\Phi(X')+\Phi(X')\Phi(X)|0\rangle\\&=\sum_i[u_i(X)\bar{u}_i(X')+u_i(X')\bar{u}_i(X)] \, .\end{aligned} \tag{2.38}$$

The Hadamard function in the physical spacetime M is connected with the Hadamard function in the universal covering space $\tilde{M}$ by the relation[15,16]

$$G^1(X,X')=\sum_{\gamma\in\Gamma}a(\gamma^{-1})\tilde{G}^1(X,\gamma X') \, . \tag{2.39}$$

The renormalized stress-energy tensor $\langle T_{\mu\nu}\rangle^{\text{ren}}$ in a physical spacetime can be obtained by applying the operator

$$D_{\mu\nu}=\tfrac{1}{2}(\tfrac{1}{2}-\xi)(\nabla_{\mu'}\nabla_{\nu}+\nabla_{\mu}\nabla_{\nu'})+(\xi-\tfrac{1}{4})g_{\mu\nu}\nabla_{\rho}\nabla^{\rho'}$$
$$-\tfrac{1}{2}\xi(\nabla_{\mu}\nabla_{\nu}+\nabla_{\mu'}\nabla_{\nu'})+\tfrac{1}{8}\xi g_{\mu\nu}(\nabla_{\rho}\nabla^{\rho}+\nabla_{\rho'}\nabla^{\rho'})$$
$$+\tfrac{1}{2}\xi[R_{\mu\nu}-\tfrac{1}{2}(1-3\xi)g_{\mu\nu}R] \qquad (2.40)$$

to $G^{1}(X,X')$ and taking the coincidence limit after subtracting the necessary divergences.[22–24] In what follows we consider a nontwisting scalar field and put $a(\gamma)=1$.

III. QUANTUM EFFECTS IN A TWO-DIMENSIONAL SPACETIME WITH A TIME MACHINE

A. "Standard model" of a two-dimensional locally static spacetime

In order to illustrate the main features of the above described general approach to the quantum field theory in a locally static multiply connected spacetime and to study the vacuum polarization in a spacetime with a time machine we consider here a two-dimensional model which allows a complete and detailed investigation. Namely, we consider a locally static two-dimensional spacetime $\overline{M}=T\times W$, which is formed by W-set of Killing trajectories. We assume that $W=S^{1}$. The proper distance coordinate along W (which we denote by l) changes from $l=0$ to $l=L$, the boundary points 0 and L being considered identical. The velocity vector of a Killing observer u^{μ} is tangent to T while its acceleration w^{μ} is orthogonal to T and can be written in the form

$$w_{\mu}=-w(l)\delta^{l}_{\mu}\,, \qquad (3.1a)$$

$$w(0)=w(L)\,. \qquad (3.1b)$$

To make consideration more concrete we assume that $w(l)\geq 0$.

For $\overline{M}$ one has

$$\pi_1(\overline{M})=H_1(\overline{M})=\mathbb{Z}\,, \qquad (3.2)$$

the closed path C with the winding number $n=1$ being the generator of both groups. The period $I[w]$ for this generator reads

$$I[w]=\oint_{C}w_{\mu}dX^{\mu}=-\int_{0}^{L}w(l)dl\,. \qquad (3.3)$$

The spacetime $\overline{M}$ can be obtained from the manifold $\overline{M}'=T\times[0,L]$ by identifying its boundaries: γ^{-}, $l=0$; and γ^{+}, $l=L$. The spacetime metric $d\overline{s}^{2}$ on $\overline{M}'$ reads

$$d\overline{s}^{2}=-\alpha^{2}(l)dt^{2}+dl^{2}\,, \qquad (3.4)$$

where

$$\alpha(l)=\exp\left[-\int_{0}^{l}w(l')dl'\right]. \qquad (3.5)$$

The regularity of the spacetime $\overline{M}$ requires that the internal metrics and external curvatures at both lines γ^{-} and γ^{+} are identical. The identity of the external curvatures is guaranteed by Eq. (3.1b), while the other condition means that the time parameters of the points $(t,0)$ of γ^{-} and (t',L) of γ^{+}, which are to be identified, are related as follows:

$$t'=At,\quad A\equiv\alpha(0)/\alpha(L)=\exp(-I[w])\geq 1\,. \qquad (3.6)$$

Our aim is studying quantum effects and, in particular, the vacuum polarization in a locally static spacetime. In two-dimensional spacetime the local contribution to the vacuum polarization can be separated from the part which depends only on the global spacetime structure. In order to show this we rewrite metric (3.4) in the form

$$d\overline{s}^{2}=b^{2}(q)ds^{2}\,, \qquad (3.7a)$$

$$ds^{2}=-a^{2}(q)dt^{2}+dq^{2}\,. \qquad (3.7b)$$

By comparing Eqs. (3.7) with Eq. (3.4) one gets

$$b(q)a(q)=\alpha(l),\quad b(q)dq=dl\,. \qquad (3.8)$$

Denote by Q the value of q which corresponds to $l=L$:

$$Q=\int_{0}^{L}\frac{dl}{b}\,. \qquad (3.9)$$

We call Eq. (3.7) the canonical form of the spacetime metric if the following conditions are satisfied:

$$b(0)=b(Q)=1\,, \qquad (3.10a)$$

$$a^{-1}\frac{da}{dq}=-W=\text{const}\,. \qquad (3.10b)$$

The first condition guarantees that the period $I[w]$ for the metric ds^{2} is the same as for the metric $d\overline{s}^{2}$, while the second condition means that the acceleration of Killing observers is constant in a locally static spacetime M with the metric ds^{2}. By using Eqs. (3.8)–(3.10) one gets

$$a(q)=\exp(-Wq)\,, \qquad (3.11a)$$

$$W^{-1}=(A-1)^{-1}\int_{0}^{L}\alpha^{-1}(l)dl\,, \qquad (3.11b)$$

$$Q=W^{-1}\ln A\,, \qquad (3.11c)$$

$$b(q)=\alpha(l)\left[1+W\int_{0}^{l}\alpha^{-1}(l')dl'\right]. \qquad (3.11d)$$

Equations (3.1b) and (3.10a) guarantee that the identification of lines $q=0$ and $q=Q$ does not create any discontinuities in the metric ds^{2} on M. We refer to the spacetime M with the metric (3.7b), (3.11a) and with the identification rule $(t,0)\leftrightarrow(At,Q)$ of its boundaries as the standard model. It should be noted that not only the topologies but also the causal structures of $\overline{M}$ and of the standard-model M are identical. Denote

$$u=\int_{0}^{l}\frac{dl}{\alpha}-t+W^{-1}\,, \qquad (3.12a)$$

$$v=\int_{0}^{l}\frac{dl}{\alpha}+t+W^{-1}\,. \qquad (3.12b)$$

The equations $u=\text{const}$ and $v=\text{const}$ determine null geodesics in both conformally related spaces $\overline{M}$ and M. The lines $u=0$ and $v=0$ are closed null geodesics, which form Cauchy horizons H_{+} and H_{-} correspondingly. In the region $R_{+}:uv>0$ lying between the Cauchy horizons

H_+ and H_- there are no closed causal curves. The closed timelike lines are possible in the regions lying beyond R_+.

Consider now a conformal massless scalar field Φ obeying the equation

$$\Box\Phi=0 \ . \tag{3.13}$$

The renormalized vacuum expectation values $\langle T_{\mu\nu}\rangle^{\rm ren}$ in two conformally related spaces $\bar{M}$ and M are related as follows:[24,25]

$$\langle \bar{T}_{\mu\nu}\rangle^{\rm ren}=\langle T_{\mu\mu}\rangle^{\rm ren}+T_{\mu\nu} \ , \tag{3.14a}$$

$$T_{\mu\nu}=\frac{1}{24\pi}\left[2\frac{b_{;\mu\nu}}{b}-4\frac{b_{;\mu}b_{;\nu}}{b^2}+g_{\mu\nu}\left[-2\frac{b_{;\epsilon}{}^{;\epsilon}}{b}+3\frac{b^{;\epsilon}_{;\epsilon b}}{b^2}\right]\right] . \tag{3.14b}$$

The quantity on the left-hand side of Eq. (3.14a) is referred to the spacetime $\bar{M}$ while the quantities standing on the right-hand side of this equation are to be calculated in the spacetime M of a standard model. If $G^1(X,X')$ is the corresponding Hadamard function in M, then (see, e.g., Refs. 22 and 23)

$$\langle T_{\mu\nu}\rangle^{\rm ren}=\lim_{X'\to X}D_{\mu\nu}G^{1,{\rm ren}}(X,X') \ , \tag{3.15a}$$

where

$$D_{\mu\nu}=\tfrac{1}{4}(\nabla_\mu\nabla_{\nu'}+\nabla_{\mu'}\nabla_\nu-g_{\mu\nu}g^{\sigma\rho'}\nabla_\sigma\nabla_{\rho'}) \ . \tag{3.15b}$$

The function $b(X)$ is regular in M; that is why the tensor $t_{\mu\nu}$ is also regular. Its nonvanishing components in (t,q) coordinates are

$$T_t^t=\frac{1}{24\pi}\left[-2\frac{b''}{b}+3\frac{b'^2}{b^2}\right] , \tag{3.16a}$$

$$T_q^q=\frac{1}{24\pi}\left[-\frac{b'^2}{b^2}+2W\frac{b'}{b}\right] . \tag{3.16b}$$

where a prime denotes d/dq. The regular contribution $T_{\mu\nu}$ is local and does not depend on the topological and causal structure of the spacetime. That is why for studying the quantum effects connected with a time-machine formation it is sufficient to calculate $\langle T_{\mu\nu}\rangle^{\rm ren}$ in the standard model M. In particular it will be shown that these quantities are divergent at the Cauchy horizons. This result means immediately that the analogous quantities are divergent at the Cauchy horizons in any locally static spacetime.

