PHYSICAL REVIEW D VOLUME 43, NUMBER 12 15 JUNE 1991

Numerical relativistic hydrodynamics: Local characteristic approach

José Ma. Martí, José Ma. Ibáñez, and Juan A. Miralles
Departamento de Física Teórica, Universidad de Valencia, 46100 Burjassot, Valencia, Spain
(Received 9 August 1990)

We extend some recent *shock capturing methods* designed to solve nonlinear hyperbolic systems of conservation laws and which avoid the use of artificial viscosity for treating strong discontinuities to a relativistic hydrodynamics system of equations. Some standard shock-tube problems and radial accretion onto a Schwarzschild black hole are used to calibrate our code.

INTRODUCTION

The term relativistic hydrodynamics refers both to those flows in which the bulk Lorentz factor $W \equiv (1-v^2)^{-1/2}$ exceeds one in more than a few percent (v is the flow velocity in units of the speed of light) or to those where the effects of the background gravitational field, or the one generated by the matter itself, are so important that a description in terms of the Einstein theory of gravity must be taken into account.

Relativistic hydrodynamics plays a major role in the realm of astrophysics. High-velocity outflows can be found in galactic jets,[1] binary neutron stars or binary white dwarfs[2] as well as being associated with star-forming regions.[3] In the first case the material fluid reaches the ultrarrelativistic regime ($W \approx 10$). On one hand velocities higher than 15% of light speed are reached in the stellar collapse of iron cores of massive stars which precludes supernovae II explosions;[4] and, on the other hand, the general-relativistic effects have been pointed out by several authors[5] many times.

As is well known by astrophysicists interested in theoretical models for type-II supernovae, the early mechanism of hydrodynamical bounce (the so-called "prompt mechanism") implies a strong shock wave. To discover the precise conditions under which the shock forms and the precise value of its strength is one of the numerical problems involved in this field.

Relativistic shocks are, from the mathematical point of view, one possible solution — a so-called "weak solution"— of the hydrodynamical equations given its hyperbolic character. But from the physical point of view they are a very important feature in several problems arising in several areas. (i) Astrophysics. We have already discussed it concerning the theoretical models of type-II supernovae: (ii) Cosmology. In theories of pregalactic fluctuations.[6] (iii) Plasma physics. Magnetoacoustic shock waves where speeds of up to 4×10^8 cm/sec have been achieved in laboratory.[7] (iv) Nuclear physics. In collisions among heavy ions.[7]

From the numerical point of view the correct modeling of shocks has been considered, basically, by two approaches: the so-called *shock tracking* and the *shock capturing* methods.

The shock tracking methods make use of an artificial viscosity Q put forward by von Neumann in the 1950's.[8] These methods have two advantages: (i) low cost, since they are easy to implement and do not require much CPU time, and (ii) efficiency, since, by experimenting with a few parameters, the user can spread out the shock into a small number of zones and damp down spurious oscillations behind the shock. Recently, Noh [9] has pointed out several errors induced by the artificial viscosity which are intrinsic to the method.

Although refined versions of Q methods, combined with adaptive mesh techniques,[10] have been widely used in the last years, a new generation of techniques, the so-called *shock capturing* methods, have gained the attention of people working on hydrodynamical problems. These methods have been specifically designed for solving, numerically, nonlinear hyperbolic systems of conservation laws.

A one-dimensional hyperbolic system of conservation laws is

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \xi} = \mathbf{s}(\mathbf{u}) \,, \qquad (1)$$

where $\mathbf{u}$ is the N-dimensional vector of unknowns and $\mathbf{f}(\mathbf{u})$ are N vector-valued functions. Strictly speaking a conservation law implies that the source term $\mathbf{s}(\mathbf{u})$ be zero. The above system is hyperbolic if the Jacobian matrix

$$\mathbf{A} = \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}} \qquad (2)$$

has real and distinct eigenvalues.

Originally, the shock capturing methods were based on Godunov's idea[8] of involving the jump conditions of Newtonian hydrodynamics for an ideal gas (the Rankine-Hugoniot relations) in order to solve, in each interface separating each numerical cell, the breakup of an initial discontinuity (*Riemann problem*). This approach, which avoids the use of artificial viscosity, has the advantage of incorporating the presence of a physical shock consistently and without numerical oscillations. Later modifications or extensions of this idea have led to the creation of a vast body of scientific literature that has coined the term *Godunov-type methods*.[11,12]

We have used a Godunov-type method in Newtonian spherically symmetric stellar-collapse calculations.[13] In Ref. 13 we have compared the results with those obtained by using a standard $\mathbf{Q}$ method, by fixing the initial model and the equation of state, and we have found differences in the behavior of the velocity field and the global energetics involved.

