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How does inflation isotropize the Universe?

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The lack of a preferred frame in the de Sitter metric leads one to argue that the end of inflation should result in an inhomogeneous rather than a homogeneous universe. Statements to the contrary have relied on scalar perturbation theory around an isotropic background. We show that to analyze properly the problem requires the introduction of velocities, implying an inhomogeneous scalar field ϕ and minimally an anisotropic background. We find by both analytic and numerical calculations that inflation does not isotropize the Universe in the short-wavelength limit $\lambda R \ll H^{-1}$. (R is a typical scale equal to the radius of the Universe.) However, if inflation persists λR becomes larger than the "horizon" H^{-1} and the wave behavior freezes in. At this point all velocities go to zero pointwise at a rate $\tanh\beta \sim 1/R$, where β is the relativistic tilt angle, a measure of velocity. For periodic boundary conditions the periodicity forces the average $\tanh\beta$ to zero faster, $\sim 1/R^4$. The results are purely classical, showing that the invocation of quantum fluctuations is not the relevant answer to the problem.

I. MOTIVATION

In a short paper Ellis and Rothman[1] (ER) pointed out that because of the symmetries inherent in the de Sitter metric, a de Sitter universe shows no preferred timelike direction. One can, for example, write the de Sitter metric in an exponentially expanding form (the inflationary universe) or in a manifestly static form (the original de Sitter model).

This being the case, the following question arises regarding the inflationary scenario: Suppose inflation takes place and establishes a truly de Sitter spacetime. Then, given the arbitrariness of the timelike direction, how does a parcel of matter at point A know to have the same four-velocity as a parcel of matter at point B? It cannot know, since the constant-time surfaces are arbitrary by any action of any element of the de Sitter group of isometries. One can always perform a Lorentz boost to an indistinguishable frame with a different four-velocity. In other words, since a de Sitter universe displays no preferred direction of time, the four-velocity of matter at any given point is not uniquely defined. A de Sitter universe has no memory (a point made clear when one considers the manifestly static form of the metric).

Naively, then, one would expect that when inflation ends, each primordial clump of matter would go its own way and the universe would end up inhomogeneous, rather than homogeneous and isotropic as inflation seeks to establish.

An alternative way to view the problem is to recall that the equation of state in a de Sitter universe is $p+\rho=0$; that is, the inertial mass density vanishes. Therefore, arbitrarily small perturbations of a given matter parcel should result in arbitrarily large accelerations. Again, the final product should be an inhomogeneous universe, not an isotropic one.

The ER paper prompted a number of responses. The first was that inflation does not establish a true de Sitter spacetime. The inflationary equation of state is $p+\rho=\dot{\phi}^2/2+p_{\rm rad}+\rho_{\rm rad}$, which is not zero (as we shall see below, this turns out to be the relevant response, at least if quantum fluctuations are ignored). Alternatively, there was the "wisp of a hair" hypothesis proposing that $p_{\rm rad}+\rho_{\rm rad}$ selects the new frame, despite the enormous redshift during inflation. True, the energy density of the radiation decreases as R^{-4}, where R is the scale factor, and the radiation density becomes overwhelmingly negligible compared to the vacuum energy of the inflation field. Nevertheless, when $v \ll 1$, the net drift velocity v of a group of photons satisfies $v \sim$const when calculated in an already homogeneous background (see Sec. IV below), and given that the *inertia* of the inflation field goes to zero in the de Sitter phase, it is conceivable that $p_{\rm rad}+\rho_{\rm rad}$ selects the new frame. However, we show in this paper that the "wisp of a hair" hypothesis fails under large enough inflationary expansion.

A third response invoked power-law inflation. Since a de Sitter universe is not established, the problem is obviated. It was also pointed out that the perturbation analysis has established that the de Sitter universe is stable to perturbations, and so the question has already been answered. Finally, there was the suggestion that quantum fluctuations preserve the quasi de Sitter memory. In the regime where quantum fluctuations are

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important,[2] $\rho+p\approx\dot{\phi}^2_{\rm QM}$, where $\dot{\phi}_{\rm QM}\sim H^3/(8\pi^2\phi)$ for a Coleman-Weinberg potential.

The variety of responses to the ER proposal suggests that the true answer, if obvious, is at least not widely disseminated. In regard to perturbation analysis, all such investigations of which we are aware concern themselves with scalar perturbations around an isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) background. No velocities are involved. In terms of the last two suggestions, we point out that quantum fluctuations arise from the de Sitter spacetime and to a large extent share its invariance properties (cf. Birrell and Davies[3(a)]). In the case of a system at the "top of the potential" where the effective potential is exactly flat, the scalar field is in fact massless. In this case, as Allen[3(b)] has shown, the natural ground state obeys $\langle\phi^2\rangle\propto\tau-\tau_0$, where the metric is $-d\tau^2+e^{2H\tau}d\sigma^2$. $\langle\phi^2\rangle$ is thus not de Sitter invariant, but grows with time. This behavior does not occur in the $m^2>0$ case where $\langle\phi^2\rangle$ is constant in time and $\langle\phi^2\rangle$ is de Sitter invariant in the Birrell-Davies vacuum. What is happening in the massless case is that random fluctuations are walking away from the point $\phi=0$, because there is no "restoring force" which in the $m^2>0$ case holds the field near $\phi=0$ via the presence of the $m^2\phi^2$ term in the Lagrangian. However, $\langle\phi^2\rangle$ is not observable when $V(\phi)$ is independent of $\langle\phi^2\rangle$, e.g., when it is flat. It is the effective potential $V(\phi)$ which describes the self- and other interactions of the field ϕ. To the extent that $V(\phi)$ is completely flat, the scalar field remains physically and effectively de Sitter invariant and cannot specify a particular four-velocity by picking out a preferred timelike direction.

Regardless of the actual resolution to the ER proposal, it behooves us to investigate velocity disturbances in the inflationary scenario. Because the FLRW cosmology does not admit velocities, an investigation beyond perturbation theory requires going minimally to an anisotropic background and an inhomogeneous scalar field ϕ, as we now demonstrate.

We take the usual stress-energy tensor for ϕ,

$$T^{\mu\nu}=\phi^{,\mu}\phi^{,\nu}-[V(\phi)+\tfrac{1}{2}\phi_{,\alpha}\phi^{,\alpha}]g^{\mu\nu}\ , \tag{1.1}$$

and assume that the zero-momentum-density frame associated with this stress tensor has four-velocity u^α. As we show below [Eq. (3.10)], $\tanh\beta\equiv u^1/u^0\propto T^1_0$, where β is the usual hyperbolic tilt angle from special relativity and where we have assumed the velocity is along one axis only ($u^2=u^3=0$).

One sees that to get a nonzero velocity for ϕ, for $g^{\mu\nu}$ diagonal, requires $T^1_0\neq 0$ or that $\phi^{,1}\neq 0$. Hence $\phi=\phi(z,t)$ is inhomogeneous. (We take $x^1\equiv z$.) For our investigation ϕ is assumed to be initially sinusoidal, although a sinusoidal ϕ is *not* in any sense a stationary solution to the spatially inhomogeneous wave equation (3.3). The velocity u^α may be thought of as the group velocity of the wave.

At first sight it appears from (1.1) that $T^1_0\neq 0$ is possible for spatially *homogeneous* ϕ, $\phi=\phi(t)$, so long as $g^{0i}\neq 0$. However, if we considered a cosmology consisting of a homogeneous space evolving in time but with homogeneous nonzero g^{0i} (nonzero shift), we would just in fact be considering a time-dependent coordinate transformation; we have not investigated this option further.

Some simplification of the problem can be achieved. Certain Bianchi types do admit velocities or tilt angles. The simplest of these is the locally rotationally symmetric (LRS) type V, which is an anisotropic generalization of the $k=-1$ FLRW model and which we describe more fully below. In that case we can impress an inhomogeneous ϕ field on an anisotropic, homogeneous background. This will necessarily require averaging inhomogeneous quantities, but saves us the difficulty of going to a fully inhomogeneous model. Many of the results below are in fact independent of the homogeneous background assumed and can be imagined to take place in a FLRW cosmology. Using type V allows us to include a kind of averaged back reaction from the nonzero velocities. We note again that in a type-V cosmology, a nonzero scalar field momentum requires inhomogeneous ϕ, and so only by correct averaging is it consistent to take the homogeneous type-V background.