B. Vacuum polarization in a standard model

For the calculations of the vacuum polarization in a standard model we use the covering space approach. The universal covering space $\tilde{M}$ for the standard model is a spacetime with the metric

$$ds^2=-e^{-2Wq}dt^2+dq^2 \ , \tag{3.17}$$

where $t\in(-\infty,\infty)$, $q\in(-\infty,\infty)$. By using the dimensionless coordinates

$$\eta=Wt, \quad \xi=\exp(Wq) \ , \tag{3.18}$$

this metric can be rewritten in the form

$$ds^2=W^{-2}d\sigma^2 \ , \tag{3.19}$$

$$d\sigma=\xi^{-2}(-d\eta^2+d\xi^2) \ . \tag{3.20}$$

The action of $\gamma_n\equiv(\gamma)^n$, where γ is a generator of the fundamental group Γ, on $\tilde{M}$ is described by the relations

$$\gamma_n\eta=A^n\eta, \quad \gamma_n\xi=A^n\xi \ , \tag{3.21}$$

the strip $\xi\in(1,A)$, $\eta\in(-\infty,\infty)$ being a fundamental domain. The spacetime with the metric (3.20) is a homogeneous space of constant curvature

$$R_{\mu\nu\alpha\beta}=-(g_{\mu\alpha}g_{\nu\beta}-g_{\mu\beta}g_{\nu\alpha}) \ , \tag{3.22}$$

which is isometric to the surface of a two-dimensional hyperboloid described by the equation $X^2+Y^2-Z^2-1=0$ in a space with a metric $dS^2=dX^2+dY^2-dZ^2$. Equation (3.15a) shows that the renormalized vacuum expectation value of the stress-energy tensor remains unchanged under the conformal rescaling (3.19) of a metric provided that $W=$const, and we can calculate $\langle T_{\mu\nu}\rangle^{\rm ren}$ directly in the spacetime with the metric (3.20).

The field equation (3.13) in null coordinates

$$\zeta_-\equiv Wu=\xi-t \ , \quad \zeta_+\equiv Wv=\xi+t \tag{3.23}$$

reads

$$\partial_{+-}\Phi\equiv\partial_+\partial_-\Phi=0 \ , \tag{3.24}$$

where $\partial_\pm=\partial/\partial\zeta_\pm$. The general solution of this equation is

$$\Phi=F_+(\zeta_+)+F_-(\zeta_-) \ , \tag{3.25}$$

$F_\pm$ being two arbitrary functions. The scalar product (2.30) in the space of complex solutions of Eq. (3.24) can be written in the form

$$\langle\langle u_1,u_2\rangle\rangle=-i\int_{\substack{\eta={\rm const}\\ \xi>0}}(u_1\dot{\bar{u}}_2-\dot{u}_1\bar{u}_2)d\xi \ , \tag{3.26}$$

where an overdot denotes $d/d\eta$. This scalar product is conserved provided the following boundary condition is imposed at $\xi=0$:

$$\left[a_1u+a_2\frac{du}{d\xi}\right]_{\xi=0}=0 \ , \quad {\rm Im}(a_2/a_1)=0 \ . \tag{3.27}$$

It is easy to show that regular solutions of massive scalar field equations vanish at $\xi=0$ for arbitrarily small value of mass. Having this in mind we chose $a_2=0$. A general positive-frequency solution in $\tilde{M}$ obeying this boundary conditions can be written in the form

$$u(\eta,x)=\int_0^\infty\frac{d\omega}{\omega}a(\omega)[\exp(i\omega\zeta_-)-\exp(-i\omega\zeta_+)] \ . \tag{3.28}$$

The positive-frequency solutions

$$U_\omega=(4\pi\omega)^{-1/2}[\exp(i\omega\zeta_-)-\exp(-i\omega\zeta_+)] \quad (3.29)$$

obeying the normalization conditions

$$\langle\langle U_\omega, U_{\omega'}\rangle\rangle=\delta(\omega-\omega') \quad (3.30)$$

form a basis in $H_{\tilde{M}}$. The Hadamard function $\tilde{G}^1$ in $\tilde{M}$ defined by Eq. (2.38) is

$$\tilde{G}^1(X,X')=-\frac{1}{4\pi}\ln\left[\frac{(\zeta_- -\zeta'_-)^2(\zeta_+ -\zeta'_+)^2}{(\zeta_+ +\zeta'_-)^2(\zeta_- +\zeta'_+)^2}\right] . \quad (3.31)$$

Positive-frequency solutions (3.28) are automorphic provided

$$a(\omega)=a(A\omega) . \quad (3.32)$$

The general solution of this functional equation is

$$a(\omega)\equiv\sum_n c_n\omega^{-2\pi i\beta n} , \quad (3.33)$$

where $\beta=(\ln A)^{-1}$, c_n are constants, and the summation is taken over all integer numbers n. By using the relations

$$\int_0^\infty \frac{d\omega}{\omega}\exp(\pm i\omega\zeta)=-\gamma\pm i\pi/2-\ln(\zeta\pm i0) , \quad (3.34a)$$

$$\int_0^\infty \frac{d\omega}{\omega}\omega^{2\pi i\beta n}\exp(\pm i\omega\zeta) =\pm ie^{\pm i\pi/2}e^{\mp\pi^2\beta n}\Gamma(2\pi i\beta n)(\zeta\pm i0)^{-2\pi i\beta n} \quad (3.34b)$$

one can show that a positive-frequency automorphic solution allows the representation

$$u=\sum_n \tilde{c}_n u_n , \quad (3.35)$$

where

$$u_0=b_0\{\ln[(\zeta_+ -i0)/(\zeta_- +i0)]+i\pi\} , \quad (3.36a)$$

$$u_n=b_n[e^{\pi^2\beta n}(\zeta_+ -i0)^{-2\pi i\beta n} -e^{-\pi^2\beta n}(\zeta_- +i0)^{-2\pi i\beta n}] . \quad (3.36b)$$

As the scalar product (2.36) in M does not depend on the particular choice of the Cauchy surface in the fundamental domain, we may choose the section $\eta=0$ as such a surface and write

$$\langle u^1,u^2\rangle=-i\int_1^A d\xi(u^1\dot{\bar{u}}^2-\dot{u}^1\bar{u}^2)_{\eta=0} . \quad (3.37)$$

The solutions (3.36) form an orthonormal basis in H_M:

$$\langle u_n,u_m\rangle=\delta_{nm} \quad (3.38)$$

provided

$$b_0=(\beta/4\pi)^{1/2},\quad b_n=[8\pi n\sinh(\chi n)]^{-1/2} , \quad (3.39)$$

where $\chi\equiv 2\pi^2\beta$. The Hadamard function G^1 defined in M by Eq. (2.38) is

$$G^1(X,X')=b_0^2[\ln(\zeta_+/\zeta_-)+i\pi][\ln(\bar{\zeta}'_+/\bar{\zeta}'_-)-i\pi] + \sum_n{}' b_n^2[e^{\chi n}(\bar{\zeta}'_+/\zeta_+)^{i\mu n}+e^{-\chi n}(\bar{\zeta}'_-/\zeta_-)^{i\mu n}-(\bar{\zeta}'_+/\zeta_-)^{i\mu n}-(\bar{\zeta}'_-/\zeta_+)^{i\mu n}]+(\text{complex conjugate}) , \quad (3.40)$$

where $\mu=2\pi\beta$, $\zeta_\pm\equiv\zeta_\pm\mp i0$, $\bar{\zeta}_\pm\equiv\zeta_\pm\mp i0$, and the prime indicates that the summation is taken over all integer numbers n excluding $n=0$. For points X and X' lying in R_+ where $\zeta_\pm>0$ and $\zeta'_\pm>0$, the distinction between $\zeta_\pm$, $\zeta'_\pm$ and their complex conjugate is not important. The Hadamard function for this case can be rewritten as

$$G^1(X,X')=\frac{\beta}{2\pi}[\ln(\zeta'_+/\zeta'_-)\ln(\zeta_+/\zeta_-)+\pi^2]+\frac{1}{2\pi}\sum_{n=1}^\infty\frac{1}{n}\coth(\chi n)\{\cos[\mu n\ln(\zeta'_+/\zeta_+)]+\cos[\mu n\ln(\zeta'_-/\zeta_-)]\} -\frac{1}{2\pi}\sum_{n=1}^\infty\frac{1}{n\sinh(\chi n)}\{\cos[\mu n\ln(\zeta'_+/\zeta_-)]+\cos[\mu n\ln(\zeta'_-/\zeta_+)]\} .$$