In the present paper, we are mainly interested in showing that an extension of Godunov-type methods for solving the equations of hydrodynamics in the special- and general-relativity theories is feasible. Our procedure rests on two points. (i) To identify, in each case, what are the particular forms for $\mathbf{u}$, $\mathbf{f}(\mathbf{u})$, and $\mathbf{s}(\mathbf{u})$. This is not a trivial step. The presence of pressure in these variables (see below) introduces algebraic difficulties in the analysis of the Jacobian matrix. (ii) To define a set of local characteristic variables, at each grid point, in terms of which the original system can be rewritten as a new one of uncoupled scalar equations (*local characteristic approach*[14]).

LOCAL CHARACTERISTIC APPROACH FOR THE SPECIAL-RELATIVISTIC HYDRODYNAMICS

The basic ingredients of our algorithm are the following.

(1) *Advancing in time.* Let $\mathbf{u}_j^n$ be the cell average of $\mathbf{u}$ over the cell j, having interfaces $\xi_{j-1/2}$ and $\xi_{j+1/2}$. At the next time level is

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \lambda[\widehat{\mathbf{f}}(\mathbf{u}_j, \mathbf{u}_{j+1}) - \widehat{\mathbf{f}}(\mathbf{u}_{j-1}, \mathbf{u}_j)] + \mathbf{\Delta t}\widehat{\mathbf{s}}_j(\mathbf{u}) , \tag{3}$$

where $\lambda = \Delta t/\Delta\xi_j$ and $\Delta\xi_j = \xi_{j+1/2} - \xi_{j-1/2}$. Quantities $\widehat{\mathbf{f}}$ are the numerical fluxes (see below). The source terms $\widehat{\mathbf{s}}_j$ are calculated, to linear accuracy, from the values of the variables at the zone centers and at the previous time step.

(2) *Cell reconstruction.* To obtain interface values from the cell-averaged quantities $\mathbf{u}_j$ different interpolation techniques have been used.[15] The order of the interpolation depends on the degree of spatial accuracy one wants to achieve.

(3) *Numerical fluxes.* The numerical fluxes are evaluated by extending the numerical fluxes of Roe's first-order upwind method[16] for nonlinear scalar hyperbolic conservation laws to systems, via a local characteristic approach.[14] Roe's prescription can be applied to each one of the scalar uncoupled equations. In this way, the numerical fluxes can be written in terms of the original variables as

$$\widehat{\mathbf{f}}(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}\left(\mathbf{f}_\mathrm{L} + \mathbf{f}_\mathrm{R} - \sum_{\alpha=1}^{3} \mid \widetilde{\lambda}_\alpha \mid \Delta\widetilde{\omega}_\alpha \widetilde{\mathrm{e}}_\alpha\right) , \tag{4}$$

where L and R stand for the left and right states of a given interface. $\widetilde{\lambda}_\alpha$ and $\widetilde{\mathrm{e}}_\alpha$ ($\alpha = 1, 2, 3$) are the eigenvalues (*characteristic speeds*) and the eigenvectors of the Jacobian $\mathbf{A}$, respectively, and the quantities $\Delta\widetilde{\omega}_\alpha$, the jumps of the local variables across each characteristic, are obtained from

$$\mathbf{u}_R - \mathbf{u}_L = \sum_{\alpha=1}^{3} \Delta\widetilde{\omega}_\alpha \widetilde{\mathrm{e}}_\alpha \ . \tag{5}$$

$\widetilde{\lambda}_\alpha$, $\widetilde{\mathrm{e}}_\alpha$, and $\Delta\widetilde{\omega}_\alpha$ as functions of $\mathbf{u}$ are evaluated at each interface and, therefore, they depend on the particular values $\mathbf{u}_L$ and $\mathbf{u}_R$. The tilde stands for the arithmetic average of the data of the problem in the present application.

Crucial to this local analysis is, then, the knowledge of the spectral decomposition of the Jacobian matrix of the system.

The one-dimensional equations of special-relativistic hydrodynamics (SRH), in planar symmetry, can be covered by taking

$$\mathbf{u} = (\rho W, \rho h W^2 v, \rho h W^2 - p)^T, \tag{6}$$

and the flux vector $\mathbf{f}$,

$$\mathbf{f} = (\rho W v, \rho h W^2 v^2 + p, \rho h W^2 v)^T, \tag{7}$$

where ρ is the density, p is the pressure, h is the specific enthalpy, $h = 1 + \epsilon + p/\rho$, and, ϵ is the specific internal energy.

Each component of the above system expresses, respectively, the conservation of the fluid rest mass, momentum, and total energy.

As is well known the Jacobian matrix has three real and distinct eigenvalues: v, and $(v \pm c_s)/(1 \pm v c_s)$ (see, for example, Ref. 17), where c_s is the sound velocity.

The equation of state (EOS), as usual, closes the system.

From a computational point of view, the third equation of the above system is not very useful since, in the Newtonian limit ($p \simeq \rho\epsilon << \rho$, $v << 1$) and for all practical purposes, this equation is identical to the first one. In practice, the third equation has been substituted by the equation that results when we subtract the first one from it.