The general procedure is as follows: We consider a one-dimensional inhomogeneous ϕ on a LRS type-V background. An initial velocity (tilt angle) is given to ϕ, and the system is evolved through the inflationary epoch. We find that the velocities have a complicated behavior as inflation proceeds, but do eventually go to zero if inflation succeeds in carrying all the spatial variations outside the "horizon size" H^{-1}. In terms of the above discussion, inflation first carries all the spatial variations in the scalar field outside the horizon. It then makes the net photon drift irrelevant because, when the wavelength exceeds the horizon size, the $\dot{\phi}^2$ and $(\phi'/R)^2$ terms in the scalar stress tensor fall off slower than does the R^{-4} behavior of the radiation field. When the quantity $(\phi'/R)^2$ gets small enough, then it and the $\dot{\phi}^2$ terms in the stress tensor fall off no faster than $\sim R^{-2}$, slowly enough that they still dominate the radiation content. Hence the "wisp of a hair" hypothesis cannot be said to hold. The inflation eventually takes the net velocity to zero.

The suggestion is frequently made that one should identify $t=$const with scalar field $\phi=$const. We have not done so principally because this would make the geometry very difficult to handle, even when it is possible to impose such a gauge. (For strongly inhomogeneous scalar fields, the defined surface may not be everywhere spacelike). At any rate, the general coordinate invariance of general relativity ensures we can work in the chosen coordinate system.

Similarly, the results here do not in any way depend on the surfaces of homogeneity of the background type-V cosmology. Averaging across a region larger than the initial horizon size allows the type-V model better to follow the dynamics than would a homogeneous-isotropic (FLRW) model which could not in any way be made consistent with velocities.

Moreover, this is a completely classical result which has nothing to do with quantum fluctuations. We justify these claims with both numerical and analytic calculations in the sections below.

II. BACKGROUND METRIC

For the background we choose the LRS type V. In a synchronous coordinate frame the metric may be written in the diagonal form

$$ds^2=-dt^2+X^2dz^2+Y^2e^{-2az}(dx^2+dy^2) . \quad (2.1)$$

Since $g_{00}=-1$ and $g_{0\alpha}=0$, the coordinate t measures proper time along the geodesics normal to the spatially homogeneous hypersurfaces of simultaneity. These coordinates, however, are not necessarily comoving with the fluid since in tilted cosmologies the perfect-fluid vector **v** does not parallel the normal vector $\mathbf{n}\equiv\partial/\partial t$. The constant a in Eq. (2.1) defines the scale of spatial curvature. Note that X^2 is analogous to $e^{2\alpha}e^{-2\beta_+}$ and $Y^2=e^{2\alpha}e^{2\beta_+}$ in the Misner[4] notation for the LRS case ($\beta_-=0$). The qualitative R of the previous discussion is a typical X or Y.

If the only source of matter has $p>0$ (particularly if it is radiation with $\gamma=\frac{4}{3}$, then there are two qualitatively distinct types of evolution:[5] The first evolves from a cigar matter singularity and expands forever without isotropizing (matter lines become null at infinity), and the second evolves from a timelike conformal singularity and approaches isotropy (FLRW) at large times and possesses a Cauchy horizon. Here we consider only the latter cases, since they are the ones which lead to an observationally consistent cosmology. The critical parameter responsible for the two different behaviors is

$$\lambda'=-\frac{8\pi T_1^0}{3a[(8\pi T_1^0/6a)^2-8\pi T_0^0/3+a^2/X^2]^{1/2}} , \quad (2.2)$$

which measures the importance of shear in the geodesic normals.[5] If $\lambda'<0$, we will get the correct behavior, i.e., asymptotic FRW evolution. For a radiation fluid ($\mu=3p$), we have $T_1^0=4pv^0v^1X^2$, where p is the fluid pressure and $v^0\geq 1$, and v^1 are the two nontrivial components of the fluid four-velocity related through the normalization $v_\alpha v^\alpha=-1$ as $v^1=\pm[(v^0)^2-1]^{1/2}/X$. For numerical work it is important to realize that $v^1\geq 0$ implies that $a>0$ is required to obtain $\lambda'<0$. Hence only certain parameter choices evolve to asymptotic FRW behavior. (For our numerical work we set $a=1$.)

The independent Einstein equations for the metric (2.1) are

$$-\frac{\dot Y^2}{Y^2}-2\frac{\dot X}{X}\frac{\dot Y}{Y}+3\frac{a^2}{X^2}=8\pi T_0^0 \quad (2.3a)$$

and

$$-2a\frac{\dot Y}{Y}+2a\frac{\dot X}{X}=8\pi T_1^0 , \quad (2.3b)$$

where we have used an overdot to mean time derivative. Equations (2.3) are the mixed (G_0^0) and (G_1^0) initial-value equations, respectively, containing only first-order derivatives. They are all we need to solve for the LRS type-V geometry. We may simplify these equations further by multiplying (2.3b) by $\dot Y/(aY)$ and adding the result to (2.3a). This eliminates the cross terms $(\dot X\dot Y)/(XY)$, and one gets a quadratic equation involving only the variable Y, which has the solution

$$\frac{\dot Y}{Y}=-\frac{8\pi T_1^0}{6a}\pm\left[\left[\frac{8\pi T_1^0}{6a}\right]^2-\frac{8\pi T_0^0}{3}+\frac{a^2}{X^2}\right]^{1/2} . \quad (2.4a)$$

Also, from (2.3b),

$$\frac{\dot X}{X}=\frac{8\pi T_1^0}{2a}+\frac{\dot Y}{Y} . \quad (2.4b)$$

Equations (2.4) and the matter-conservation equations are the only equations that need to be solved. We consider only the cases in which the universe is expanding, which requires the positive root in Eq. (2.4a).

Our model contains two forms of matter as sources to the Einstein equations: a radiation fluid ($\gamma=\frac{4}{3}$) and a scalar field ϕ. First, we discuss the radiation. In the frame of Eq. (2.1) the four-velocity is chosen to be

$$v^\alpha=(v^0,v^1,0,0) . \quad (2.5)$$

With (2.5) the nonvanishing components of the perfect-fluid energy tensor

$$T_\beta^\alpha=(\mu+p)v^\alpha v_\beta+pg_\beta^\alpha \quad (2.6)$$

become

$$T_{0r}^0=-4p(v^0)^2+p , \quad (2.7a)$$

$$T_{1r}^1=4pX^2(v^1)^2+p , \quad (2.7b)$$

$$T_{1r}^0=4pv^0v^1X^2 , \quad (2.7c)$$

$$T_{2r}^2=T_{3r}^3=p . \quad (2.7d)$$

Here the subscript r denotes radiation. The tensor components must satisfy the energy- and momentum-conservation equations $T^{\alpha\beta}{}_{;\beta}=0$. The energy equation ($\alpha=0$) is

$$(\dot\mu+\dot p)(v^0)^2-\dot p+(\mu+p)\left[2v^0\dot v^0+(v^0)^2\left[\frac{\dot X}{X}+2\frac{\dot Y}{Y}\right]+(v^1)^2X\dot X-2av^0v^1\right]=0 , \quad (2.8a)$$

and the only nontrivial momentum equation ($\alpha=3$) is

$$(\dot\mu+\dot p)v^0v^1+(\mu+p)\left[v^0\dot v^1+v^1\dot v^0+3v^0v^1\frac{\dot X}{X}+2v^0v^1\frac{\dot Y}{Y}-2a(v^1)^2\right]=0 . \quad (2.8b)$$

Substituting for the case of radiation ($\mu=3p$) and rewriting (2.8a) by using (2.8b) to eliminate the explicit appearance of p and $\dot p$ in the equation for $\dot v^0$ gives

$$\dot v^0=\frac{c_3}{c_4} , \quad (2.9a)$$

$$\dot{p}=-\frac{4p}{4(v^0)^2-1}(2v^0\dot{v}^0+c_1)\ , \qquad (2.9b)$$

where

$$c_1=(v^0)^2\left[\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}\right]+(v^1)^2\dot{X}X-2v^0v^1a\ , \qquad (2.10a)$$

$$c_2=3\frac{\dot{X}}{X}+2\frac{\dot{Y}}{Y}-2a\frac{v^1}{v^0}\ , \qquad (2.10b)$$

$$c_3=c_2-\frac{\dot{X}}{X}-\frac{4c_1}{4(v^0)^2-1}\ , \qquad (2.10c)$$

$$c_4=\frac{8v^0}{4(v^0)^2-1}-\frac{1}{v^0}-\frac{v^0}{(v^0)^2-1}\ . \qquad (2.10d)$$

In addition,

$$v^1=\frac{[(v^0)^2-1]^{1/2}}{X} \qquad (2.11)$$

is derived from the normalization condition $v^\alpha v_\alpha=-1$ and (assuming $a>0$) has the correct sign for asymptotic FRW behavior.