The summation in this expression can be done by using the relations (see Ref. 26)

$$\sum_{n=1}^\infty\frac{1}{n}\cos(2yn)=-\ln(2\sin y) , \quad (3.41a)$$

$$\sum_{n=1}^\infty\frac{\kappa^n}{n(1-\kappa^{2n})}\cos(2yn)=\tfrac{1}{2}[\ln Q_0-\ln\theta_4(y/\pi)] , \quad (3.41b)$$

$$\sum_{n=1}^\infty\frac{\kappa^{2n}}{n(1-\kappa^{2n})}\cos(2yn)=\tfrac{1}{2}[\ln(2Q_0)+\tfrac{1}{4}\ln\kappa+\ln\sin(y)-\ln\theta_1(y/\pi)] , \quad (3.41c)$$

where $\theta_i(z)\equiv\theta_i(z,\kappa)\equiv\theta_i(z|\tau)$ are the Jacobi functions and

$$Q_0=\prod_{n=1}^\infty(1-\kappa^{2n}),\quad \kappa=\exp(2\pi i\tau) .$$

Finally, we get for the Hadamard function $G^1(X,X')$ the explicit expression

$$G^1(X,X')=\frac{1}{2\pi}\left[\beta\ln(\zeta'_+/\zeta'_-)\ln(\zeta_+/\zeta_-)-\ln\left[\frac{\theta_1[\beta\ln(\zeta'_+/\zeta_+)]\theta_1[\beta\ln(\zeta'_-/\zeta_-)]}{\theta_4[\beta\ln(\zeta'_+/\zeta_-)]\theta_4[\beta\ln(\zeta'_-/\zeta_+)]}\right]\right], \tag{3.42a}$$

where $\theta_i(z)\equiv\theta_i(z,\kappa)$ and $\kappa=\exp(-2\pi^2\beta)$. It is worthwhile noting that this expression for the Hadamard function can be also obtained by using the representation (2.39), which in our case has the form

$$G^1(X,X')=-\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\ln\left[\frac{(\zeta_- - A^n\zeta'_-)^2(\zeta_+ - A^n\zeta'_+)^2}{(\zeta_+ + A^n\zeta'_-)^2(\zeta_- + A^n\zeta'_+)^2}\right]. \tag{3.42b}$$

The proof of the identity of the expressions (3.42a) and (3.42b) is given in the Appendix.

In order to calculate the renormalized stress-energy tensor one can use the standard point-splitting method.[22–24] But in our case due to the high symmetry of the universal covering space it is possible to simplify the calculations. The space M is locally isometric to $\tilde{M}$. Thus the terms which are to be subtracted in order to renormalize the vacuum expectation value of the stress-energy tensor are the same in the both spaces and we can write

$$\langle T_{\mu\nu}\rangle_M^{\rm ren}=T^1_{\mu\nu}+\langle T_{\mu\nu}\rangle_{\tilde{M}}^{\rm ren}, \tag{3.43}$$

where $\langle T_{\mu\nu}\rangle_M^{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\tilde{M}}^{\rm ren}$ are the renormalized stress-energy tensors in M and $\tilde{M}$ correspondingly and

$$T^1_{\mu\nu}=\lim_{x'\to x}D_{\mu\nu}[G^1(X,X')-\tilde{G}^1(X,X')]. \tag{3.44}$$

The expression in the square brackets on the right-hand side of (3.44) is a regular function. It is easy to see that $T^1_{+-}=T^1_{-+}=0$ while the nonvanishing components T^1_{++} and T^1_{--} are

$$T^1_{\pm}=\lim_{x'\to x}\partial_\pm\partial'_\pm[G^1(X,X')-\tilde{G}^1(X,X')]. \tag{3.45}$$

The calculations give

$$\partial_\pm\partial'_\pm\tilde{G}^1(X,X')=-\frac{1}{2\pi}\frac{1}{(\zeta'_\pm-\zeta_\pm)^2}, \tag{3.46a}$$

$$\partial_\pm\partial'_\pm G^1(X,X')=\frac{\beta}{2\pi\zeta_\pm\zeta'_\pm}+\frac{\beta^2}{2\pi\zeta_\pm\zeta'_\pm}\left[-\frac{\pi^2}{\sin^2\pi z_\pm}+8\pi^2\sum_{n=1}^{\infty}\frac{n\kappa^{2n}}{1-\kappa^{2n}}\cos(2\pi n z_\pm)\right], \tag{3.46b}$$

where $z_\pm=\beta\ln(\zeta'_\pm/\zeta_\pm)$. By using these relations we get

$$T^1_\pm=-\frac{1}{\zeta_\pm^2}F(\beta), \tag{3.47a}$$

where

$$F(\beta)=\frac{1}{48\pi}-\frac{\beta}{4\pi}+\frac{\beta^2\pi}{12}-2\pi\beta^2\sum_{n=1}^{\infty}\frac{n\exp(-4\pi^2\beta n)}{1-\exp(-4\pi^2\beta n)}. \tag{3.47b}$$

The series in (3.47b) is slowly convergent for small values of β. In the Appendix it is shown that $F(\beta)$ allows another representation that is especially useful for small β:

$$F(\beta)=\frac{1}{8\pi}\sum_{n=1}^{\infty}\frac{1}{\sinh^2(n/2\beta)}. \tag{3.47c}$$

The quantity $\langle T_{\mu\nu}\rangle_{\tilde{M}}^{\rm ren}$ can be calculated by using Eq. (3.14b) for the conformal transformation (3.20), relating $\tilde{M}$ with a flat half-plane $x>0$ and by taking into account that the renormalized stress-energy tensor in the latter space vanishes. Simple calculations give

$$\langle T_{\mu\nu}\rangle_{\tilde{M}}^{\rm ren}=-\frac{1}{24\pi}g_{\mu\nu}, \tag{3.48}$$

where $g_{\mu\nu}$ is a metric in $\tilde{M}$. Combining these results we finally get

$$\langle T_{\mu\nu}\rangle_M^{\rm ren}=-\left[\frac{1}{\zeta_+^2}\delta_\mu^+\delta_\nu^+ +\frac{1}{\zeta_-^2}\delta_\mu^-\delta_\nu^-\right]F(\beta)-\frac{1}{24\pi}g_{\mu\nu}. \tag{3.49}$$

The first term of this expression describes null fluxes and is traceless, while the second term has the trace which correctly reproduces the conformal trace anomaly in M:

$$\langle T^\mu_\mu\rangle_M^{\rm ren}=-\frac{1}{12\pi}\equiv\frac{1}{24\pi}R. \tag{3.50}$$

The leading term of $\langle T_{\mu\nu}\rangle_M^{\rm ren}$ near the Cauchy horizon $H_+:\zeta_-=0$ is

$$\langle T_{\mu\nu}\rangle_M^{\rm ren}=-\frac{F(\beta)}{u^2}k_\mu k_\nu,\quad k_\mu=\nabla_\mu u. \tag{3.51}$$

This relation shows that near H_+ there exists an infinitely growing flux of negative energy density which is propagating in the direction of the gravitational potential decrease. This conclusion is evidently valid not only for a standard model but also for any two-dimensional locally static spacetime with a nonpotential gravitational field. This result can be interpreted as an evidence of possible quantum instability of the Cauchy horizon in such spaces. In the next section we argue that this result is also valid in a more general situation when the metric is time dependent, provided a time machine is formed in such a space.

Before considering this more general case we make some remarks concerning the weak-field limit of $\langle T_{\mu\nu}\rangle_M^{\rm ren}$. Assume that the nonpotential gravitational field is weak ($\delta \equiv A-1 \ll 1$); then we have $W \simeq \delta/L$ and $\beta \equiv (\ln A)^{-1} \simeq \delta^{-1}$. In this limit, the term of (3.47b) that contains a series is of the order $\delta^{-2}\exp(-4\pi^2/\delta)$, and it can be neglected. Thus we have $F(\beta) \simeq \pi^2/12\delta^2$. For fixed values of l and t, one has $u \simeq v \simeq L/\delta$, and hence up to the terms which vanish in the $\delta \to 0$ limit one has

$$\langle T_{uu}\rangle_M^{\rm ren} = \langle T_{vv}\rangle_M^{\rm ren} = -\frac{\pi}{12L^2}\ ,$$
$$\langle T_{uv}\rangle_M^{\rm ren} = \langle T_{vu}\rangle_M^{\rm ren} = 0\ . \qquad (3.52)$$

These expressions correctly reproduce the value of the renormalized stress-energy tensor for the conformal massless scalar field in a two-dimensional cylindrical spacetime (see, e.g., Ref. 24). This is true not only for $\langle T_{\mu\nu}\rangle_M^{\rm ren}$ but also for the Hadamard function G^1. In order to show this, it is sufficient to note that, in the weak-field limit,

$$\beta \ln(\zeta'_+/\zeta_+) \simeq (u-u')/L,\quad \beta \ln(\zeta'_-/\zeta_-) \simeq (v-v')/L\ , \qquad (3.53)$$

and the θ functions $\theta_i(z,\kappa)$ have the following asymptotic behavior for small values of κ:

$$\theta_1(z,\kappa) \simeq 2\kappa^{1/4}\sin(\pi z)\ ,\quad \theta_4(z,\kappa) \simeq 1\ .$$

By using these relations we obtain the following expression for the Hadamard function in the weak-field approximation:

$$G^1(X,X') \simeq -(4\pi)^{-1}\ln\{16\sin^2[\pi(u-u')/L] \times \sin^2[\pi(v-v')/L]\}\ .$$

C. Renormalized stress-energy tensor in a two-dimensional spacetime with a time machine

We showed that the time-machine formation in a locally static two-dimensional spacetime is accompanied by the divergence of the renormalized stress-energy tensor of a quantum field near the Cauchy horizon. The following two questions naturally arise: (i) does this result depend on a particular model we have chosen, and (ii) does it depend on a particular choice of the vacuum state we have made? In order to answer these questions we consider a more general two-dimensional model describing the time-machine formation.