Let us define the quantities

$$r \equiv \rho W \ , \tag{8}$$

$$m \equiv \rho h W^2 v \ , \tag{9}$$

$$e \equiv \rho h W^2 - p \ , \tag{10}$$

and rewrite system (1),

$$\frac{\partial \mathbf{u}}{\partial \mathrm{t}}+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathrm{x}}=0\ , \tag{11}$$

where now

$$\mathbf{u}=(r, m, e-r)^T \tag{12}$$

and the flux vector $\mathbf{f}$ is

$$\mathbf{f}=\left(\frac{rm}{e+p}, \frac{m^2}{e+p}+p, m\left(1-\frac{r}{e+p}\right)\right)^T\ . \tag{13}$$

The special role played by pressure into the set of new variables $\aleph=\{r, m, e\}$ should be noted. People working with these SRH equations (see, for example, Ref. 19) and with $\mathbf{Q}$ methods have chosen other sets of variables and have treated those terms containing the pressure as sourcelike terms. Our approach leads to some, merely algebraic, difficulties. Nevertheless, these difficulties are largely compensated by the great advantage arising from the fact that this set $\aleph$ of variables allows us to show up the conservative character of the system and to apply to it a particular Godunov-type method.

Finally, the knowledge of the set of physical variables $\wp=\{\rho, v, \epsilon\}$ at each time step requires merely to solve an implicit equation in p to which powerful methods can be applied.

LOCAL CHARACTERISTIC APPROACH FOR THE GENERAL-RELATIVISTIC HYDRODYNAMICS

The general-relativistic hydrodynamics equations in the one-dimensional case (for example, spherical symmetry) can be written[18] after some algebraic manipulation in a form well suited to our numerical applications:

$$\frac{\partial \mathbf{u}}{\partial \mathrm{t}}+\frac{1}{\sqrt{\gamma}} \frac{\partial \sqrt{\gamma} \mathbf{f}(\mathbf{u})}{\partial \mathrm{r}}=\mathbf{s}(\mathbf{u})\ , \tag{14}$$

where

$$\mathbf{u}=(D, S, \tau)^T \tag{15}$$

and the flux vector $\mathbf{f}$ is

$$\mathbf{f}=\left(\alpha \frac{DS}{\tau+D+p}, \alpha\left(\frac{S^2}{\tau+D+p}+p \gamma^{rr}\right), \alpha S\left(\frac{\tau+p}{\tau+D+p}\right)\right)^T\ , \tag{16}$$

and the source terms $\mathbf{s}(\mathbf{u})$ are

$$\mathbf{s}(\mathbf{u})=\left(-\frac{1}{2} D \frac{\partial \ln \gamma}{\partial t}, -\frac{1}{2} \alpha\left(\frac{S^2}{\tau+D+p}\right) \frac{\partial \ln \gamma_{rr}}{\partial r}+\frac{\alpha}{\gamma_{rr}} p \frac{\partial \ln (\sqrt{\gamma} / \gamma_{rr})}{\partial r}-\frac{\tau+D}{\gamma_{rr}} \frac{\partial \alpha}{\partial r}-S \frac{\partial \ln (\gamma_{rr} \sqrt{\gamma})}{\partial t},\right.$$
$$\left.-\frac{1}{2} \frac{S^2}{\tau+D+p} \frac{\partial \gamma_{rr}}{\partial t}-S \frac{\partial \alpha}{\partial r}-(D+p) \frac{\partial \ln \sqrt{\gamma}}{\partial t}\right)^T\ . \tag{17}$$

The above equations are the expression of the local laws of conservation of baryon-number density and energy-momentum in a space-time $\mathcal{M}$ where the four-dimensional metric $g_{\mu\nu}$ has been split into the objects (α, γ_{ij})

$$ds^2=-\alpha^2 dt^2+\gamma_{ij} dx^i dx^j \tag{18}$$

(latin indices run from 1 to 3), and the energy-momentum tensor $T_{\mu\nu}$ corresponds to a perfect fluid

$$T_{\mu\nu}=\rho h u_\mu u_\nu+p g_{\mu\nu} \tag{19}$$

(greek indices run from 0 to 3).
We have introduced the variables

$$D \equiv \rho W\ , \tag{20}$$

$$S \equiv \alpha T^{0r}=\rho h W^2 v / \sqrt{\gamma_{rr}}\ , \tag{21}$$

$$\tau \equiv \alpha^2 T^{00}-D=\rho h W^2-p-\rho W\ . \tag{22}$$

The quantity γ is the determinant of the matrix γ_{ij}. v is defined by $v \equiv \sqrt{\gamma_{rr}} u^r / \alpha u^0$ (indices 0 and r stand for the temporal and radial components, respectively) and represents the fluid velocity relative to an inertial observer at rest in the coordinate frame. The Lorentz-like factor is $W \equiv \alpha u^0$ and satisfies the relation

$$W=1 / \sqrt{1-v^2}\ . \tag{23}$$

In the Newtonian limit, the set of new variables $\aleph=\{D, S, \tau\}$ tends to the set $\{\rho, \rho v, \rho \epsilon+\frac{1}{2} \rho v^2\}$.