III. SCALAR FIELD AND VELOCITY

The scalar field is described by the tensor (1.1). The inflationary potential is chosen as

$$V(\phi)=\sigma(\phi^2-\phi_{\min}^2)^2/2\ . \qquad (3.1)$$

Explicitly writing the nonvanishing components of the stress-energy tensor for a one-dimensional space (we consider inhomogeneities along the z axis only) gives

$$T^0_{0s}=-\frac{\dot{\phi}^2}{2}-\frac{\phi'^2}{2X^2}-V(\phi)\ , \qquad (3.2a)$$

$$T^1_{1s}=\frac{\dot{\phi}^2}{2}+\frac{\phi'^2}{2X^2}-V(\phi)\ , \qquad (3.2b)$$

$$T^0_{1s}=-\dot{\phi}\phi'\ , \qquad (3.2c)$$

$$T^2_{2s}=T^3_{3s}=\frac{\dot{\phi}^2}{2}-\frac{\phi'^2}{2X^2}-V(\phi)\ , \qquad (3.2d)$$

where the subscript s is used to denote the scalar field and we have used a prime to denote differentiation with respect to z. Taking $T^{\mu\nu}{}_{;\nu}=0$ gives the partial differential equation

$$\ddot{\phi}-\frac{\phi''}{X^2}+\dot{\phi}\left[\frac{2\dot{Y}}{Y}+\frac{\dot{X}}{X}\right]+\frac{2a\phi'}{X^2}+\frac{\partial V}{\partial\phi}=0\ , \qquad (3.3)$$

for both the momentum and energy equations. Our development of the Einstein equations for X and Y assume we are dealing with a spatially homogeneous universe, but we have explicitly considered an inhomogeneous ϕ field. To reconcile this difference we assume that the average energy content of the spacetime is the source for the Einstein equations. We define average energy-tensor components for the scalar field as

$$T^0_{0s}\equiv\langle T^0_{0s}\rangle=-\frac{\langle\dot{\phi}^2\rangle}{2}-\frac{\langle\phi'^2\rangle}{2X^2}-\langle V(\phi)\rangle\ , \qquad (3.4a)$$

$$T^1_{1s}\equiv\langle T^1_{1s}\rangle=\frac{\langle\dot{\phi}^2\rangle}{2}+\frac{\langle\phi'^2\rangle}{2X^2}-\langle V(\phi)\rangle\ , \qquad (3.4b)$$

$$T^0_{1s}\equiv\langle T^0_{1s}\rangle=-\langle\dot{\phi}\phi'\rangle\ , \qquad (3.4c)$$

$$\begin{aligned}T^2_{2s}&=T^3_{3s}\\&\equiv\langle T^2_{2s}\rangle\\&=\langle T^3_{3s}\rangle\\&=\frac{\langle\dot{\phi}^2\rangle}{2}-\frac{\langle\phi'^2\rangle}{2X^2}-\langle V(\phi)\rangle\ ,\end{aligned} \qquad (3.4d)$$

where the notation $\langle f(z)\rangle$ is introduced for the spatial average of $f(z)$ defined as

$$\langle f(z)\rangle=\frac{\int f(z)dz}{\int dz}\ . \qquad (3.5)$$

The complete solution of the tilted LRS type-V metric (2.1) with uncoupled radiation and scalar fields is given by Eqs. (2.4), (2.7), (2.9)–(2.11), (3.3), and (3.4) with the total-energy tensor given by $T^\mu_\nu=T^\mu_{\nu r}+T^\mu_{\nu s}$. By "uncoupled" we mean there is no microphysical interaction between the two sources. They interact only gravitationally, and their energy tensors are separately conserved.

As mentioned in Sec. I, we measure the velocity by the tilt angle. The tilt angle is defined as the hyperbolic angle of rotation needed to diagonalize the mixed energy tensor in the orthonormal frame oriented along the x,y,z,t directions. The energy tensor in the orthonormal frame is written as

$$\tilde{T}^\mu_\nu=\begin{bmatrix}\tilde{T}^0_0 & \tilde{T}^0_1 & 0 & 0\\ \tilde{T}^1_0 & \tilde{T}^1_1 & 0 & 0\\ 0 & 0 & \tilde{T}^2_2 & 0\\ 0 & 0 & 0 & \tilde{T}^3_3\end{bmatrix}\ , \qquad (3.6)$$

where a tilde represents components in the orthonormal frame. Since the metric (2.1) is diagonal and we are transforming a mixed tensor, we have the simple result $\tilde{T}^\alpha_\beta=T^\alpha_\beta$ for $\alpha=\beta$. The only components affected are the off-diagonal terms which have the form $\tilde{T}^0_1=T^0_1/X$ and $\tilde{T}^1_0=XT^1_0$. The eigenvalues for the submatrix (0,1) of (3.6) are

$$\lambda_\pm=\frac{\tilde{T}^0_0+\tilde{T}^1_1}{2}\pm\frac{1}{2}[(\tilde{T}^1_1-\tilde{T}^0_0)^2+4\tilde{T}^1_0\tilde{T}^0_1]^{1/2}\ , \qquad (3.7)$$

with eigenvectors defined by the equations

$$(\tilde{T}^0_0-\lambda_\pm)u^0+\tilde{T}^0_1u^1=0\ , \qquad (3.8a)$$

$$\tilde{T}^1_0u^0+(\tilde{T}^1_1-\lambda_\pm)u^1=0\ . \qquad (3.8b)$$

Equations (3.8) give the equivalent expressions for the tilt angle:

$$\bar{\beta} \equiv \frac{u^1}{u^0} = -\frac{\tilde{T}^0_0 - \lambda_\pm}{\tilde{T}^0_1} \tag{3.9a}$$

or

$$\bar{\beta} \equiv \frac{u^1}{u^0} = -\frac{\tilde{T}^1_0}{\tilde{T}^1_1 - \lambda_\pm} . \tag{3.9b}$$

In Eqs. (3.9) we have used the notation $\bar{\beta}$ to mean $\bar{\beta} \equiv \tanh\beta$, where β is the usual angle of tilt.[5] Note that $\bar{\beta} \to 0$ when $\tilde{T}^i_0 = 0$. In this limit $\lambda_+ = \tilde{T}^1_1$ and $\lambda_- = \tilde{T}^0_0 < 0$. We therefore choose the smallest eigenvalue λ_- with eigenvector defined by (3.8) as the only combination producing the correct result for the case of vanishing tilt.

Substitution of (3.7) into (3.9a) using λ_- gives the general expression

$$\bar{\beta} = \frac{-2\tilde{T}^1_0}{\tilde{T}^1_1 - \tilde{T}^0_0 + [(\tilde{T}^1_1 - \tilde{T}^0_0)^2 + 4\tilde{T}^1_0 \tilde{T}^0_1]^{1/2}} . \tag{3.10}$$

Here

$$\tilde{T}^\mu_\nu = \tilde{T}^\mu_{\nu r} + \langle \tilde{T}^\mu_{\nu s} \rangle \tag{3.11}$$

includes both the radiation and scalar fields such that using (2.7) and (3.4) for the energy-tensor components gives

$$\tilde{T}^1_1 - \tilde{T}^0_0 = \langle \dot{\phi}^2 \rangle + \frac{\langle (\phi')^2 \rangle}{X^2} + 8p(v^0)^2 - 4p , \tag{3.12}$$

$$\tilde{T}^1_0 = \frac{\langle \dot{\phi}\phi' \rangle}{X} - 4pv^0[(v^0)^2 - 1]^{1/2} , \tag{3.13}$$

and

$$\tilde{T}^1_0 \tilde{T}^0_1 = -\left[\frac{\langle \dot{\phi}\phi' \rangle}{X} - 4pv^0[(v^0)^2 - 1]^{1/2} \right]^2 . \tag{3.14}$$

Note that, for radiation only, we have the result

$$\bar{\beta}_r = [(v^0)^2 - 1]^{1/2} / v^0 , \tag{3.15}$$

and for the *inhomogeneous* (not averaged) scalar field alone we have

$$\bar{\beta}_s = -\frac{2}{X} \left[\frac{\dot{\phi}\phi'}{\dot{\phi}^2 + (\phi'/X)^2 + |\dot{\phi}^2 - (\phi'/X)^2|} \right] , \tag{3.16}$$

which depends on both time and space. If $\dot{\phi}/(\phi'/X) \le 1$, then Eq. (3.16) becomes

$$\bar{\beta}_s = -\frac{\dot{\phi}}{\phi'/X} \le 1 , \tag{3.17}$$

while if $\dot{\phi}/(\phi'/X) > 1$,

$$\bar{\beta}_s = -\frac{\phi'/X}{\dot{\phi}} \le 1 , \tag{3.18}$$

so that the tilt is always bounded by the speed of light. We do see, however, the possibility of tilt growing momentarily to the speed of light. Equations (3.10) and (3.16) have no appearance of the potential $V(\phi)$; it is only the deviation from the de Sitter behavior that contributes to the inertia of the scalar field. It will turn out that the overall $1/X$ factor in (3.16) is crucial in determining the behavior of the velocity.