Consider a two-dimensional manifold $-\infty < t < \infty$, $0 \le x \le B$, with the metric

$$ds^2 = \alpha^2(t,x)ds_0^2 = \alpha^2(t,x)(-dt^2+dx^2)\ , \qquad (3.54)$$

and consider the spacetime M which is obtained from this manifold by gluing its boundaries: γ_-, $x=0$; and γ_+, $x=B$. In order that spacetime M be regular, the following two conditions are to be satisfied: (i) the internal geometries of the lines γ_- and γ_+ must be isometric, and (ii) the external curvature of these lines must have no jumps. These conditions imply the relations

$$\alpha_-(t)dt = \alpha_+(t')dt'\ , \qquad (3.55a)$$

$$\partial_x[\alpha^{-1}(t,0)] = \partial_x[\alpha^{-1}(t',B)]\ , \qquad (3.55b)$$

where $\alpha_-(t) \equiv \alpha(t,0)$ and $\alpha_+(t) \equiv \alpha(t,B)$.

The first condition relates the time coordinate t at the line γ_- and time coordinate t' at line γ_+ for the points which are to be identified. This condition means simply that the proper time parameters $\tau_\pm$ for these points must coincide. We assume that in the remote past the spacetime was a static cylindrical one and that the gravitational field was absent so that we have, in this region,

$$\alpha(t,x) \simeq 1,\quad t' \simeq t\ . \qquad (3.56)$$

The proper time parameters $\tau_\pm$ along the lines $\gamma_\pm$ can be written in the form

$$\tau_\pm = \lim_{t_0 \to -\infty}\left[\int_{t_0}^{t}\alpha_\pm(t)dt + t_0\right]\ . \qquad (3.57)$$

The condition of the point identification $\tau_- = \tau_+$ or equivalently

$$\lim_{t_0\to-\infty}\left[\int_{t_0}^{t}\alpha_-(t)dt + t_0\right] = \lim_{t_0\to-\infty}\left[\int_{t_0}^{t'}\alpha_+(t)dt + t_0\right] \qquad (3.58)$$

allows one to relate the corresponding time parameters $t' = T(t)$. As earlier we assume that

$$A(t) \equiv \alpha_-(t)/\alpha_+(t) \ge 1\ . \qquad (3.59)$$

Under this assumption we have

$$dT/dt \equiv A(t) \ge 1,\quad \lim_{t_0\to-\infty}\frac{dT}{dt} = 1\ . \qquad (3.60)$$

The second condition of the spacetime regularity (3.55b) does not greatly restrict the class of models under consideration. For example it can be satisfied if one chooses $\alpha(t,x) = F(f(t)g(x))$, where F, f, and g are arbitrary functions, provided that $g'(0) = g'(B) = 0$.

We assume now that there exists a solution of the equation

$$T(t) = t + B\ , \qquad (3.61)$$

which we denote by t_-. This condition means that a null ray, leaving γ_- at a point p_- at the time moment t_-, enters γ_+ at a point p_+ at time $T(t_-)$. These points $p_\pm$ are to be identified so that the null ray forms a closed null geodesic in M. This geodesic coincides with the future Cauchy horizon H_+. Under the assumptions we made, there is no past Cauchy horizon H_- in M. In other words, our model describes a time-machine formation in an originally cylindrical spacetime due to a rising of a nonpotential gravitational field.

Consider a conformal massless scalar field Φ in the spacetime M. The field equation $\Box\Phi=0$ in the (t,x) coordinates takes the form

$$(-\partial_t^2+\partial_x^2)\Phi=0 , \tag{3.62}$$

while the boundary conditions read

$$\Phi(t,0)=\Phi(T(t),B) , \tag{3.63a}$$

$$\partial_x\Phi(t,0)=\partial_x\Phi(T(t),B) . \tag{3.63b}$$

Denote $z_\epsilon=t+\epsilon x$; then the general solution of Eq. (3.62) can be written as

$$\Phi(X)=f_+(z_+)+f_-(z_-) , \tag{3.64}$$

where functions $f_\pm(z_\pm)$ obey the conditions

$$f_\epsilon(z_\epsilon)=f_\epsilon[T(z_\epsilon)+\epsilon B], \quad \epsilon=\pm 1 . \tag{3.65}$$

In order to find a solution of this functional equation we consider mapping

$$F_\epsilon : z_\epsilon \to T(z_\epsilon)+\epsilon B . \tag{3.66}$$

Denote by F_ϵ^{-1} the operator which is inverse to F_ϵ and denote $F_\epsilon^n=[F_\epsilon]^n$ and $F_\epsilon^{-n}=[F_\epsilon^{-1}]^n$. Let us choose two arbitrary constants z_ϵ^0 and consider the two sets of points $z_\epsilon^n=F_\epsilon^n[z_\epsilon^0]$ numerated by an integer number n. The sequence z_+^n is monotonically increasing and $\lim_{n\to\pm\infty}z_+^n=\pm\infty$, while the sequence z_-^n is monotonically decreasing (provided $z_-^0<t_-$) and $\lim_{n\to+\infty}z_-^n=-\infty$, $\lim_{n\to-\infty}z_-^n=t_-$. We denote as R_ϵ^n the regions (strips) where $z_\epsilon^n<z_\epsilon<z_\epsilon^{n+1}$.

Consider a function $g_\epsilon(z_\epsilon)$ determined in the strip R_ϵ^0 and obeying the conditions

$$g_\epsilon(z_\epsilon^0)=g_\epsilon(z_\epsilon^1), \quad \frac{dg_\epsilon}{dz_\epsilon}(z_\epsilon^0)\frac{dF_\epsilon}{dz_\epsilon}(z_\epsilon^1)=\frac{dg_\epsilon}{dz_\epsilon}(z_\epsilon^1) , \tag{3.67}$$

and define the value of the functions $f_\epsilon(z_\epsilon)$ in a strip R_ϵ^n by the relation

$$f_\epsilon(z_\epsilon)=g_\epsilon(F_\epsilon^{-n}(z_\epsilon)) . \tag{3.68}$$

The functions $f_\epsilon(z_\epsilon)$ are smooth solutions of the functional equation (3.65) and they describe wave propagation in the left ($\epsilon=+1$) and in the right ($\epsilon=-1$) directions. For such waves it is sufficient to specify their values only in the initial strips R_ϵ^0. The values of the waves in other strips are determined due to the identification of the boundaries γ_- and γ_+.

It is convenient to put $z_-^0=z_+^0\equiv z_0$ and choose such a value of this parameter z_0 for which the initial strips $R_\pm^0$ are located in the remote past where $\alpha(t,x)\simeq 1$ and $t\simeq t'$. One can use the freedom in the choice of z_0 in order to guarantee that $H_+\subset R_+^{n_+}$ for some finite number n_+. The section Σ_0 defined by the equation $t=t_{\text{in}}=z_0$ lies in both initial strips $R_\pm^0$, and, instead of giving the initial functions $g_\epsilon(z_\epsilon)$ in these strips, one can single out a solution of Eqs. (3.62) and (3.63) by specifying its initial Cauchy data $f_\epsilon(t_{\text{in}},x)$ and $\partial_t f_\epsilon(t_{\text{in}},x)$.

The spacetime M is static in the remote past and it allows a natural in-vacuum state definition. The corresponding positive-frequency in-basis $u_{\epsilon n}$ can be defined as follows. In the initial strips R_ϵ^0,

$$u_{\epsilon n}(X)=p_{\epsilon n}(z_\epsilon)\equiv(4\pi n)^{-1/2}\exp(-2\pi B^{-1}z_\epsilon i) , \tag{3.69a}$$

while in the strips R_ϵ^n,

$$u_{\epsilon n}(X)=p_{\epsilon n}(F_\epsilon^{(-n)}(z_\epsilon)) . \tag{3.69b}$$

Let $X\equiv(t,x)$ be an arbitrary point lying below the Cauchy horizon H_+ and $R_+^{n_+}$ and $R_-^{n_-}$ be the strips where this point is located. We shall refer to the numbers n_ϵ as to the *strip index* of X. The Hadamard function

$$G_{\text{in}}^1(X,X')=\sum_{\epsilon,n}[u_{\epsilon n}(X)\bar{u}_{\epsilon n}(X')+u_{\epsilon n}(X')\bar{u}_{\epsilon n}(X)] \tag{3.70}$$

for such a vacuum state can be written in the form

$$G_{\text{in}}^1(X,X')=-(4\pi)^{-1}\ln\left[\prod_\epsilon Q_\epsilon(z_\epsilon,z'_\epsilon)\right] , \tag{3.71a}$$

$$Q_\epsilon(z_\epsilon,z'_\epsilon)=4\sin^2\{\pi B^{-1}[Z_\epsilon(z_\epsilon)-Z_\epsilon(z'_\epsilon)]\} , \tag{3.71b}$$

$$Z_\epsilon(z_\epsilon)=F_\epsilon^{-n_\epsilon}(z_\epsilon), \quad Z_\epsilon(z'_\epsilon)=F_\epsilon^{-n'_\epsilon}(z'_\epsilon) , \tag{3.71c}$$

where n_ϵ and n'_ϵ are the strip indices of X and X' correspondingly, while z_ϵ and z'_ϵ are null coordinates of these points.