As in the special-relativistic case, from the primary variables $\aleph=\{D, S, \tau\}$ we must obtain the set of physical variables $\wp=\{\rho, v, \epsilon\}$ at each time step.

In the present paper we consider the so-called *test relativistic fluid approximation*, where the motion of the fluid is assumed to occur in a given space-time $\mathcal{M}$. Within this approach the gravitational field produced by the fluid itself is neglected in relation to the background. This procedure is strictly correct in, for example, models of accretion onto compact objects.

From the numerical point of view a *local characteristic* *approach* to the above system needs, as we have said before, to know the spectral decomposition of the Jacobian matrix.

We have obtained the characteristic speeds associated with the system by applying the theory of characteristic hypersurfaces for the quasilinear hyperbolic system of equations describing a test relativistic fluid [7] and found

$$\lambda_0 = \frac{\alpha}{\sqrt{\gamma_{rr}}} v \ , \tag{24}$$

$$\lambda_\pm = \frac{\alpha}{\sqrt{\gamma_{rr}}}(v \pm c_s)/(1 \pm v c_s) \tag{25}$$

(see Appendix for details).

SOME TESTS

We are going to comment on several of the tests that our code has overcome: (1) the reflection shock; (2) the relativistic Sod's tube problem; (3) the relativistic blast wave; and, finally, (4) radial accretion onto a Schwarzschild black hole.

As a first test we have computed the *shock reflection* problem, that is, the shock thermalization of cold, relativistically moving gas hitting a wall.[10,19] The initial conditions are $\rho = \rho_1$, $v = v_1$, and $\epsilon = 0$. An ideal-gas law of $\Gamma = \frac{5}{3}$ has been assumed.

We have made a sample of runs varying the value of v_1 in such a way that all regimes are covered: Newtonian, relativistic, and ultrarelativistic (we have arrived at $W \approx 23$). The qualitative behavior is the same as in Figs. 1 and 2. These figures show the state of the variables velocity and pressure, respectively, in a given instant of their evolution, when the shock is well formed, and for the values of initial velocity and density $v_1 = 0.8 (W = 5/3)$ and $\rho_1 = 1$. For the sake of comparison we have displayed the analytical solution (continous line). An Eulerian mesh of 100 grid points was used and no cell reconstruction has been made, so the algorithm applied in this test is only of first-order accuracy.

Our results differ from the theoretical ones by less than 0.02% for pressure; the density is more sensitive to the boundary conditions and the relative error is less than 0.2%, with the exception of a few cells nearest the wall where the relative error is less than 3.5%. These relative errors are better than the ones published by Centrella and Wilson[19] for their run No. 7. As can be seen in Figs. 1 and 2, the shock is sharply solved in typically two or three numerical points and, unlike the *mono scheme* of Hawley *et al.*,[19] is free of spurious oscillations. It should be noted that the resolution of the shock is poorer, and needs eight or nine points, when $W >> 1$, that is, when the difference between the light velocity and the initial velocity verifies $1 - v_1 = O(\Delta x^p)$, where p stands for the global accuracy of the algorithm. Later refinements (linear or parabolic cell reconstruction, etc.) can improve these results even more.

As a second test of our code, we computed the breakup of an initial pressure discontinuity in a shock-tube into its three constituent nonlinear waves (*Sod's shock-tube* problem[20]). Initial conditions are $\{p_L = 1, \rho_L = 1, v_L = 0\}$ for $0 \leq x \leq 0.5$ and $\{p_R = 0.1, \rho_R = 0.125, v_R = 0\}$ for $0.5 < x \leq 1$. An ideal-gas law of $\Gamma = 1.4$ at each side of the discontinuity is assumed. Let us point out the fact that the maximum velocity reached is ≈ 0.42, which is roughly less than one-half the Newtonian value; therefore, even though the initial conditions are the corresponding ones of the standard Sod problem we prefer to

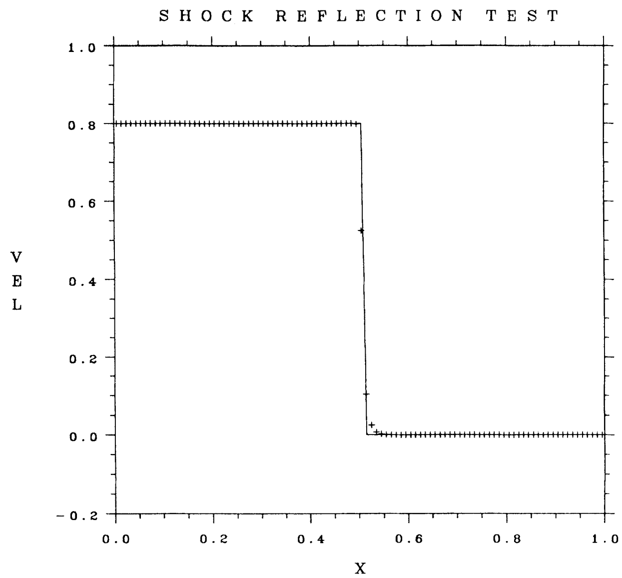

FIG. 1. Velocity (in units of the speed of light) in the relativistic *shock wall* test.