IV. PERTURBATIVE EQUATIONS

The calculations presented in this paper were stimulated by a derivation which shows that the tilt angle in the radiation fluid approaches a constant as the universe inflates. This may be seen using perturbative methods and denoting deviations in the four-velocity by $(v^0)^2 = 1 + \epsilon^2$, where $\epsilon \ll 1$. The zeroth-order solution is obtained from (2.9) and (2.10) by ignoring terms proportional to ϵ and to $1/X$ since $X \to \infty$. In addition, we have the late-time approximate relations

$$\frac{\dot{X}}{X} = \frac{\dot{Y}}{Y} \equiv H \sim \text{const} , \tag{4.1}$$

in the inflationary regime, which show that $X, Y \sim e^{Ht}$. With these simplifications and noting that from (2.11) $v^1 = \epsilon/X \ll 1$, Eqs. (2.10) reduce to $c_1 = H[4(v^0)^2 - 1]$, $c_2 = 5H$, $c_3 = 0$, and $c_4 \to \infty$, yielding $\dot{v}^0 = 0$ and $v^0 = 1$, corresponding to the isotropic case with zero tilt. Equation (2.9b) for the pressure then has the solution $p = p_0 e^{-4Ht}$, which shows that $p \sim X^{-4}$.

Corrections from terms due specifically to the type-V geometry modify the evolution of the four-velocity and pressure. In this case we keep terms proportional to ϵ and write, for (2.10),

$$c_1 = 4H(v^0)^2 - H - 2v^0 v^1 a , \tag{4.2a}$$

$$c_2 = 5H - \frac{2av^1}{v^0} , \tag{4.2b}$$

$$c_3 = \frac{2av^1}{v^0[4(v^0)^2 - 1]} , \tag{4.2c}$$

and

$$c_4 = \frac{8v^0}{4(v^0)^2 - 1} - \frac{1}{v^0} - \frac{v^0}{(v^0)^2 - 1} . \tag{4.2d}$$

At this point the only approximation is Eq. (4.1). With $v^1 = \epsilon/X$ and using the further approximation that $\epsilon \ll 1$, we have $c_3 \approx 2a\epsilon/(3X)$ and $c_4 \approx -\epsilon^{-2}$, which together give, for (2.9a),

$$\dot{v}^0 \approx \epsilon\dot{\epsilon} \approx -\frac{2a\epsilon^3}{3X} . \tag{4.3}$$

Equation (4.3) has the solution (using $X \propto e^{Ht}$)

$$\epsilon = \frac{1}{\epsilon_0^{-1} - 2a/3HX} , \tag{4.4}$$

which quickly approaches a constant ($=\epsilon_0$) for large X. Perturbative modifications to the pressure at this lowest order are, from (2.9b),

$$\frac{\dot{p}}{p} = -4H + \frac{8a\epsilon}{3X} , \tag{4.5}$$

with the solution

$$p = p_0 e^{-4Ht - (8a\epsilon)/(3HX)} . \tag{4.6}$$

The radiation tilt (3.15) to lowest order is given by

$$\bar{\beta} \sim \epsilon \ , \tag{4.7}$$

which, according to (4.4), approaches a constant exponentially as the universe inflates ($X \rightarrow \infty$). Hence inflation cannot remove the tilt in the photon field. The density in the photons goes to zero as $X^{-4} \sim e^{-4Ht}$, however.

We now investigate the scalar wave equation (3.3) and the behavior in the tilt when both radiation and scalar fields are introduced as sources of matter. Perturbative methods are used to obtain approximate solutions to (3.3) for the scalar field ϕ near $\phi \sim 0$, where the potential is very nearly flat. Linearization of (3.3) around $\phi = 0$ results in

$$\ddot{\phi} - \frac{\phi''}{X^2} + \dot{\phi}\left[2\frac{\dot{Y}}{Y} + \frac{\dot{X}}{X}\right] + \frac{2a\phi'}{X^2} - 2\sigma\phi = 0 \ , \tag{4.8}$$

where, because $\phi \ll 1$, we have approximated $\partial V/\partial\phi \approx -2\sigma\phi$ to first order in ϕ. In what follows we will consider solutions to the zeroth- ($\partial V/\partial\phi \approx 0$) as well as the first-order equations [the zeroth-order differential equation is just (4.8) without the term $-2\sigma\phi$].

V. A UNIVERSE WITH HIGH-FREQUENCY WAVES DOES NOT ISOTROPIZE DURING INFLATION

It is possible to solve Eq. (4.8) during inflation (H=const) via the WKB approximation in the high-frequency regime. We take short-wavelength ripples in ϕ, so that for these high-frequency waves the $\partial V/\partial\phi$ term in Eq. (4.8) can be neglected. (However, the homogeneous background is still potential dominated so that inflation takes place.) Then

$$\phi = e^{-3Ht/2} u(t) e^{ikz} \ , \tag{5.1}$$

where u solves

$$\ddot{u} + u\left[\frac{k^2 + 2iak}{X^2} - \frac{9H^2}{4}\right] = 0 \ . \tag{5.2}$$

We see that u will be oscillatory in time so long as k/X exceeds approximately $3H/2$, that is, so long as the physical wavelength is much smaller than the "horizon" size $\sim H^{-1}$. The coefficient

$$f^2 \equiv \left[\frac{k^2 + 2iak}{X^2} - \frac{9H^2}{4}\right] \ , \tag{5.3}$$

of the undifferentiated term in (5.2), approximates

$$f^2 \sim \frac{k^2}{X_0^2 e^{2Ht}} \tag{5.4}$$

(X_0=const), in such a situation, and the WKB approximate solution is then

$$\phi \sim e^{-Ht} e^{ik/(HX)} e^{ikz} \ . \tag{5.5}$$

Although the two directions of propagation are different because of the $a\phi'$ in Eq. (4.8), in the high-frequency limit the propagation is the same for both directions.

Now, from (5.5),

$$|\dot{\phi}| \sim \left|\phi \frac{ik}{HX}(-H)\right| \sim \left|\phi \frac{k}{X}\right| \tag{5.6}$$

and

$$\left|\frac{\phi'}{X}\right| \sim \left|\phi \frac{k}{X}\right| \ . \tag{5.7}$$

Hence, in this WKB limit, the tilt is constant [cf. Eqs. (3.17) and (3.18)], and inflation does not remove the velocity. However, if inflation persists, the requirement for validity of the WKB approximation fails, because X grows and eventually $k/X < H$. In that case the wave behavior will become frozen in and the solution can no longer be described as "high frequency." We investigate such behavior now.

VI. LONG-WAVELENGTH TILT BEHAVIOR

A. Zero- and first-order solutions

To derive approximate long-wavelength frozen-in fluctuations, we assume that ϕ in Eq. (4.8) has the periodic form

$$\phi = \phi_0(t) + c(t)\sin[\omega(t) + kz] \ . \tag{6.1}$$

Equation (4.8) is linear, and so $\phi_0(t)$ evolves independently of the $c(t)$ terms. Inserting the ansatz (6.1) into (4.8) and equating the sine and cosine terms independently to zero leads to

$$\ddot{c} + \dot{c}\left[2\frac{\dot{Y}}{Y} + \frac{\dot{X}}{X}\right] + c\left[\frac{k^2}{X^2} - \dot{\omega}^2 - 2\sigma\right] = 0 \ , \tag{6.2a}$$

$$\ddot{\omega} + \dot{\omega}\left[2\frac{\dot{Y}}{Y} + \frac{\dot{X}}{X} + 2\frac{\dot{c}}{c}\right] + \frac{2ak}{X^2} = 0 \ , \tag{6.2b}$$

and

$$\ddot{\phi}_0 + \dot{\phi}_0\left[2\frac{\dot{Y}}{Y} + \frac{\dot{X}}{X}\right] - 2\sigma\phi_0 = 0 \ . \tag{6.2c}$$

The zeroth-order equations for ϕ (obtained by simply dropping terms involving σ) reduce in the inflationary regime to

$$\ddot{c} + \dot{c}\left[2\frac{\dot{Y}}{Y} + \frac{\dot{X}}{X}\right] - c\dot{\omega}^2 = 0 \ , \tag{6.3a}$$

$$\ddot{\omega} + \dot{\omega}\left[2\frac{\dot{Y}}{Y} + \frac{\dot{X}}{X} + 2\frac{\dot{c}}{c}\right] = 0 \ , \tag{6.3b}$$

and

$$\ddot{\phi}_0 + \dot{\phi}_0\left[2\frac{\dot{Y}}{Y} + \frac{\dot{X}}{X}\right] = 0 \ , \tag{6.3c}$$

by ignoring terms $\propto 1/X^2$ since $X \rightarrow \infty$. This amounts to ignoring spatial gradients and is appropriate after a long period of inflation. Equations (6.3) are exactly solvable.