Now we are ready to obtain the desired expression for the renormalized stress-energy tensor $\langle T_{\mu\nu}\rangle^{\text{ren}}$ in M. For this purpose we choose a point X lying below the Cauchy horizon H_+ and two regions U_1 and U_2, $X\in U_1\subset U_2$, which are small enough so that they lie in the same strips $R_\pm^{n_\pm}$ as X and do not intersect the curve $\gamma_-\equiv\gamma_+$. Consider a new smooth metric on M,

$$d\hat{s}^2=\Omega^2(X)ds^2 , \tag{3.72}$$

which coincides with the original metric ds^2 everywhere beyond U_2 and is flat inside U_1:

$$\Omega(X)=\begin{cases}1, & X\in M\setminus U_2 ,\\ \alpha^{-1}, & X\in U_1 .\end{cases} \tag{3.73}$$

By using Eq. (3.14) one can get

$$\langle T_{\mu\nu}\rangle^{\text{ren}}=\langle \hat{T}_{\mu\nu}\rangle^{\text{ren}}+\hat{t}_{\mu\nu} , \tag{3.74a}$$

$$\hat{t}_{\mu\nu}=\frac{1}{24\pi}\left[-2\frac{\alpha_{;\mu\nu}}{\alpha}+\eta_{\mu\nu}\left[\frac{\alpha_{;\epsilon}{}^{\epsilon}}{\alpha}-\frac{\alpha_{;\epsilon}\alpha^{;\epsilon}}{\alpha^2}\right]\right] . \tag{3.74b}$$

All the operations on the right-hand side of Eq. (3.74b) are assumed to be done in the flat metric $\eta_{\mu\nu}=\text{diag}(-1,1)$.

The quantity $\langle \hat{T}_{\mu\nu}\rangle^{\text{ren}}$ can be easily calculated because the null geodesics in both spaces ds^2 and $d\hat{s}^2$ are the same and hence Eq. (3.71) gives also the Hadamard function in a spacetime with a new metric. On the other hand the metric $\hat{g}_{\mu\nu}$ is flat in the vicinity of X and in order to renormalize $G^1_{\rm in}$ it is sufficient to subtract a flat space Hadamard function from it. Thus we have

$$\langle \hat{T}_{\mu\nu} \rangle^{\rm ren} = \lim_{X' \to X} D_{\mu\nu} G^{1,\rm ren}_{\rm in}(X,X') , \quad (3.75a)$$

$$G^{1,\rm ren}_{\rm in}(X,X') = -(4\pi)^{-1} \ln \left[\prod_\epsilon \hat{Q}_\epsilon(z_\epsilon, z'_\epsilon) \right] , \quad (3.75b)$$

where

$$\hat{Q}_\epsilon(z_\epsilon, z'_\epsilon) = \frac{4\sin^2\{\pi B^{-1}[Z_\epsilon(z_\epsilon) - Z_\epsilon(z'_\epsilon)]\}}{(z_\epsilon - z'_\epsilon)^2} . \quad (3.75c)$$

The coincidence limits $z'_\epsilon \to z_\epsilon$ of the derivatives of $\hat{Q}_\epsilon(z_\epsilon, z'_\epsilon)$ which are needed for the calculations of the renormalized vacuum expectation value $\langle \hat{T}_{\mu\nu} \rangle^{\rm ren}$ can be easily obtained, and one gets

$$\langle \hat{T}_{\epsilon\epsilon}(X) \rangle^{\rm ren} = \frac{1}{4\pi} \left[\frac{Z'''_\epsilon(z_\epsilon)}{6Z'_\epsilon(z_\epsilon)} - \frac{[Z''_\epsilon(z_\epsilon)]^2}{[Z'_\epsilon(z_\epsilon)]^2} - \frac{\pi^2}{3L^2}[Z'_\epsilon(z_\epsilon)]^2 \right] , \quad (3.76)$$

$$\langle T_{+-}(X) \rangle^{\rm ren} = 0 .$$

We use these expressions to analyze the properties of the vacuum polarization near the Cauchy horizon H_+. It is easy to see that the function $Z_+(z_+)$ is regular near H_+. It means the regularity of $\langle \hat{T}_{++} \rangle^{\rm ren}$ and hence of $\langle T_{++} \rangle^{\rm ren}$ near the Cauchy horizon H_+. We show now that the quantity $\langle \hat{T}_{--} \rangle^{\rm ren}$ (and hence $\langle T_{--} \rangle^{\rm ren}$) is divergent near H_+. For this purpose we consider in more detail the behavior of mapping (3.66) for $\epsilon = -1$ near the limiting point t_-. This mapping in the vicinity of this point (for large negative n) can be linearized and we have

$$t_- - z_-^{n+1} \simeq D(t_- - z_-^n) , \quad (3.77)$$

where

$$D = \frac{dT}{dt}(t_-) .$$

For points lying in the initial strip $z_- \in R^0_-$ one can always choose such a big positive number N that for $k > 0$ the following relation is valid:

$$F_-^{-N-k}(z_-) \simeq t_- + D^{-k}[F_-^{-N}(z_-) - t_-] . \quad (3.78)$$

We fix this value N. The following approximate expression takes place for $z_- \in R_-^{-(N+k)}$:

$$Z_-(z_-) \simeq F^N[p(z_-)] , \quad (3.79a)$$

where

$$p_- \equiv p(z_-) \equiv D^k(z_- - t) + t_- \in R^N_- . \quad (3.79b)$$

The relations (3.79) allow one to show that, for $z_- \in R_-^{-(N+k)}$, one has

$$Z'_-(z_-) \simeq (dF^N[p_-]/dp_-)D^k ,$$

$$Z''_-(z_-) \simeq (d^2F^N[p_-]/dp_-^2)D^{2k} ,$$

$$Z'''_-(z_-) \simeq (d^3F^N[p_-]/dp_-^3]D^{3k} ,$$

and so on. For fixed N the function $F^N[p_-]$ and its derivatives are regular and bounded in R^N_-. Near the limiting point $z_- \to t_-$ this function remains bounded while D^k infinitely increases:

$$D^k \simeq (t_- - z_-)^{-1} .$$

Combining these results it is easy to show that $\langle \hat{T}_{--} \rangle^{\rm ren}$ is divergent near H_+ and the leading part of the divergence is of the form

$$\langle \hat{T}_{--}(X) \rangle^{\rm ren} \simeq (t_- - z_-)^{-2} . \quad (3.80)$$

We proved the divergence of $\langle \hat{T}_{\mu\nu}(X) \rangle^{\rm ren}$ for a special choice of the initial state. It is easy to see from the proof that this result is valid for any other state provided the corresponding Hadamard function is regular in the initial strip $R^0_\pm$. That is, the divergence of the renormalized stress-energy tensor near the Cauchy horizon H_+ is a generic property of two-dimensional time-machine models.

IV. VACUUM POLARIZATION IN FOUR-DIMENSIONAL SPACETIME WITH A TIME MACHINE

A. Polarized hypersurfaces in a locally static multiply connected spacetime

The main result of the previous sections is that in two-dimensional spacetimes the formation of the time machine is always accompanied by infinite increase of the renormalized stress-energy tensor. The calculations in four dimensions are much more complicated, but nevertheless it is possible to argue that the arising of the divergences of the renormalized stress-energy tensor is also a quite general feature of a locally static spacetime with a nonpotential gravitational field.

Denote as earlier $\tilde{M} = T \times \tilde{W}$ the universal covering space of the spacetime $M = T \times W$. The space W may be identified with a fundamental domain in $\tilde{W}$. To make consideration more concrete we assume that there exist only one wormhole. In this case the universal covering space $\tilde{W}$ consists of an infinite number of sheets $W^n = \gamma_n W$ numerated by an integer winding number n. The Hadamard function $G^1(X,X')$ in a locally static spacetime can be written in the form

$$G^1(X,X') = \sum_n \tilde{G}^1(X, \gamma_n X') , \quad (4.1)$$

where the summation is taken over the winding number n. The divergent terms which are needed to be subtracted for the renormalization are the same in M and in its universal covering space $\tilde{M}$, and hence we have

$$\langle T_{\mu\nu} \rangle^{\rm ren}_M = \langle T_{\mu\nu} \rangle^{\rm ren}_{\tilde{M}} + T_{\mu\nu} , \quad (4.2a)$$

where

$$T_{\mu\nu}(X)=\sum_n{}' T^n_{\mu\nu}(X) , \quad (4.2b)$$

$$T^n_{\mu\nu}(X)=\lim_{X'\to X} D_{\mu\nu} G^1(X,\gamma_n X') . \quad (4.2c)$$

The prime in this formula indicates that the term with $n=0$ is omitted from the summation. The differential operator $D_{\mu\nu}$ defined by Eq. (2.40) for a conformal invariant massless scalar field ($\xi=\frac{1}{6}$) is of the form

$$D_{\mu\nu}=\tfrac{1}{6}(\nabla_{\mu'}\nabla_\nu+\nabla_\mu\nabla_{\nu'}-\tfrac{1}{2}g_{\mu\nu}\nabla_\rho\nabla^{\rho'}) -\tfrac{1}{12}[\nabla_\mu\nabla_\nu+\nabla_{\mu'}\nabla_{\nu'}-\tfrac{1}{4}g_{\mu\nu}(\nabla_\rho\nabla^\rho+\nabla_{\rho'}\nabla^{\rho'}) -(R_{\mu\nu}-\tfrac{1}{4}g_{\mu\nu}R)] . \quad (4.3)$$

We have seen that in two-dimensional case $\langle T_{\mu\nu}\rangle^{\rm ren}_{\tilde M}$ is regular and finite while $T_{\mu\nu}$, being dependent on the global causal structure of M, is divergent at the future Cauchy horizon H_+. It is evident that in the four-dimensional case the analogous divergency will also arise for those points X for which a pair X and $\gamma_n X$ can be connected by a null geodesic line in $\tilde M$. In the latter case the point X is located at the characteristic conoid with the vertex at $\gamma_n X$ and thus $G^1(X,\gamma_n X')$ is divergent. Such a geodesic connecting X and $\gamma_n X$ in the universal covering space $\tilde M$ is projected into a closed null geodesic in M which begins and ends at a point X and possesses a winding number n.

