Quadrupole moments of the decuplet baryons

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An explicit analytical expression is derived and numerical estimates are given for the quadrupole moments of the decuplet baryons in the framework of the constituent quark model up to first order in the quark-gluon coupling constant. The possibility for measurements of the quadrupole moments is also discussed.

Tensor forces between quarks are predicted by QCD and displayed in the mass splitting of orbitally excited mesons and baryons, in the $E2$ radiative decay $\Delta \Longrightarrow p\gamma$, and also in the occurrence of static quadrupole moments of the decuplet baryons. The quadrupole moments of the decuplet have been discussed in the constituent quark model[1–4] and in other approaches.[5,6] In this Brief Report we show that, in addition to the numerical estimates of Refs. 1–4, the problem of calculating the quadrupole moments of the decuplet admits of an explicit analytical solution in the first order to the quark-gluon coupling constant.

We use the method applied earlier to the calculation of the neutron charge radius in the constituent quark model.[7] Let us introduce an auxiliary potential

$$V = C_{\alpha\beta} Q_{\alpha\beta} \ , \tag{1}$$

where $C_{\alpha\beta}$ is a small traceless tensor, $Q_{\alpha\beta}$ is the operator for the quadrupole moment of a baryon,

$$Q_{\alpha\beta} = \sum_i e_i [3(x_i - R)_\alpha (x_i - R)_\beta - (\mathbf{x}_i - \mathbf{R})^2 \delta_{\alpha\beta}] \ , \tag{2}$$

where e_i are the quark charges in units of the proton charge, x_i the quark coordinates, and R defines the center-of-mass frame. The energy shift to first order in $C_{\alpha\beta}$ is given by the well-known expression

$$\Delta W = \langle \Psi | V | \Psi \rangle \ . \tag{3}$$

If $W = W(\alpha_s, C_{\alpha\beta})$ represents the energy of the system exact to all orders in α_s and includes also first-order corrections to $C_{\alpha\beta}$, then the equality

$$Q_{\alpha\beta} = \frac{dW}{dC_{\alpha\beta}} \tag{4}$$

is to be correct to all orders in α_s. We wish to find the first-order one-gluon-exchange contribution to the quadrupole moments, and so it is sufficient to restrict the expansion of the energy W up to the term $\sim \alpha_s C_{\alpha\beta}$:

$$W = W_1 + \alpha_s W_2 + C_{\alpha\beta} W_3^{\alpha\beta} + \alpha_s C_{\alpha\beta} W_4^{\alpha\beta} + \cdots \ . \tag{5}$$

In agreement with Eq. (4), we have, to zeroth-order in α_s,

$$Q_{\alpha\beta}^{[0]} = W_3^{\alpha\beta} \ , \tag{6}$$

and, to first order in α_s,

$$Q_{\alpha\beta}^{[1]} = \alpha_s W_4^{\alpha\beta} \ . \tag{7}$$

In Eq. (5) the cross term $\sim \alpha_s C_{\alpha\beta}$ can be computed in two different ways, that is, according to the scheme $O(1) \Longrightarrow O(\alpha_s) \Longrightarrow O(\alpha_s C_{\alpha\beta})$ or to the scheme $O(1) \Longrightarrow O(C_{\alpha\beta}) \Longrightarrow O(\alpha_s C_{\alpha\beta})$. This means that we may compute first the corrections of order α_s or $C_{\alpha\beta}$ to the quark wave function and, at the final stage, the energy shift $\sim \alpha_s C_{\alpha\beta}$. The potential (1) represents some kind of oscillator potential, and so the first-order correction $\sim C_{\alpha\beta}$ to the quark wave function can be obtained by simple redefinition of the model parameters.

Equation (4) implies

$$Q_{\alpha\beta} = \frac{d}{dC_{\alpha\beta}} \langle \Psi(C_{\alpha\beta}) | V_T | \Psi(C_{\alpha\beta}) \rangle \Big|_{C_{\alpha\beta}=0} \ , \tag{8}$$

where V_T is tensor part of the Fermi-Breit potential:

$$V_T = \sum_{i<j} \frac{\alpha_s n_{\alpha\beta} \sigma_{\alpha i} \sigma_{\beta j}}{6 m_i m_j r_{ij}^3} \ . \tag{9}$$

m_i are the constituent quark masses, $n_{\alpha\beta} = 3 n_\alpha n_\beta - \delta_{\alpha\beta}$, and $\mathbf{n} = (\mathbf{x}_i - \mathbf{x}_j)/|\mathbf{x}_i - \mathbf{x}_j|$.