They admit the first integrals

$$\dot{\omega}=\frac{D}{XY^2c^2}\ , \tag{6.4a}$$

$$\dot{\phi}_0=\frac{E}{XY^2}\ , \tag{6.4b}$$

and

$$\dot{c}^{\,2}=\frac{1}{X^2Y^4}\left[K-\frac{D^2}{c^2}\right]\ , \tag{6.4c}$$

where D, E, and K are constants. The solution to (6.4c) is found by multiplying both sides by c^2 and taking the square root to give

$$\frac{1}{(Kc^2-D^2)^{1/2}}d(c^2)=\frac{2}{XY^2}dt\ , \tag{6.5}$$

which has the solution

$$c^2=\frac{D^2}{K}+K\left[\int\frac{dt}{XY^2}\right]^2\ , \tag{6.6}$$

and is solved once the behavior of X and Y are known. To find X and Y, we note that, during inflation, the matter source terms in the mixed energy tensor are potential dominated, thus allowing us to simplify Eqs. (2.4) to

$$\begin{aligned}\frac{\dot{X}}{X}=\frac{\dot{Y}}{Y}&=\left[-\frac{8\pi T_0^0}{3}\right]^{1/2}\\ &=\left[\frac{8\pi V(\phi=0)}{3}\right]^{1/2}\\ &=\left[\frac{4\pi\sigma\phi_{\min}^4}{3}\right]^{1/2}\ ,\end{aligned} \tag{6.7}$$

which, as already noted, has for solutions $X=X_0e^{Ht}$ and $Y=Y_0e^{Ht}$, where $H=(4\pi\sigma\phi_{\min}^4/3)^{1/2}$. Substituting (6.7) into (6.6) gives

$$c^2=\frac{D^2}{K^2}+K\left[F^2+\frac{1}{9H^2X^2Y^4}-\frac{2F}{3HXY^2}\right]\ , \tag{6.8}$$

where F is another constant of integration. Solutions for ϕ_0 and ω are now found to be

$$\omega\approx\omega_0-\frac{D}{3HXY^2c_0^2} \tag{6.9}$$

and

$$\phi_0=\phi_{00}-\frac{E}{3HXY^2}\ , \tag{6.10}$$

with ω_0 and ϕ_{00} as constants of integration and $c_0^2=KF^2+D^2/K^2$, and where we have included only the lowest-order terms in the solution (6.9). In this zeroth-order solution, it is evident from (6.8)–(6.10) that c, ω, and ϕ_0 approach constant values in the inflationary regime.

We now consider first-order solutions to Eqs. (6.2) with $\sigma\neq0$. Equations (6.2) can be simplified using the inflationary limit (4.1). In addition, since both X and Y are very large, we can ignore terms proportional to $1/X^2$, thus reducing Eqs. (6.2) to

$$\ddot{c}+3H\dot{c}-c(\dot{\omega}^{\,2}+2\sigma)=0\ , \tag{6.11a}$$

$$\ddot{\omega}+\dot{\omega}\left[3H+2\frac{\dot{c}}{c}\right]=0\ , \tag{6.11b}$$

and

$$\ddot{\phi}_0+3H\dot{\phi}_0-2\sigma\phi_0=0\ . \tag{6.11c}$$

Equation (6.11b) is unchanged from the zeroth-order solution and admits the first integral given by (6.4a). It is evident that $\dot{\omega}\to0$ as the universe expands, thereby "freezing" any spatial inhomogeneities. This allows us to derive simple approximations to Eqs. (6.11) as

$$c=c_0e^{\chi t}\ , \tag{6.12a}$$

$$\phi_0=\phi_{00}e^{\chi t}\ , \tag{6.12b}$$

$$\omega=\omega_0\ , \tag{6.12c}$$

where c_0, ϕ_{00}, and ω_0 are constants and where

$$\chi=-\frac{3H}{2}+\frac{1}{2}[(3H)^2+8\sigma]^{1/2} \tag{6.13}$$

represents the dominant or growing mode solution only. The lesson from Eqs. (6.1)–(6.12) is that, if the spatial gradients can be ignored (e.g., if the spatial scale is longer than the horizon scale), then each point in space evolves separately and independently as a homogeneous cosmology.

Now to ascertain the behavior in the tilt angle (3.10) for these zeroth- and first-order solutions, we compute the averaged energy tensor of the combined radiation and scalar fields $\tilde{T}^{\alpha}_{\beta}=\tilde{T}^{\alpha}_{\beta r}+\langle\tilde{T}^{\alpha}_{\beta s}\rangle$. The appropriate averaged energy-tensor components assuming $|\phi|\ll\phi_{\min}$ are

$$\langle\tilde{T}^0_{0s}\rangle=-\frac{1}{2}\dot{\phi}_0^2-\frac{1}{4}\dot{c}^{\,2}-\frac{1}{4}c^2\dot{\omega}^{\,2}-\frac{1}{4}\frac{c^2k^2}{X^2}-\frac{\sigma\phi_{\min}^4}{2}\ , \tag{6.14a}$$

$$\langle\tilde{T}^0_{1s}\rangle=-\frac{c^2k\dot{\omega}}{2X}\ , \tag{6.14b}$$

$$\langle\tilde{T}^1_{0s}\rangle=\frac{c^2k\dot{\omega}}{2X}\ , \tag{6.14c}$$

$$\langle\tilde{T}^1_{1s}\rangle=\frac{1}{2}\dot{\phi}_0^2+\frac{1}{4}\dot{c}^{\,2}+\frac{1}{4}c^2\dot{\omega}^{\,2}+\frac{1}{4}\frac{c^2k^2}{X^2}-\frac{\sigma\phi_{\min}^4}{2}\ . \tag{6.14d}$$

From these

$$\tilde{T}^1_1-\tilde{T}^0_0=\dot{\phi}^{\,2}+\frac{\dot{c}^{\,2}}{2}+\frac{c^2\dot{\omega}^{\,2}}{2}+\frac{c^2k^2}{2X^2}+8p(v^0)^2-4p\ , \tag{6.15}$$

$$\tilde{T}^1_0\tilde{T}^0_1=-\left[-\frac{c^2k\dot{\omega}}{2X}+4pv^0[(v^0)^2-1]^{1/2}\right]^2\ , \tag{6.16}$$

and

$$\tilde{T}_0^1 = \frac{c^2 k\dot{\omega}}{2X} - 4pv^0[(v^0)^2-1]^{1/2} . \tag{6.17}$$

The zeroth-order case exhibits the following behavior: $(\tilde{T}_1^1-\tilde{T}_0^0)\sim 1/X^2$, $\tilde{T}_0^1\tilde{T}_1^0\sim 1/X^8$, and $\tilde{T}_0^1\sim 1/X^4$. The averaged tilt then vanishes as $\bar{\beta}\sim 1/X^2$. Similarly, for the first-order solutions: $(\tilde{T}_1^1-\tilde{T}_0^0)\sim e^{2\chi t}$, $\tilde{T}_0^1\tilde{T}_1^0\sim 1/X^8$, and $\tilde{T}_0^1\sim 1/X^4$. Since $e^{2\chi t}\ll X=e^{Ht}$, the tilt is seen to go to zero very rapidly as $\bar{\beta}\sim 1/X^4$. Although the radiation tilt (4.7) is constant during inflation, the tilt with radiation and scalar fields together vanishes as the universe inflates. This is due to the fact that the radiation contribution to the energy tensor diminishes as $p\sim 1/X^4$ and eventually becomes negligible compared to the $\dot{\phi}^2$ and $(\phi'/X)^2$ scalar field terms.