We prove now that for any chosen spatial point $\mathbf{x}\in W$ there exists such a time moment $t=T_n(\mathbf{x})$ that the points $X=(t,\mathbf{x})$ and $\gamma_n X$ in $\tilde M$ are connected by a null geodesic. For this purpose we note that null geodesics remain invariant under conformal transformations and for their study we can use the ultrastatic spacetime $\hat M = T\times \hat W$ with a metric

$$d\hat s^2=-dt^2+d\lambda^2 , \quad d\lambda^2=H_{ij}dx^i dx^j , \quad H_{ij}=\alpha^{-2}h_{ij} , \quad (4.4)$$

which is conformally related with the metric ds^2 of $\tilde M$ given by Eq. (2.20):

$$ds^2=\alpha^2 d\hat s^2 . \quad (4.5)$$

Null geodesics in $\hat M$ allow a simple description. Let $x^i=x^i(\lambda)$ be a geodesic line in a three-dimensional space with the metric $d\lambda^2$ and λ be a proper length parameter along this line. Then the curve $X^\mu=X^\mu(\lambda)=(\lambda,x^i(\lambda))$ is a null geodesic in $\hat M$ (and hence in $\tilde M$).

The next step in our proofs is to show that any two points of $\hat W$ can be connected by a geodesic lying in $\hat W$. There is a well-known theorem (see, e.g., Ref. 21) which declares that in a complete connected Riemannian manifold any two points can be connected by a geodesic of a minimal length. This result cannot be directly applied to our case because the space $\hat W$ is not complete. The incompleteness of $\hat W$ can be easily demonstrated. Choose a point $\mathbf{y}$ and consider a sequence $\mathbf{y}_n\equiv\gamma_{-n}\mathbf{y}$. Let C be an arbitrary curve connecting $\mathbf{y}_0=\mathbf{y}$ and $\mathbf{y}_1$ and let L_C be a length of C. Then the distance $d=d(\mathbf{y}_0,\mathbf{y}_1)$ between these points which is defined as an exact low boundary of lengths of curves connecting the points is evidently less than L_C: $d\le L_C$. The transformation property of α, $\gamma\alpha(\mathbf{x})\equiv\alpha(\gamma\mathbf{x})=A^{-1}\alpha(\mathbf{x})$, implies that

$$d_n\equiv d(\mathbf{y}_n,\mathbf{y}_{n+1})\le A^{-n}L_C .$$

It means that $\{\mathbf{y}_n\}$ is a Cauchy sequence, but there is no limiting point of this sequence in $\hat W$ and hence $\hat W$ is incomplete.

Nevertheless, it is possible to show that there always exists a geodesic of a minimal length connecting two points $\mathbf{y}_0$ and $\mathbf{y}_n$ in $\hat W$ provided that a distance L between the mouths in W is much larger than the size of a mouth. In order to prove this we show at first that such a curve C of a minimal length which begins at $\mathbf{y}_0$ and ends at $\mathbf{y}_n\in\hat W^{-n}$ cannot intersect a sheet $\hat W^{-(n+2)}$. Consider a path C' which begins at $\mathbf{y}_0$ and ends at $\mathbf{y}_n$ but which possesses a part lying in $\hat W^{-(n+2)}$. This path contains a fragment $\mathbf{b}\mathbf{b}'\mathbf{b}''\cdots\mathbf{c}''\mathbf{c}'\mathbf{y}_n$ for which $\mathbf{b},\mathbf{y}_n\in\hat W^{-n}$, $\mathbf{b}',\mathbf{c}'\in\hat W^{-(n+1)}$ and $\mathbf{b}'',\mathbf{c}''\in\hat W^{-(n+2)}$. It is evident that the length of a part $\mathbf{b}'\mathbf{b}''\cdots\mathbf{c}''\mathbf{c}'$ of this curve is larger than $2LA^{-(n+1)}$, where L is a distance between the mouths in the fundamental domain $\hat W^0$. It is possible to transform the curve C' into a new shorter curve C'' by omitting its part $\mathbf{b}'\mathbf{b}''\cdots\mathbf{c}''\mathbf{c}'$ and connecting the points $\mathbf{b}'$ and $\mathbf{c}'$ by a curve lying in $\hat W^{-n}$ and passing near the mouth. If b is a characteristic size of a mouth in $\hat W^0$, then the length of $\mathbf{b}'\mathbf{c}'$ is of the order of bA^{-n}, and hence the length of C'' is less than the lenght of C', provided $2L > Ab$. In other words, in order to find a curve of a minimal length connecting $\mathbf{y}_0$ and $\mathbf{y}_n$, it is sufficient to consider only curves which do not intersect the sheet $\hat W^{-(n+2)}$. That is why one can change the geometry in the part of $\hat W$ which consists of the sheets $\hat W^{-k}$ for $k\ge n+2$ without any influence on the existence of a curve of a minimal length connecting $\mathbf{y}_0$ and $\mathbf{y}_n$. In particular, we can make a conformal transformation with a conformal factor equal to 1 in $\hat W^{-k}$ for $k\le n+1$ [and hence preserving the metric (4.3) there] and which is equal to $\alpha^2 A^{-2(n+3)}$ for $k\ge n+3$. The space with this conformally transformed metric is a complete one so that the theorem mentioned above guarantees the existence of a geodesic of a minimal length in this space. But we have seen that this geodesic cannot enter the sheet $\hat W^{-(n+2)}$, and thus it is a required geodesic in $\hat W$.

Denote by $C_n(\mathbf{x})$ the geodesic of a minimal length connecting $\mathbf{x}$ and $\mathbf{x}_n\equiv\gamma_n\mathbf{x}$ in $\hat W$ and denote its proper length by $\lambda_n(\mathbf{x})$. Consider now a null geodesic in $\hat M$ generated by $C_n(\mathbf{x})$. Suppose that it begins at a point $\mathbf{x}$ at a time moment t then it reaches $\mathbf{x}_n$ at a time moment $t'=t+\lambda_n(\mathbf{x})$. The end point $(t',\mathbf{x}_n)$ of this geodesic coincides with $\gamma_n X=(A^n t,\mathbf{x}_n)$ provided

$$t=T_n(\mathbf{x})\equiv\lambda_n(\mathbf{x})/(A^n-1) . \quad (4.6)$$

This result shows that, for a given winding number n and for an arbitrary spatial point $\mathbf{x}\in W$ of a locally static spacetime M with a nonpotential gravitational field, there exists a closed null geodesic which begins at this point, passes through the wormhole exactly n times, and returns into the same spacetime point. In the general case there is only one such closed geodesic but in principle there may exist points for which there are more than one or even an infinite number of closed null geodesics. It happens for example when the initial and the end points of a closed geodesic are conjugated along it. A point which allows more than one geodesic we shall call a singular one. We assume that singular points are isolated.

The above proved results can be summarized as follows. In a multiply connected locally static spacetime M with a nonpotential gravitational field, there exists a system of hypersurfaces H_n formed by points through which closed null geodesics with a winding number n pass. Each Killing observer intersects each of these hypersurfaces. At the nonsingular points the projection of H_n on W along the Killing trajectories is a regular one. Following Ref. 11 we refer to H_n as to the "nth polarized hypersurfaces."

Consider a nonsingular point of a polarized hypersurface. At this point it is possible to define two vectors k_i^μ and k_f^μ: a vector k_i^μ being a null vector tangent to a closed null geodesic at its initial point X, and a vector k_f^μ being the corresponding tangent vector at the end of the geodesic, normalized by conditions

$$u_\mu k_i^\mu = u_\mu k_f^\mu = -1 \ , \tag{4.7}$$

where u^μ is a four-velocity of a Killing observer at this point. In a general case the geometrical properties of polarized hypersurfaces and of fields of pairs (k_i^μ, k_f^μ) depend on the details of the spacetime metric.

According to the above arguments one may expect that the renormalized stress-energy tensor $\langle T_{\mu\nu}\rangle^{\rm ren}$ is divergent near the nth polarized hypersurface H_n. In the next section we argue that it does really happen and describe the structure of the divergencies.

B. Asymptotic behavior of the renormalized stress-energy tensor $\langle T_{\mu\nu}\rangle^{\rm ren}$ near the polarized hypersurface

Now we show that the contribution $T^n_{\mu\nu}$ to the renormalized stress-energy tensor $\langle T_{\mu\nu}\rangle^{\rm ren}$ defined by Eq. (4.2) is divergent near the nth polarized hypersurface H_n. For this purpose we choose a nonsingular point X_0 of H_n and consider a null geodesic C_n in $\widetilde{M}$ connecting X_0 and $X_{0n} \equiv \gamma_n X_0$. We assume that C_n lies in some causal domain $\Omega[C_n]$, i.e., in a connected open set where any pair of points can be connected by a geodesic and where for any points p and q $J^+(p)\cap J^-(q)$ is a compact subset of $\Omega[C_n]$, or void. The Hadamard function $\widetilde{G}^1$ allows the following expansion in $\Omega[C_n]$:[27–29]

$$\widetilde{G}^1(X,X') = \frac{\Delta^{1/2}}{4\pi^2}\left[\frac{1}{\sigma(X,X')} + v(X,X')\ln|\sigma(X,X')| + w(X,X')\right] , \tag{4.8}$$

where $\sigma(X,X') = \frac{1}{2}s^2(X,X')$ is a geodetic interval, and $\Delta = \Delta(X,X')$ is a Van Vleck–Morette determinant. The functions v and w are regular in the coincidence limit. The function v is uniquely defined and depends only on the local geometry of $\widetilde{M}$, while w depends on a quantum state.