We consider three quarks with constituent masses, μ, μ, and μ', and charges e_1, e_2, e', for quarks 1, 2, and 3, respectively, and interacting through the potential

$$U = k/2 \sum_{i<j} (\mathbf{x}_i - \mathbf{x}_j)^2 \ .$$

In standard variables

$$\boldsymbol{\rho} = (\mathbf{x}_1 - \mathbf{x}_2)/\sqrt{2} \ ,$$

$$\boldsymbol{\lambda} = (2\mathbf{x}_3 - \mathbf{x}_1 - \mathbf{x}_2)/\sqrt{6} \ ,$$

the operator of the quadrupole moment (2) takes the form

$$Q_{\alpha\beta} = e\rho_{\alpha\beta} + q\lambda_{\alpha\beta} + p(\rho\lambda)_{\alpha\beta} \ , \tag{10}$$

where

$$e=(e_1+e_2)/2\ ,$$
$$q=3e[\mu'/(2\mu+\mu')]^2+6e[\mu/(2\mu+\mu')]^2\ ,$$
$$p=\sqrt{3}(e_2-e_1)\mu'/(2\mu+\mu')\ ,$$

and

$$\rho_{\alpha\beta}=3\rho_\alpha\rho_\beta-\rho^2\delta_{\alpha\beta}\ ,$$
$$\lambda_{\alpha\beta}=3\lambda_\alpha\lambda_\beta-\lambda^2\delta_{\alpha\beta}\ , \tag{11}$$
$$2(\rho\lambda)_{\alpha\beta}=3(\rho_\alpha\lambda_\beta+\lambda_\alpha\rho_\beta)-2(\boldsymbol{\rho\lambda})\delta_{\alpha\beta}\ .$$

The perturbed wave function in the potential (1) can easily be found to be

$$\Psi(\boldsymbol{\rho},\boldsymbol{\lambda})=(m_\rho\omega_\rho/\pi)^{3/4}(m_\lambda\omega_\lambda/\pi)^{3/4}\times\exp[-m_\rho\omega_\rho\boldsymbol{\rho}^2/2-m_\lambda\omega_\lambda\boldsymbol{\lambda}^2/2-eC_{\alpha\beta}\rho_{\alpha\beta}/(2\omega_\rho)-C_{\alpha\beta}\lambda_{\alpha\beta}/(2\omega_\lambda)-C_{\alpha\beta}(\rho\lambda)_{\alpha\beta}/(\omega_\rho+\omega_\lambda)]\ , \tag{12}$$

where

$$m_\rho=\mu,\quad m_\lambda=3\mu'\mu/(2\mu+\mu')\ ,\qquad \omega_\rho^2=3k/\mu,\quad \omega_\lambda^2=k(2\mu+\mu')/\mu'\mu\ . \tag{13}$$

The values $n_{\alpha\beta}/r^3$ entering the tensor part of the Fermi-Breit potential are to be averaged over the wave function (12). For the decuplet baryons the calculations give

$$Q_{33}=-\frac{2\alpha_s}{15\sqrt{2\pi}}\left[\left[\frac{m_\rho}{\omega_\rho}\right]^{1/2}\frac{e}{\mu\mu}+\frac{e/(4m_\rho^2\omega_\rho^3)+3q/(4m_\lambda^2\omega_\lambda^3)}{[1/(4m_\rho\omega_\rho)+1/(4m_\lambda\omega_\lambda)]^{5/2}}\frac{1}{\mu\mu'}\right], \tag{14}$$

and, particularly for the Δ isobar, we get

$$Q_{33}(\Delta)=\tfrac{2}{5}\langle r_n^2\rangle e_\Delta\ , \tag{15}$$

where e_Δ is the Δ charge, $\langle r_n^2\rangle$ is the neutron charge radius in the constituent model,[7,8]

$$\langle r_n^2\rangle=-\frac{\alpha_s}{3(2\pi m_n^3\omega)^{1/2}}\ , \tag{16}$$

with m_n being the nonstrange-quark mass, and $\omega=\omega_\rho$ is defined according to Eq. (13) for $\mu=m_n$. Equation (16) can also be used to eliminate the quark-gluon coupling constant α_s and the oscillator potential strength k from expression (14). As a result, the quadrupole moments of the decuplet are expressed in terms of the neutron charge radius and the ratio between strange- and nonstrange-quark masses only. In Table I we give predictions for the quadrupole moments obtained in this way. The states with zero charge acquire nonzero quadrupole moments only in the presence of quarks with different masses. For the charged particles variations of the ratio m_s/m_n in the range 1.3–1.5 result in changes of the predictions from 0.003 fm^2 for the Σ^{*+} to 0.007 fm^2 for the Ω^-.