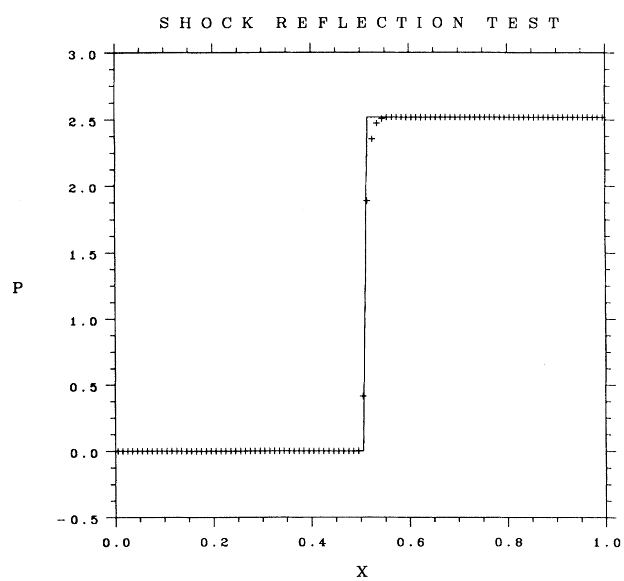

FIG. 2. Pressure in the relativistic shock wall test.

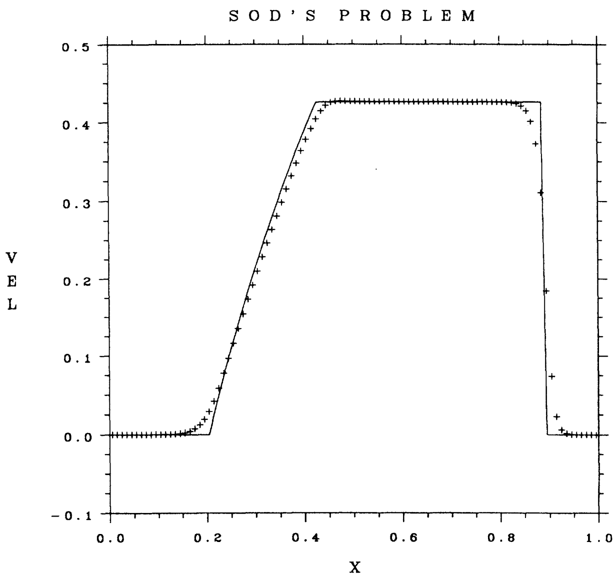

FIG. 3. Velocity (in units of the speed of light) in the relativistic *Sod's tube* test.

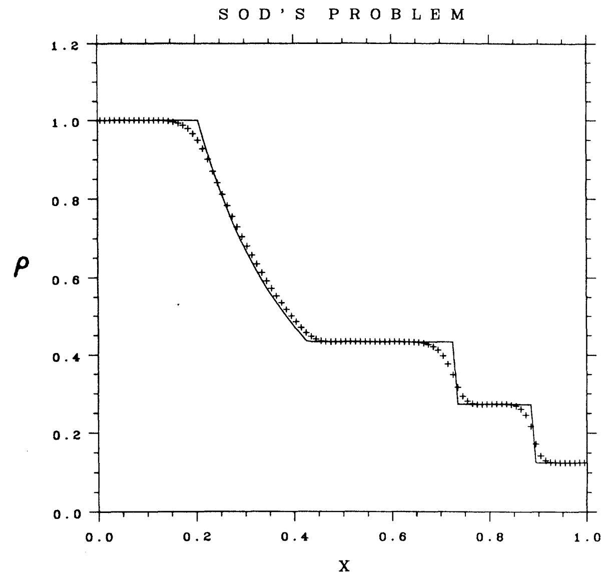

FIG. 5. Density in the relativistic Sod's tube test.

emphasize the above property by calling this test the *relativistic Sod shock-tube problem* . As before, results displayed in Figs. 3 – 5 have been obtained taking a Eulerian mesh of 100 grid points and without cell reconstruction. Numerical points are made explicit in these figures over the analytical solution (continuous line). From these figures we think that we can be confident of our code: there are no spurious oscillations, there is no overshooting in the region of the contact discontinuity, it covers the analytical solution quite well, and a finer griding or linear reconstruction would give even better results. Those interested in this can compare it with the large sample of schemes studied by Hawley *et al.* [19]

As a third test of our numerical code, we computed the formation of a *relativistic blast wave* . The initial data are $\{p_L = 10^3, \rho_L = 1, v_L = 0\}$ for $0 \leq x \leq 0.5$ and $\{p_R = 10^{-2}, \rho_R = 1, v_R = 0\}$ for $0.5 < x \leq 1$. An ideal gas law of $\Gamma = \frac{5}{3}$ at each side of the discontinuity is assumed. These initial conditions are similar to those used in the previous problem, but now we are allowing an initial pressure jump of several orders of magnitude. Figures 6–8 show the numerical points compared with the analytical solution (continuous line). We have succeeded in solving this problem free of spurious oscillations