Consider points $X^\mu = X_0^\mu - \delta t\, \xi^\mu(X_0)$ and $X_n^\mu = \gamma_n X^\mu = X_{0n}^\mu - \delta t\, A^n \xi^\mu(X_{0n})$ located on the Killing trajectories passing through X_0 and X_{0n} and choose a parameter δt to be so small that both points belong to $\Omega[C_n]$. For a geodetic interval $\sigma(X,X_n)$ between these points we have

$$\sigma(X,X_n) = -[\xi^\mu(X_0)\sigma_{,\mu} + A^n \xi^{\mu'}(X_{0n})\sigma_{,\mu'}]\delta t \ , \tag{4.9}$$

where

$$\sigma_{,\mu} = \frac{\partial\sigma(X_0,X_{0n})}{\partial X_0^\mu} \ , \quad \sigma_{,\mu'} = \frac{\partial\sigma(X_0,X_{0n})}{\partial X_{0n}^\mu} \ . \tag{4.10}$$

Let $X^\mu = X^\mu(\tilde\lambda)$ be a null geodesic connecting X_0 and X_{0n}, $\tilde k^\mu = \tilde k^\mu(\tilde\lambda) = dX^\mu/d\tilde\lambda$ and $\tilde\lambda$ be an affine parameter along the geodesic $[X_0{}^\mu = X^\mu(0),\ X_{0n}{}^\mu = X^\mu(\Lambda_n)]$; then

$$\sigma_{,\mu} = -\Lambda_n \tilde k_{i\mu}, \quad \sigma_{,\mu'} = \Lambda_n \tilde k_{f\mu} \ , \tag{4.11}$$

where $\tilde k_i^\mu = \tilde k^\mu(0)$ and $\tilde k_f^\mu = \tilde k^\mu(\Lambda_n)$. The value $\xi_\mu \tilde k^\mu$ is constant along a geodesic. Thus we have

$$\sigma(X,X_n) = -\Lambda_n (A^n - 1)\xi_\mu(X_0)\tilde k^\mu \delta t \ . \tag{4.12}$$

It should be stressed that an affine parameter $\tilde\lambda$ is defined up to a linear transformation $\tilde\lambda \to c\tilde\lambda$ while the value of $\Lambda_n \tilde k^\mu$ does not depend on this ambiguity. We fix the choice of the affine parameter by the condition

$$\xi_\mu(X_0)\tilde k^\mu = -1 \ , \tag{4.13}$$

and fix the normalization of the Killing vector ξ^μ by the relation

$$\xi_\mu(X_0)\xi^\mu(X_0) = -1 \ . \tag{4.14}$$

For this choice $\delta t = \delta\tau$, i.e., it is a proper time distance from X to the Cauchy horizon along a Killing trajectory. The spacetime metric ds^2 in $\widetilde{M}$ is conformal to the metric $d\hat s^2$ in $\widehat{M}$ and the affine parameter $\tilde\lambda$ in $\widetilde{M}$ is connected with the affine parameter λ in $\widehat{M}$ along the same null geodesic by the relation

$$d\tilde\lambda = \alpha^2 d\lambda \ . \tag{4.15}$$

Thus we have

$$\Lambda_n = \int_{\mathbf{x}}^{\mathbf{x}_n} \alpha \left[h_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right]^{1/2} d\lambda \ , \tag{4.16}$$

where the integration is taken over a three-dimensional geodesic in $\widehat{W}$. By using these relations we can rewrite Eq. (4.12) in the form

$$\sigma(X,X_n) = \Lambda_n (A^n - 1)\delta t \ , \tag{4.17}$$

where Λ_n is a "redshifted spatial distance" between $\mathbf{x}$ and $\mathbf{x}_n$ along a geodesic.

The leading (divergent at $\delta t \to 0$) part of $T^n_{\mu\nu}$ is of the form

$$\begin{aligned} T^n_{\mu\nu}(X) &\simeq \frac{\Delta_n^{1/2}}{12\pi^2\sigma^3(X,X_n)}[\sigma_{\mu'}\sigma_\nu + \sigma_\mu\sigma_{\nu'} - \tfrac{1}{2}g_{\mu\nu}\sigma_\rho\sigma^{\rho'} - \tfrac{1}{2}(\sigma_\mu\sigma_\nu + \sigma_{\mu'}\sigma_{\nu'})] \\ &= -Q_n(\delta t)^{-3}K_{n\mu\nu} \ , \end{aligned} \tag{4.18}$$

where

$$Q_n = \frac{\Delta_n^{1/2}}{12\pi^2 \Lambda_n (A^n - 1)^3} , \quad (4.19a)$$

$$K_n^{\mu\nu} = \tilde{k}_i^{\mu}\tilde{k}_f^{\nu} + \tilde{k}_i^{\nu}\tilde{k}_f^{\mu} - \tfrac{1}{2}g^{\mu\nu}\tilde{k}_{i\rho}\tilde{k}_f^{\rho} + \tfrac{1}{2}(\tilde{k}_i^{\mu}\tilde{k}_i^{\nu} + \tilde{k}_f^{\mu}\tilde{k}_f^{\nu})$$

$$= A^n(k_i^{\mu}k_f^{\nu} + k_i^{\nu}k_f^{\mu} - \tfrac{1}{2}g^{\mu\nu}k_{i\rho}k_f^{\rho}) \quad (4.19b)$$

$$+ \tfrac{1}{2}(k_i^{\mu}k_i^{\nu} + A^{2n}k_f^{\mu}k_f^{\nu}) .$$

In the latter relation we use null vectors k_i^{μ} and k_f^{μ} which are connected with $\tilde{k}_i^{\mu}$ and $\tilde{k}_f^{\mu}$ by $k_i^{\mu} = \tilde{k}_i^{\mu}$, $k_f^{\mu} = A^{-n}\tilde{k}_f^{\mu}$ and which are normalized as follows:

$$k_i^{\mu}u_{\mu} = k_f^{\mu}u_{\mu} = -1 ,$$

where u^{μ} is a four-velocity of a Killing observer. If $(A-1)$ is small, then $\Lambda_n \sim nL$, where L is a distance between the mouths. The Van Vleck–Morette determinant Δ_n which enter Eq. (4.19a) can be found by considering the behavior of a infinitely small beam of null rays emitted at the initial point X. The estimations in the geometric optics approximation give[11] $\Delta_n^{1/2} \sim (b/L)^{j(n)}$, where b is a size of a mouth, and $j(n) \sim n$ or $(n+1)$.

Equation (4.18) shows that $T_{\mu\nu}^{n}(X)$ (and hence $\langle T_{\mu\nu}(X)\rangle^{\text{ren}}$) is divergent at the polarized hypersurface H_n in a locally static spacetime M with a nonpotential gravitational field. The local energy density $\epsilon(X)$ at a point X corresponding to the renormalized stress-energy tensor (4.18) is

$$\epsilon(X) \equiv T_{\mu\nu}^{n}(X)u^{\mu}u^{\nu}$$

$$= -\tfrac{1}{2}Q_n(\delta t)^{-3}[1 + A^{2n} + A^n(4 + k_{i\rho}k_f^{\rho})] . \quad (4.20)$$

By using the inequality $-2 \le k_{i\rho}k_f^{\rho} \le 0$ we can see that the local energy density at the points lying just below a polarized hypersurface H_n $(n > 0)$ is always negative and is divergent as a third power of a proper time distance to the polarized hypersurface. For large n the leading term of $T_{\mu\nu}^{n}$ is of the form

$$T^{n\mu\nu} \simeq -\tfrac{1}{2}Q_n(\delta t)^{-3}A^{2n}k_f^{\mu}k_f^{\nu} , \quad (4.21)$$

and it describes a flux of null fluid with a negative energy density.