B. Effect of boundary conditions

The rapid disappearance of the average of $\bar{\beta}$ across the grid is not only a result of inflation, but of the choice of boundary conditions (open versus closed universes). To understand this, note that once inflation takes place and X and Y are sufficiently large, the spatial variations "freeze" in time and so we can write a general solution to the linear equation (4.8) for ϕ in separable form $\phi=A(t)B(z)$. (This result is analogous to $\dot{\omega}\rightarrow 0$, described above; it merely freezes a phase shift into the spatial profile.) Substituting the separable form into the expression $\langle\dot{\phi}\phi'\rangle$ gives

$$\begin{aligned}\langle\dot{\phi}\phi'\rangle &= \langle \dot{A}BAB'\rangle\\ &= A\dot{A}\langle BB'\rangle = \frac{A\dot{A}}{2}\left\langle\frac{d}{dz}B^2\right\rangle .\end{aligned} \tag{6.18}$$

This last expression shows the spatial average of a total space derivative of B^2. Implicit in the ansatz (6.1) is a periodicity of the solution along the z axis. Because B, and therefore B^2, is periodically identified over space, the spatial average of its derivative necessarily vanishes. This has nothing to do with inflation per se; it assumes only that ϕ is periodic and can be written in separable form.

To examine the tilt behavior with an ansatz *not* periodically identified, we again assume $\phi=A(t)B(z)$ and substitute into (4.8) to get

$$\frac{\ddot{A}}{A}X^2+3\frac{\dot{A}}{A}HX^2-2\sigma X^2=\frac{B''}{B}-2a\frac{B'}{B}=b\equiv\text{const} . \tag{6.19}$$

Solutions to the spatial part of ϕ are

$$B=B_0e^{[a\pm(a^2+b)^{1/2}]z} . \tag{6.20}$$

Note that solutions may be exponential as well as sinusoidal, depending on the sign and relative difference of the constant b with the scale of spatial curvature a. Since we just discussed sinusoidal solutions above, we now concentrate on the exponential cases.

The time part of ϕ satisfies

$$\ddot{A}+3H\dot{A}-A\left[\frac{b}{X^2}+2\sigma\right]=0 . \tag{6.21}$$

For $b/X^2\ll 2\sigma$ we have the same solution derived above, namely, $A=A_0e^{\chi t}$ with χ defined by Eq. (6.13). Alternatively, we can take the usual inflationary limit[2] and neglect the $\ddot{A}$ term to give

$$\frac{\dot{A}}{A}=\frac{1}{3H}\left[\frac{b}{X^2}+2\sigma\right] , \tag{6.22}$$

which has the solution

$$A=A_0e^{2\sigma t/(3H)-b/(6H^2X^2)} . \tag{6.23}$$

A general solution for ϕ in this regime may therefore be written as

$$\phi=\phi_0e^{2\sigma t/(3H)-b/(6H^2X^2)}e^{[a\pm(a^2+b)^{1/2}]z} , \tag{6.24}$$

where $\phi_0=A_0B_0$. The solution (6.23) for a sufficiently inflated universe $(X\rightarrow\infty)$ approaches that given in (6.12) for the applicable case $\sigma<H^2$. We thus use (6.12) for the time part of ϕ.

To analyze behavior in the tilt angle, it is once again necessary to compute the averages

$$\langle\dot{\phi}^2\rangle=d\phi_0^2\chi^2e^{2\chi t} , \tag{6.25a}$$

$$\langle\phi'^2\rangle=d\phi_0^2\rho^2e^{2\chi t} , \tag{6.25b}$$

and

$$\langle\dot{\phi}\phi'\rangle=d\phi_0^2\chi\rho e^{2\chi t} , \tag{6.25c}$$

where $\rho=a\pm(a^2+b)^{1/2}$ and d is the spatial average of B/B_0 in (6.20):

$$d=\frac{\int e^{2\rho z}dz}{\int dz}=\frac{e^{2\rho L}-1}{2\rho L} , \tag{6.26}$$

where L is some coherence length along the z axis. Note now that $\langle\dot{\phi}\phi'\rangle$ does not vanish as it did in the periodic case. Substitution of (6.25) into Eqs. (3.12)–(3.14) gives $(\tilde{T}_1^1-\tilde{T}_0^0)\sim e^{2\chi t}$, $\tilde{T}_0^1\tilde{T}_1^0\sim e^{4\chi t}/X^2$, and $\tilde{T}_0^1\sim e^{2\chi t}/X$. The averaged tilt (3.10) for this case vanishes exponentially as $\bar{\beta}\sim 1/X$, and this behavior is due entirely to inflation. From Eq. (3.16) for the tilt in the inhomogeneous scalar field, we see that the ratio of quadratic functions appearing inside the large parentheses is essentially constant during inflation and it is the factor $1/X$ outside the large parentheses which forces the tilt to zero.

Inflation does remove velocities when quantum fluctuations are ignored. The behavior of the tilt requires that spatial gradients be significant, if tilt is to remain substantial. In our assumption of homogeneous background, we guarantee that a wave decomposition in z is valid. The physical wavelength is not the constant λ_z, however, but is proportional to $X\sim e^{Ht}$. Eventually, $X\lambda_z>1/H$, where H is the Hubble parameter during inflation. Hence, even if ϕ'/X exceeds $\dot{\phi}$ initially during evolution, we are certain that once $X\lambda_z > H^{-1}$, the wave freezes in and ϕ'/X thereafter decreases; the tilt angle or velocity then goes to zero. During the time that the inflation fluctuations are short wavelength enough to act like photons, the tilt angle stays large. Once they move outside the horizon, however, fluctuations in ϕ act like any other inhomogeneity: Their wavelength continues to increase exponentially until their tilt goes to zero. We now proceed numerically to justify the above claims.

VII. NUMERICAL RESULTS

In setting up a numerical grid for the discretization of differential equations, it is necessary to define spatial scales in reference to some characteristic length such as the horizon size. The horizon is approximated as the distance a photon propagating along the z axis travels since the initial singularity in the isotropic version of the metric (2.1) in a radiation-dominated universe. The scalar field is neglected in this calculation because in order to evolve data from the *start* of inflation to its natural end, radiation-dominated initial data must be supplied so as to let the expansion dynamics evolve on its own into an inflationary model. We therefore assume that the cosmological history prior to the start of any of our simulations was approximately radiation dominated. The horizon size is written as

$$L_h = \int \frac{dt}{X} = \int \frac{dX}{X\dot{X}} , \tag{7.1}$$

where X is defined by Eqs. (2.4) with $\dot{X}/X = \dot{Y}/Y$ and $T^0_0 = T^0_{0r}$:

$$\left[\frac{\dot{X}}{X} \right]^2 = \frac{a^2}{X^2} - \frac{8\pi}{3}[p - 4p(v^0)^2] . \tag{7.2}$$

Assuming that $v^0 = 1$ and $\dot{v}^0 = 0$, Eq. (2.9b) gives for the pressure $p = p_0 X^{-4}$, with p_0 a constant. Substitution of this result into (7.2) yields

$$\dot{X} = \left[a^2 + \frac{8\pi p_0}{X^2} \right]^{1/2} , \tag{7.3}$$

which in turn is used in (7.1) to provide an estimate of the horizon size:

$$\begin{aligned} L_h &= \int_0^X \frac{dx'}{(a^2 X'^2 + 8\pi p_0)^{1/2}} \\ &= \frac{1}{a} \ln \left[\frac{(a^2 X^2 + 8\pi p_0)^{1/2} + aX}{\sqrt{8\pi p_0}} \right] . \end{aligned} \tag{7.4}$$

The grid length L_g in our simulations is chosen to be some fraction of the horizon, $L_g = rL_h$.

We discretize the scalar wave equation (3.3) using a fourth-order center difference scheme. For a grid of n nodes, this results in n coupled second-order ordinary differential equations (ODE's) in time (one for each node). The resulting ODE's are then rewritten as $2n+4$ (four additional equations are needed to solve for X, Y, v^0, and p) coupled first-order equations and are integrated with a fourth-order Runge-Kutta method.

We present numerical solutions to a number of different cases using both periodic and nonperiodic boundary conditions. Periodic boundary conditions are applied simply by introducing two dummy zones on either side of the grid and identifying them with the appropriate zones within the grid. For the nonperiodic case we linearly extrapolate ϕ beyond the grid edges by setting the value of ϕ' on the edges equal to a constant which is determined internally after solving (3.3). This effectively projects the solution for ϕ linearly along the tangent direction defined by ϕ' at the zone boundaries. One might object to the use of nonperiodic boundary conditions in solving the scalar wave equation. In particular, any solution we obtain must satisfy $\langle T^{\mu\nu}{}_{;\nu} \rangle = 0$ for consistency. As we have defined our averages by (3.5), this condition is satisfied so long as the field ϕ solves Eq. (3.3) across the entire grid, including the dummy zones used for the boundary conditions. The one such example presented in Fig. 6 satisfies this condition trivially since one has homogeneous inflation at the zone edges, thus allowing Eq. (3.3) to be solved on the dummy zones by setting $\phi' = \phi'' = 0$.