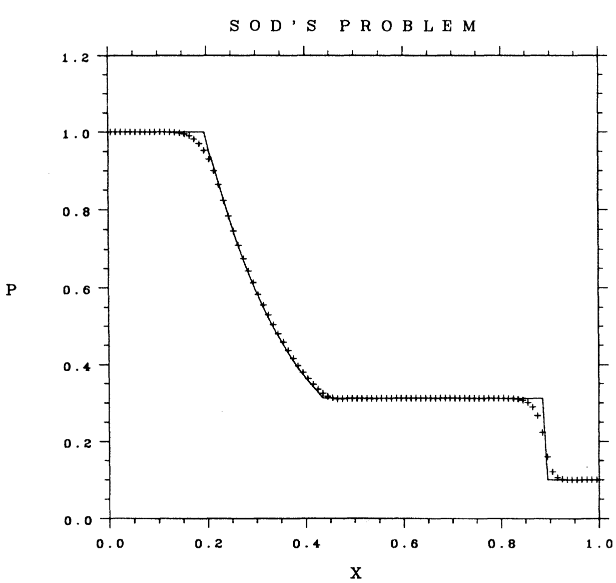

FIG. 4. Pressure in the relativistic Sod's tube test.

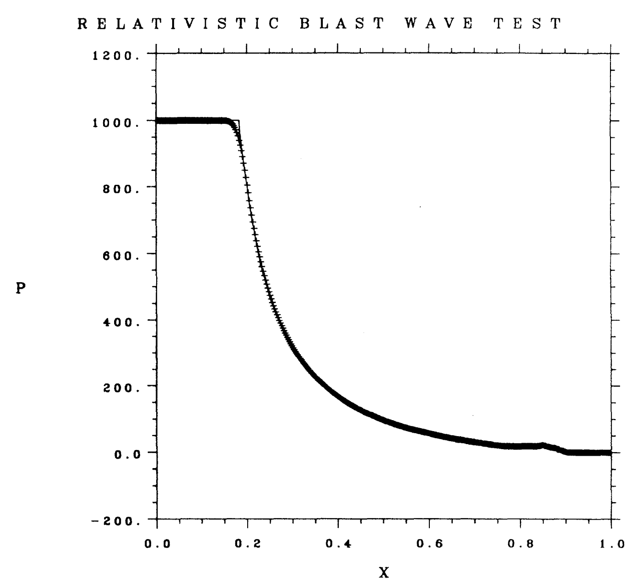

FIG. 6. Pressure in the relativistic *blast wave* test.

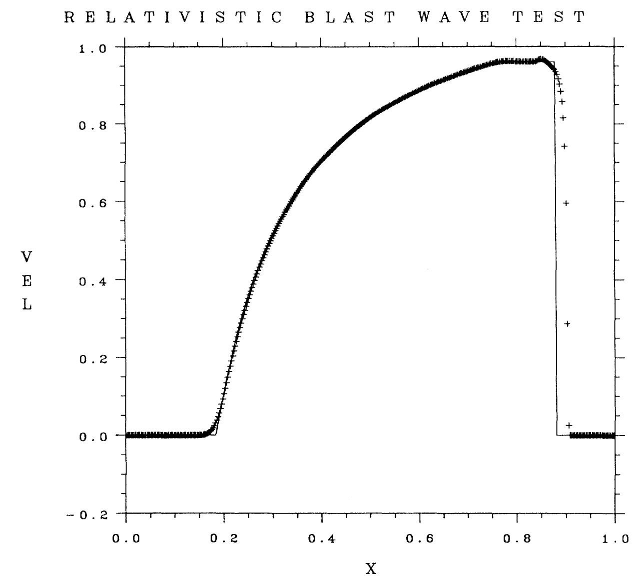

FIG. 7. Velocity (in units of the speed of the light) in the relativistic blast wave test.

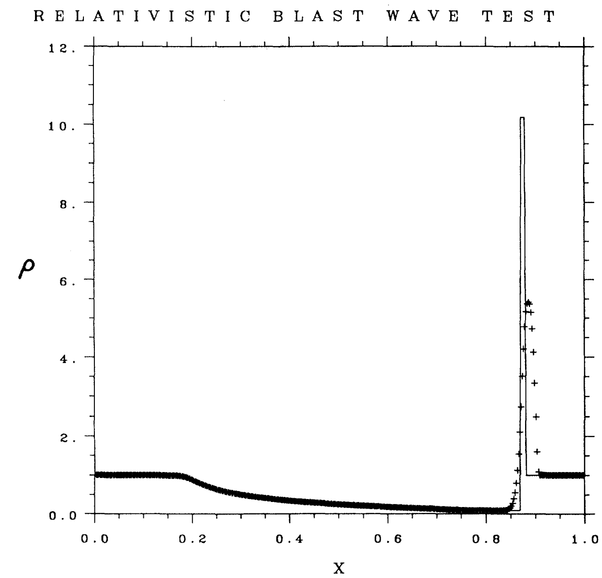

FIG. 8. Density in the relativistic blast wave test.