V. SUMMARY AND CONCLUDING REMARKS

Now we summarize the main points of the present paper. The general feature of locally static multiply connected spacetimes is that the corresponding gravitational field is nonpotential. In a spacetime with N wormholes there are N independent topological invariants (2.10) which give the measure of the nonpotentiality of the gravitational field. If any of these invariants does not vanish the spacetime allows closed null geodesics and it may be considered as a model of a time machine. Closed paths that begin and end at the same point form homotopically nonequivalent classes. In a spacetime with a single wormhole these classes are characterized by a winding number n. For a given winding number and a spatial point $\mathbf{x}$ there always exist a closed null geodesic with this winding number which begins and ends at the same spacetime point $(t,\mathbf{x})$. These points form the nth polarized hypersurface H_n. All the polarized hypersurfaces are located above the Cauchy horizon H_+ and cross it along a common null geodesic. The renormalized stress-energy tensor of free quantum fields in a locally static multiply connected spacetime background is divergent at the polarized hypersurfaces. For a massless field in a four-dimensional spacetime this divergency is of the form (4.18). This stress-energy tensor describes the flux of energy density which is negative below the polarized hypersurface and positive above it. The absolute value of the corresponding vacuum energy density is of the order $\sim B_n\hbar c/(L\delta\tau^3)$,where $\delta\tau$ is a proper time distance to the polarized hypersurface H_n, L is a distance between the mouths, and $B_n \sim (b/L)^{n \text{ or } n+1}$, b being a size of a mouth. In the vicinity of the Cauchy horizon H_+ (i.e., for large n) the renormalized stress-energy tensor describes null fluid propagation. These results are in a complete agreement with the results of Kim and Thorne.[11] In a two-dimensional case all the polarized hypersurfaces coincide with the Cauchy horizon H_+ and

$$\langle T_{\mu\nu}\rangle^{\text{ren}} \simeq -\hbar c/\delta\tau^2 k_{\mu}k_{\nu} ,$$

where k^{μ} is a null vector tangent to H_+. This result allows the following interpretation. In two-dimensional cylindrical spacetime in the absence of the gravitational field there is no vacuum energy flux, while the Casimir energy density is negative. Under the action of a nonpotential gravitational field, this energy density begins to flow. As a result of blueshift effect the energy density of this flux grows and leads to the divergency of $\langle T_{\mu\nu}\rangle^{\text{ren}}$ near the Cauchy horizon H_+.

In the general case, the negative energy flux due to the vacuum polarization is directed to those wormhole's mouth where the gravitational potential is less and it decreases the mass of this mouth. As a result, one may expect that the gravitational field becomes "more potential." The negative mass which is falling into the mouth infinitely grows as $\delta\tau \to 0$. That is why its back reaction might prevent the formation of the Cauchy horizon.

It should be stressed that the sign of the vacuum energy density may depend on the spin of a field. In particular one may expect that the contribution of fermions to the energy density has an opposite sign than the contribution of bosons, so that in the general case the sign of the total energy density depends on the number of fields of different spins. If the resulting energy density is positive, then the mass of one of the mouths grows and its gravitational potential becomes less and less. This process may be stopped by producing a black hole. In principle we do not exclude the situation when there is an exact cancellation of the leading contributions of all fields (as it happens for the vacuum energy density in flat spacetime in a supersymmetric theory). In the latter case one may expect that the divergency of the renormalized stress-energy tensor becomes slightly more mild so that $\langle T_{\mu\nu}\rangle^{\text{ren}} \sim B_n\hbar c/(L^2\delta\tau^2)$ but the mass transfer from one mouth to the other again remains unbounded as $\delta\tau \to 0$.

The described result in some respect resembles the result obtained by DeWitt,[31] who has shown that a change of spatial topology is always accompanied by infinite particle, and energy production, and if the back reaction of the created matter is taken into account one may expect the dynamical suppression of topological changes.

Until now we consider linearized quantum fields on a given classical gravitational field background. Kim and Thorne in their recent paper[11] argue that the quantum fluctuations of a metric at Planck scales may remove the singularities of $\langle T_{\mu\nu}\rangle^{\rm ren}$. It may happen that the divergencies would be cut off at $c\delta\tau \simeq l_{\rm Pl} = (hG/c^3)^{1/2}$. As a result of this cutoff the absolute maximum value of $\langle T_{\mu\nu}\rangle^{\rm ren}$ will be of order $\sim (hc/l_{\rm Pl}^4)(l_{\rm Pl}/L)$, i.e., $L/l_{\rm Pl}$ times less than the Planckian density. The corresponding total mass transfer does not exceed $\Delta M \sim b^2/GL$. A more invariant cutoff at $\delta\sigma \simeq cL\delta\tau \simeq l_{\rm Pl}^2$ is also possible for which $AM \sim M$. Both these estimations look quite reasonable, nevertheless for their proof one needs to know the physics at Planckian scales.

We conclude the paper by the following remark. From the very beginning it was expected that inside traversable wormholes and time machines there must be energy condition violations due to the quantum effects. But it appeared that due to the vacuum polarization effect these objects are internally unstable until the quantum gravity cures this instability.

ACKNOWLEDGMENTS

The author thanks Kip Thorne for many stimulating discussions and for sending a copy of Ref. 11 before its publication. He is also grateful to Werner Israel, Igor Novikov, Igor Volovich, and Bill Unruh for discussions of different subjects concerning the present paper. The author thanks the Department of Physics of the University of British Columbia (U.B.C.) and the Canadian Institute for Advanced Research (CIAR) for hospitality during the preparation of this paper, and the Natural Sciences and Engineering Research Council of Canada through the University of Alberta, U.B.C., and CIAR for financial support.

APPENDIX

In this appendix we demonstrate how the expression (3.42) for the Hadamard function in the spacetime of a two-dimensional standard model can be obtained by using the representation (2.39), which in the case under consideration takes the form

$$G^1(X,X') = \sum_{n=-\infty}^{\infty} \tilde{G}^1(X,\gamma_n X') , \tag{A1}$$

where

$$\tilde{G}^1(X,\gamma_n X') = -\frac{1}{4\pi}\ln\left[\frac{(\zeta_- - A^n\zeta'_-)^2(\zeta_+ - A^n\zeta'_+)^2}{(\zeta_+ + A^n\zeta'_-)^2(\zeta_- + A^n\zeta'_+)^2}\right] . \tag{A2}$$

Equation (A1) can be identically rewritten in the form

$$G^1(X,X') = \tilde{G}^1(X,X') - \frac{1}{4\pi}\ln\left[\frac{J_-(V_{--})J_-(V_{++})}{J_+(V_{-+})J_+(V_{+-})}\right] , \tag{A3}$$

where

$$V_{\epsilon\epsilon'} = \frac{1}{2\pi}\ln(\zeta'_{\epsilon'}/\zeta_\epsilon) \tag{A4}$$

and

$$J_\epsilon(V) = \prod_{n=1}^{\infty} [1 + 2\epsilon\cosh(2\pi V)q^{2n} + q^{4n}], \quad q \equiv A^{-1/2} . \tag{A5}$$

The quantities $J_\epsilon(V)$ can be rewritten in the terms of the Jacobi θ functions $\theta_i(z) \equiv \theta_i(z,q) = \theta_i(z|\tau)$:

$$J_-(V) = \frac{\theta_1(iV|\tau)}{2iq_0 q^{1/4}\sinh(\pi V)} ,\qquad J_+(V) = \frac{\theta_2(iV|\tau)}{2q_0 q^{1/4}\cosh(\pi V)} , \tag{A6}$$

where

$$\tau = \frac{1}{i\pi}\ln q = \frac{i}{2\pi}\ln A \equiv \frac{i}{2\pi\beta} , \tag{A7}$$

$$q_0 = \prod_{n=1}^{\infty}(1 - q^{2n}) . \tag{A8}$$

Using the properties of the θ functions under the unimodular transformations $z \to z/\tau$, $\tau \to -1/\tau$ (Ref. 30) we can write

$$\theta_1(iV|\tau) = i(2\pi\beta)^{1/2}\exp(-2\pi^2\beta V^2)\theta_1(2\pi\beta V|2\pi i\beta) ,\qquad \theta_2(iV|\tau) = (2\pi\beta)^{1/2}\exp(-2\pi^2\beta V^2)\theta_4(2\pi\beta V|2\pi i\beta) . \tag{A9}$$

It is easy to verify that

$$\frac{\cosh(\pi V_{+-})\cosh(\pi V_{-+})}{\sinh(\pi V_{++})\sinh(\pi V_{--})} = \frac{(\zeta'_+ + \zeta_-)(\zeta'_- + \zeta_+)}{(\zeta'_+ - \zeta_+)(\zeta'_- - \zeta_-)} , \tag{A10a}$$

$$V_{++}^2 + V_{--}^2 - V_{+-}^2 - V_{-+}^2 = \frac{1}{2\pi^2}\ln(\zeta'_+/\zeta'_-)\ln(\zeta_+/\zeta_-) . \tag{A10b}$$

By combining the relations (A6), (A9), and (A10), we can rewrite Eq. (A3) in the form

$$G^1(X,X') = \frac{1}{2\pi}\left[\beta\ln(\zeta'_+/\zeta'_-)\ln(\zeta_+/\zeta_-) - \ln\left[\frac{\theta_1[\beta\ln(\zeta'_+/\zeta_+)]\theta_1[\beta\ln(\zeta'_-/\zeta_-)]}{\theta_4[\beta\ln(\zeta'_+/\zeta_-)]\theta_4[\beta\ln(\zeta'_-/\zeta_+)]}\right]\right] , \tag{A11}$$

where $\theta_i(z) \equiv \theta_i(z,\kappa)$ and $\kappa = \exp(-2\pi^2\beta)$. This formula does coincide with the expression (3.42a)

The representation (A1) for the Hadamard function can be used for the calculation of $\langle T_{\mu\nu}\rangle_M^{\rm ren}$. In particular we have

$$\partial_{\pm}\partial'_{\pm}G^1(X,X')=-\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{A^n}{(\zeta_{\pm}-A^n\zeta'_{\pm})^2}\,, \qquad (A12)$$

and for $T^1_{\pm}$ given by Eq. (3.45) we obtain the expression (3.47a) with

$$F(\beta)=\frac{1}{8\pi}\sum_{n=1}^{\infty}\frac{1}{\sinh^2(n/2\beta)}\,. \qquad (A13)$$

The above proven identity of Eqs. (A1) and A11) guarantees that the function $F(\beta)$ given by Eq. (A13) coincides with Eq. (3.47b).

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