For all figures presented in this paper, the z axis runs from left to right and time increases as one looks from the back to the front. The viewing angles chosen for Figs. 1–6 were found to provide the clearest representation of the tilt behavior despite the foreshortened distortion of the z axis which is resolved with 150 zones in all cases. We have taken as initial data four-velocity $v^0 = 1.001$, pressure $p = 0.001$, and spatial curvature scale $a = 1$, and we set $\sigma = 0.01$ and the constant $\phi_{\min} = 1$ in the scalar potential. Also, two additional variables α and β_+ are introduced such that $X = e^{\alpha - 2\beta_+}$ and $Y = e^{\alpha + \beta_+}$ in keeping with the Misner notation. All our simulations begin with $\alpha = 1$ and $\beta_+ = 0$.

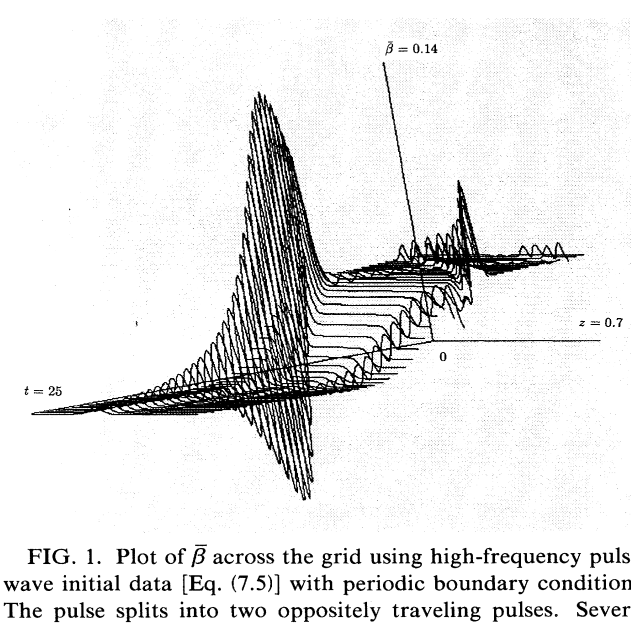

FIG. 1. Plot of $\bar{\beta}$ across the grid using high-frequency pulse-wave initial data [Eq. (7.5)] with periodic boundary conditions. The pulse splits into two oppositely traveling pulses. Several traversals of the grid by the pulses are seen. The tilt in the pulses does not decay until inflation succeeds in carrying the wavelength "outside the horizon."

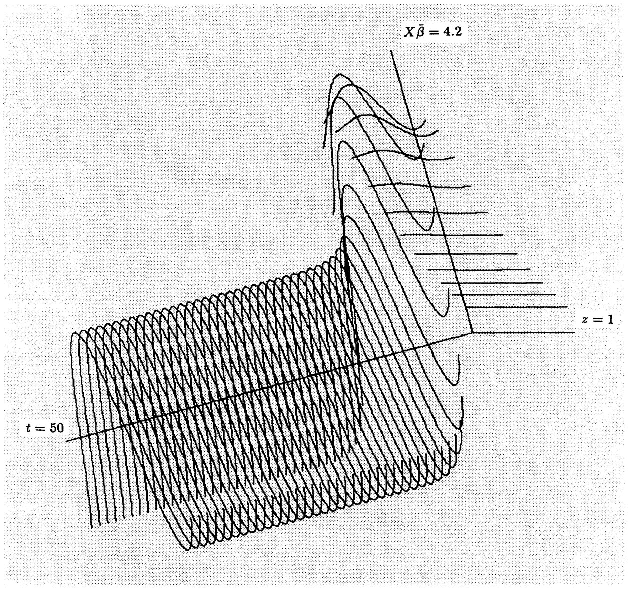

FIG. 2. Plot of $X\bar{\beta}$ across the grid using sine-wave initial data [Eq. (7.6)] with periodic boundary conditions and solving the linear scalar wave equation with $V=\sigma/2$ and $\partial V/\partial\phi=-2\sigma\phi$. Here the universe has inflated to $\alpha=12$ at $t=50$ and $\phi(z=0)=0.08$.

We show the tilt behavior of a universe with initially high-frequency waves in the form of localized pulses in Fig. 1 where we plot the spacetime evolution of $\overline{\beta}$ including both radiation and scalar fields: $\tilde{T}^{\alpha}_{\beta}=\tilde{T}^{\alpha}_{\beta r}+\tilde{T}^{\alpha}_{\beta s}$. The initial data used for the scalar field are in the form of a Gaussian wave packet:

$$\phi=\phi_0+ce^{-b^2(z-L/2)^2}\ , \tag{7.5}$$

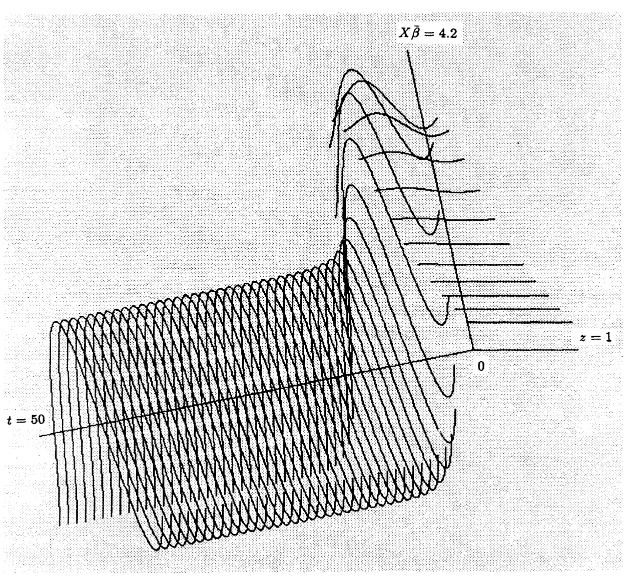

FIG. 3. As in Fig. 2 except here we solve the scalar wave equation with $V=\sigma(\phi^2-1)^2/2$ and $\partial V/\partial\phi=2\sigma\phi(\phi^2-1)$. The results are essentially identical to those of Fig. 2.

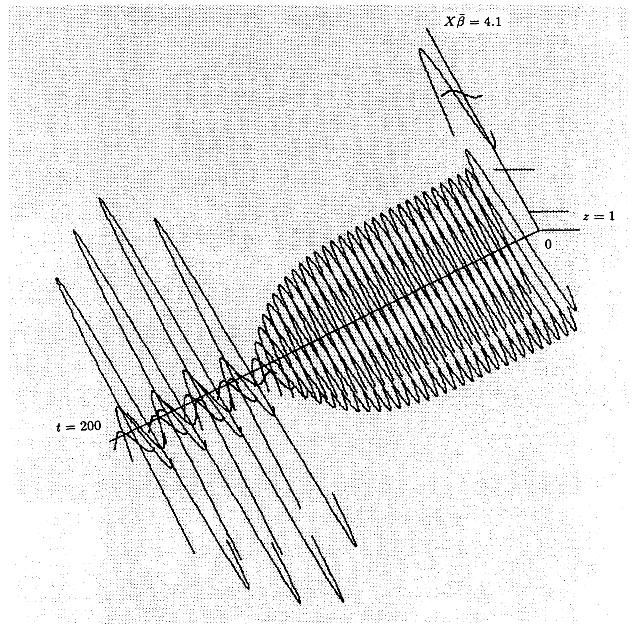

FIG. 4. As in Fig. 3 except here we follow the evolution for a longer time as $\phi(z=0)$ rolls down the potential wall and oscillates a few times inside the well. Hence the universe has inflated to $\alpha=27$ at $t=200$.

where $\phi_0=0.01$, $\dot{\phi}=0$, $c=0.001$, $b=20/L$, and $L=0.7$ is the grid length. This pulse splits subsequently into a rightward and leftward traveling pulse. The horizon, as we have defined it (7.4), is initially $L_H=1.6$ and is much greater than the pulse half-width. In this WKB limit, for which the physical wavelength is much smaller than the "horizon" size, the tilt is nonvanishing within the pulse and settles to a constant. Note that the pulse propaga-