with a mesh of 400 points and linear cell reconstruction. Our results are similar to those obtained by Norman and Winkler[10] although they have obtained both a very good resolution and the correct value of the density in the thin dense shell bounded by the leading shock front and trailing contact discontinuity due, mainly, to their adaptive-mesh technique. The refinements introduced by using a denser mesh or a cell reconstruction are very encouraging. Concerning this test, it might be worthwhile to note the following peculiarity: Because of the relativistic composition of velocities, the characteristic field to the left of the shock converges into it in a nearly parallel way smearing this side of the shock (see Fig. 7). This fact together with the proximity of a contact discontinuity, the source of diffusion, turns this experiment into a challenging test for numerical schemes.[21]

In the above three tests gravitation is absent. This allows us to be confident of the performance of the code in treating problems involving very strong shocks, even in the ultrarelativistic regime $W >> 1$.

In order to experiment with general-relativistic hydrodynamics we have considered the problem of spherical *accretion onto a black hole.* Equations (14) to (17) have quite simple forms if steady-state conditions are assumed (see Ref. 22). We have tested our code to reproduce one of the stationary solutions. We have chosen the solution corresponding to a Schwarzschild black hole of $2M_\odot$ and having a critical point located at $r_{\rm crit} = 295$ km. We have taken a mesh of 90 grid points equally spaced in variable r^2 and spanning the interval ($r_{\rm min} = 25$ km, $r_{\rm max} = 75$ km). The analytical solution has been covered; we have displayed in Table I the relative errors as a function of time in units of a hundred times the *Courant time*, which is a very conservative unit. As can be seen in Table I the relative errors are of the order of 10^{-3}. A different grid, a logarithmic one, for example, and a more careful treatment of the boundary conditions near the singular point will lead to lower errors.

TABLE I. Relative errors (accretion test).

$t/100t_c$	Maximum relative error (10^{-3})				Mean relative error (10^{-3})			
	p	ρ	ϵ	v	p	ρ	ϵ	v
1	3.8	2.2	2.8	4.5	0.8	0.4	0.4	0.4
2	5.5	3.2	3.0	4.7	1.7	0.9	0.7	1.1
3	6.9	4.1	2.8	−5.4	2.6	1.5	1.1	1.9
4	8.2	4.9	3.3	−6.4	3.7	2.1	1.6	2.8
5	9.4	5.6	3.8	−7.3	5.0	2.9	2.1	3.8

SUMMARY AND FURTHER ISSUES

We have extended some recent *shock capturing methods* designed to solve nonlinear hyperbolic systems of conservation laws and which avoid the use of artificial viscosity for treating strong discontinuities to a relativistic hydrodynamics system of equations. We have presented some standard tests which lead us to be confident of our code. As far as we know our present work is the first to explore the use of those numerical methods in the field of relativistic hydrodynamics. The value of such an exploration has already been pointed out by Hawley *et al.*[19]

At the present time we are considering astrophysical applications (e.g., stellar collapse) in which hydrodynamics is evolving coupled with the gravitational field generated by the matter itself through Einstein equations. Extension of our code to the two-dimensional case is in progress.

ACKNOWLEDGMENTS

This work has been supported by the Spanish CICYT (grant PB87-0583-C02-02). José Ma. Martí has benefited from the financial support of the Conselleria de Cultura, Educació i Ciencia de la Generalitat Valenciana. The authors are indebted to Dr. V.Giménez for his assistance with graphical software.

APPENDIX

For clarity in exposition let us summarize the main steps concerning the theory of characteristic hypersurfaces for the quasilinear hyperbolic system of equations describing a test relativistic fluid such as can be found in Anile.[7]

Let $\mathcal{M}$ be a space-time, that is, a differentiable manifold of dimension-4, endowed with a Lorentz metric g of signature $+2$. In $\mathcal{M}$ we consider a quasilinear system of N first-order partial differential equations for the unknown field $\mathbf{u}$, which in local coordinates (x^μ) is written

$$A_B^{\alpha A}(u^C)\nabla_\alpha u^B = f^A(u^C) \tag{A1}$$

$(\alpha = 0,1,2,3)$, where the field $\mathbf{u}$, representing physical quantities, has components $u^A(x^\alpha), A = 1,2,...,N$. The components of the α $N \times N$ matrixes $A_B^{\alpha A}(u^C)$ and the N vectors $f^A(u^C)$ are differentiable functions of (u^C). Finally, $\nabla_\alpha u^B$ is the covariant derivative associated with g. In what follows we will consider the above system to be hyperbolic according to the conditions given by Anile.[7]