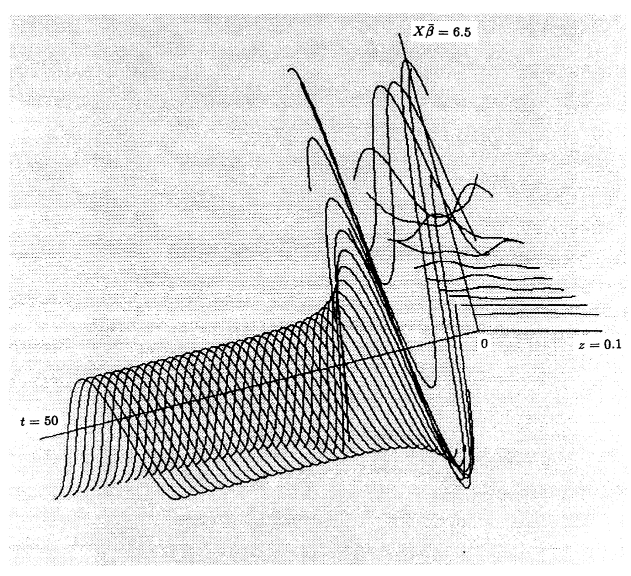

FIG. 5. As in Fig. 3 except here we use high-frequency perturbations ($L_g=0.1<L_h$). Tilt is essentially constant until the wavelength exceeds H^{-1}.

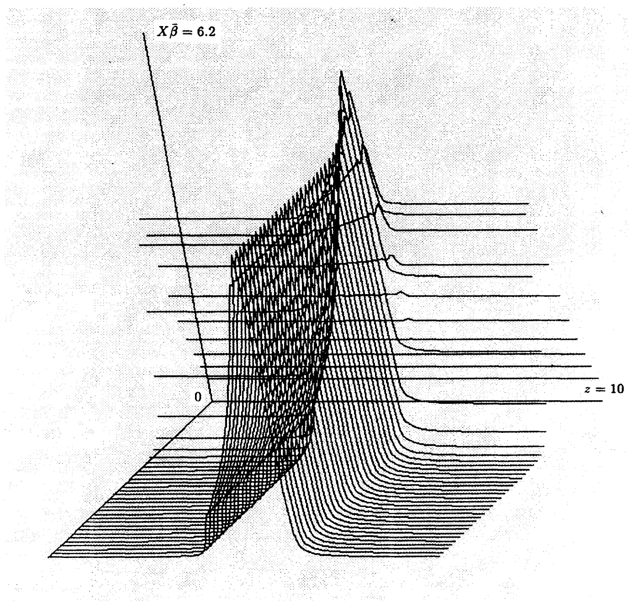

FIG. 6. Plot of $X\bar{\beta}$ using ramped initial data defined in (7.7) with nonperiodic boundary conditions.

tion in the two directions is different, because the wavelength is not short enough to make the $a\partial\phi/\partial z$ term completely negligible. However, if inflation persists as it does in Fig. 1, the WKB approximation fails and the wave behavior freezes in as the tilt is driven to zero. This is our clearest example that field velocities associated with short wavelengths do not decay away (in fact, they may increase), but as the wavelength exceeds H^{-1}, the tilt does decay away.

In Figs. 2–6 we display the spacetime evolution of the quantity $X\bar{\beta}$ including both radiation and scalar fields. The factor X removes the dominant late-time decrease and allows plotting $\bar{\beta}$. We have taken as initial data four-velocity $v^0 = 1.01$, pressure $p = 0.01$, and spatial curvature scale $a = 1.0$, and we set $\sigma = 0.01$ and the constant $\phi_{\min} = 1$ in the scalar potential. The horizon as we have defined it [Eq. (7.4)] is initially the same ($L_h = 0.68$) for all following cases.

Figures 2–5 are solutions $X\bar{\beta}$ for initial data of the form

$$\phi = \phi_0 + c\,\sin(\omega + kz) \tag{7.6a}$$

and

$$\dot{\phi} = \dot{\phi}_0 + \dot{c}\,\sin(\omega + kz) + c\dot{\omega}\cos(\omega + kz)\ , \tag{7.6b}$$

where $\phi_0 = 0.01$, $c = 0.001$, $\dot{\phi}_0 = 0.01$, and $\omega = \dot{\omega} = \dot{c} = 0$ with periodic boundary conditions and with $k = 2\pi/L_g$. Figures 2 and 3 both have a grid length $L_g = 1 > L_h$. The only difference in the two graphs is that Fig. 2 shows solutions to the linearized equations with $V(\phi) = \sigma/2$ and $\partial V(\phi)/\partial\phi = -2\sigma\phi$, while Fig. 3 solves the fully nonlinear scalar wave equation. The results are virtually identical. In both cases the universe has inflated to $\alpha \sim 12$ at $t = 50$ (the final run time) and the potential at $z = 0$ has rolled down to $\phi(z = 0) = 0.08$. The radiation tilt at the final time is equal to $\bar{\beta}_r = 0.12$ and is constant during inflation as predicted. The *averaged* scalar tilt has the same value as the combination (radiation plus scalar field) tilt $\sim 10^{-15}$ and both vanish as $\sim 1/X^4$ as predicted by our analytic work. Note that, although the spatial profile of $\bar{\beta}$ vanishes as X^{-1}, the averaged tilt goes to zero much faster as a consequence of averaging over a periodically identified grid. Our results clearly indicate that the behavior is dominantly linear, and we do not anticipate that the nonlinearities will alter our basic analytic results. In Fig. 4 we follow the evolution into the bottom of the potential well for a few oscillations. Now $\alpha = 27$ at $t = 200$ and $\bar{\beta}_r$ remains at 0.12. The averaged tilt angle oscillates about zero with amplitude $\sim 10^{-22}$.

Figure 5 shows results for the same initial data [Eq. (7.6)] as Figs. 2–4 except that here we consider disturbances of higher frequencies. The grid length is $L_g = 0.1 < L_h$. The results are essentially the same. Initially, the tilt does not decrease, but eventually the wavelength exceeds H^{-1} and the tilt decreases (pointwise) like X^{-1}. The universe at the latest time plotted has inflated to $\alpha \sim 12$. The radiation tilt $\bar{\beta}_r$ is constant and equal to 0.12. The averaged scalar and combination tilt angles are $\sim 10^{-15}$ and vanish as $\sim 1/X^4$. The difference in the high- and low-frequency cases is that the high-frequency perturbations propagate across the grid a number of times with essentially constant tilt amplitude before their spatial profiles freeze during inflation.

One final case is presented in Fig. 6. The initial data are of the form

$$\phi = \phi_0 + c\,\arctan\left[\frac{2n}{L}z - n\right]\ , \tag{7.7}$$

with $\phi_0 = 0.01$, $c = -0.001$, $\dot{\phi}_0 = 0.01$, the number of zones $n = 150$, and the grid length $L = 10$. This form was chosen as the simplest which is not periodically identified and which is flat at the grid edges so that one has homogeneous inflation at the zone edges, thus preventing any ambiguous boundary effects from entering the problem. In Fig. 6 we have $\alpha \sim 12$, $\bar{\beta}_r = 0.12$, and $\bar{\beta}_s = \bar{\beta} \sim 10^{-6}$, and its average vanishes as $\sim 1/X$ because there is no periodicity to force the average to zero.

VIII. CONCLUSIONS

We have investigated behavior in the tilt angle for a one-dimensional inhomogeneous ϕ field on a LRS type-V background. Our results show that the velocities eventually go to zero as inflation proceeds to carry all spatial variations outside the horizon. This is a classical result, and we have not pursued the interesting question of whether the presence of quantum fluctuations leads to wave fields with net tilt, i.e., net velocity.

One might object to our approach on the basis that the inhomogeneous ϕ field superimposed on a homogeneous background represents a bastard model. A fully inhomogeneous model should be explored, but because our averaging was done component by component, we do not anticipate different results.

Finally, the investigation shows, once again, the necessity of carrying out computations in nonstandard cosmological models to verify the utility of inflation.

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[1]G. F. R. Ellis, and T. Rothman, University of Cape Town report, 1986 (unpublished).

[2]J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1982).

[3](a) N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Spaces* (Cambridge University Press, Cambridge, England, 1982). Chapter V examines in detail field behavior in de Sitter backgrounds. (b) B. Allen, Phys. Rev. D **32**, 3136 (1985).

[4]C. W. Misner, Astrophys. J. **151**, 431 (1968).

[5]S. Collins and G. F. R. Ellis, Phys. Rep. **56**, 65 (1979).