Let Σ be a hypersurface in space-time $\mathcal{M}$, with the local equation in local coordinates

$$\phi(x^\mu) = 0 \ . \tag{A2}$$

Then Σ is said to be a characteristic hypersurface for the above quasilinear system if Φ satisfies (Anile,[7] Sec. 2.3) either

$$u^\mu \phi_\mu = 0 \tag{A3}$$

or

$$(u^\mu \phi_\mu)^2 - c_s^2 h^{\mu\nu}\phi_\mu\phi_\nu = 0 \ , \tag{A4}$$

where $\phi_\mu = \partial_\mu \phi$ and $h^{\mu\nu}$ is the projection tensor onto the three-space orthogonal to u^μ

$$h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \ . \tag{A5}$$

In Minkowski space and the one dimensional case is $\phi = \phi(t,x)$. If we take

$$\phi = x - \lambda t \tag{A6}$$

the above equations for ϕ lead to the well-known values for λ,

$$\lambda_0 = v \ , \tag{A7}$$

$$\lambda_\pm = (v \pm c_s)/(1 \pm vc_s) \ , \tag{A8}$$

which represent the slopes of the three families of characteristic lines corresponding, respectively, to the so-called "material waves" (λ_0) and the "acoustic waves" ($\lambda_\pm$) in the special-relativistic fluid dynamics.

Let us consider a more general space-time whose line element is the one given by (25) and restrict ourselves to the spherical symmetric case. It is easy to calculate the values of λ that result from the above equations for ϕ. These are

$$\lambda_0 = \frac{\alpha v}{\sqrt{\gamma_{rr}}} \ , \tag{A9}$$

$$\lambda_\pm = \frac{\alpha}{\sqrt{\gamma_{rr}}}(v \pm c_s)/(1 \pm vc_s) \ , \tag{A10}$$

where, now, v is the velocity defined in the text ($v \equiv \sqrt{\gamma_{rr}}u^r/\alpha u^0$).

[1] M.C. Begelman, R.D. Blandford, and M.J. Rees, Rev. Mod. Phys. **56**, 255 (1984).
[2] W. Kundt, in *Astrophysical Jets*, edited by W. Kundt (Reidel, Dordrecht, 1987).
[3] K.N.R. Taylor, J.W.V. Storey, G. Sandell, P.M. Williams, and W.J. Zealey, Nature **311**, 236 (1984).
[4] G.E. Brown, H.A. Bethe, and G. Baym, Nucl. Phys. **A375**, 481 (1982).
[5] E. Baron, J. Cooperstein, and S. Kahana, Phys. Rev. Lett., **55**, 126 (1985); J.A. Miralles, J. Ma. Ibáñez, J. Ma. Martí, and A. Pérez, Astron. Astrophys. **237**, 97 (1990).
[6] E.P.T. Liang, Astrophys. J. **216**, 206 (1977); J.C. Miller and O. Pantano, Phys. Rev D **40**, 1789 (1989).
[7] A.M. Anile, *Relativistic Fluids and Magneto-fluids* (Cambridge University Press, Oxford, 1989).
[8] R. Richtmyer and K. Morton, *Difference Methods for*

Initial-Value Problems, 2nd ed. (Interscience, New York, 1967).

[9]W.F. Noh, J. Comput. Phys. **72**, 78 (1987).

[10]M.L. Norman and K-H.A. Winkler, in *Astrophysical Radiation Hydrodynamics*, edited by M.L. Norman and K-H.A. Winkler (Reidel, Dordrecht, 1986).

[11]B.A. Fryxell, E. Müller, and W.D. Arnett, in *Numerical Methods in Astrophysics*, edited by P.R. Woodward (Academic, New York, 1991); A. Harten and S. Osher, SIAM J. Num. Anal. **24**, 279 (1987); B. Van Leer, SIAM J. Sci. Stat. Comput. **5**,1 (1984).

[12]B. Einfeldt, SIAM J. Num. Anal. **25**, 294 (1988).

[13]J. Ma. Martí, J. Ma. Ibáñez, and J.A. Miralles, Astron. Astrophys. **235**, 535 (1990).

[14]H.C. Yee, in *The Tenth International Conference on Numerical Methods in Fluid Dynamics*, edited by F.G. Zhuang and Y.L. Zhu (Springer-Verlag, Berlin, 1986).

[15]P. Colella and P.R. Woodward, J. Comput. Phys. **54**, 174 (1984); B. Van Leer, *ibid.* **23**, 276 (1977); **32**, 101 (1979).

[16]P.L. Roe, J. Comput. Phys. **43**, 357 (1981).

[17]E.P.T. Liang, Astrophys. J. **211**, 361 (1977); Ch.R. McKee and S.A. Colgate, *ibid.* **181**, 903 (1973).

[18]B. Coll, Thèse d'Etat, Université de Paris VI, 1980; C. Bona (private communication).

[19]J. Centrella and J.R. Wilson, Astrophys. J. Suppl. **54**, 229 (1984); J.F., Hawley, L.L. Smarr, and J.R. Wilson, *ibid.* **55**, 211 (1984).

[20]G.A. Sod, J. Comput. Phys. **27**, 1 (1978).

[21]Concerning this point, the authors acknowledge fruitful conversations with Professor A. Marquina and Dr. R. Donat.

[22]F.C. Michel, Astrophys. Space Sci. **15**, 153 (1972).