The calculation illustrated above uses as an unperturbed state the lowest harmonic-oscillator configuration. The spin-spin part of the Fermi-Breit potential, however mixes $0\hbar\omega$ and $2\hbar\omega$ states, and such a superposition should in principle be taken into account in Eq. (8). The resulting correction is of the second order in α_s, and numerical evaluations lead actually to a contribution of the order $\sim 30\%$.[4]

Finally, we make a few comments on the possibility of measuring the quadrupole moments of the decuplet states. As for the Δ, one can consider[13] the inelastic scattering of electrons or of polarized photons off protons. Detecting the emission of a photon prior to the Δ decay, one can extract the (small) contribution by means of a detailed analysis of the angular distribution. It should be remembered that this mechanism is very similar to the radiative pion capture used for the determination of the Δ magnetic moment.[14] The smaller widths of the Σ^* and Ξ^* states do not allow one to apply this procedure. Since Ω^- is long lived, its quadrupole moment can in principle be measured by studying energy-level splittings in hydrogenlike atoms formed by a Ω^- and a heavy nucleus. The energy of the quadrupole in an elec-

TABLE I. Quadrupole moments ($Q_{33}\times10^{-2}$ fm^2) of the decuplet baryons with $\langle r_n^2\rangle=(-11.6\pm0.5)\times10^{-2}$ fm^2 (Ref. 9) and strange-nonstrange constituent quark mass ratio $m_s/m_n=1.5$ For comparison, the results of other calculations in the nonrelativistic quark model (NRQM), Refs. 1–4, the chiral bag model (CBM, Ref. 5), and the nonrelativistic quark model with the meson couplings (Ref. 6), are shown. In Ref. 3 the ratios of the quadrupole moments of Δ and Δ charge radius are given. The absolute values in the table are obtained using the expression for the Δ charge radius from Sec. 4.3 of Ref. 10, expansion coefficients of the higher orbital and radial modes from Sec. 2, Eq. (4) of Ref. 11, and the harmonic-oscillator parameter from Ref. 12.

Baryon	Δ^{++}	Δ^+	Δ^0	Δ^-	Σ^{*+}	Σ^{*0}	Σ^{*-}	Ξ^{*0}	Ξ^{*-}	Ω^-
NRQM (Ref. 1)										1.8
NRQM (Ref. 2)	−6.6	−3.3	0.0	3.3						
NRQM (Ref. 3)	−9.8	−4.9	0.0	4.9						3.1
NRQM (Ref. 4)	−17.8	−8.9	0.0	8.9						
CBM (Ref. 5)	−12.6	−6.3	0.0	6.3						
NRQM with mesons (Ref. 6)	−6.0	−2.1	1.8	5.7	−2.2	−0.01	2.0	−0.59	1.0	0.57
NRQM (present calculation)	−9.3	−4.6	0.0	4.6	−5.4	−0.7	4.0	−1.3	3.4	2.8

tric field of a nucleus with charge Z and spin zero can be estimated to be of the order $\Delta E_Q \sim (\alpha Z)^4 Q_{33} m^3_{\Omega^-}$, where α is the fine-structure constant. As compared with the spin-orbital splitting, $\Delta E_{ls} \sim (\alpha Z)^4 l m_{\Omega^-}$, the quadrupole moment produces energy shifts of the same order of magnitude as the spin-orbit interaction in states $l \sim 1$, and its relative contribution decreases with increasing of orbital momentum as $1/l$. Exotic atoms consisting of a Σ^- and a heavy nucleus have been studied previously to determine the magnetic moment of Σ^- in an experiment performed at the Alternate Gradient Synchrotron at BNL.[15] The fine structure has been measured in x rays emitted in transitions between states with $l \sim 10$. In experiments of such a type for Ω^-, one can expect contributions $\sim 10\%$ coming from the quadrupole moment (for a more detailed discussion, see Ref. 16).

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[1]S. S. Gershtein and Yu. M. Zinoviev, Yad. Fiz. **33**, 1442 (1981) [Sov. J. Nucl. Phys. **33**, 772 (1981)].

[2]S. S. Gershtein and G. V. Dzhikiya, Yad. Fiz. **34**, 1566 (1981) [Sov. J. Nucl. Phys. **34**, 870 (1981)].

[3]N. Isgur, G. Karl, and R. Koniuk, Phys. Rev. D **25**, 2395 (1982).

[4]D. Drechsel and M. M. Giannini, Phys. Lett. **143B**, 329 (1984).

[5]M. I. Krivoruchenko, Z. Phys. C **36**, 243 (1987).

[6]W. J. Leonard and W. J. Gerace, Phys. Rev. D **41**, 924 (1990).

[7]M. I. Krivoruchenko, Yad. Fiz. **41**, 777 (1985) [Sov. J. Nucl. Phys. **41**, 495 (1985)].

[8]R. D. Carlitz, S. D. Ellis, and R. Savit, Phys. Lett. **68B**, 443 (1977).

[9]V. E. Krohn and C. E. Ringo, Phys. Rev. D **8**, 1305 (1973); L. Koester, W. Nistler, and W. Waschkowski, Phys. Rev. Lett. **36**, 1021 (1976).

[10]M. M. Giannini, Rep. Prog. Phys. **54**, 453 (1991).

[11]N. Isgur, G. Karl, and R. Koniuk, Phys. Rev. D **25**, 2394 (1982).

[12]N. Isgur and G. Karl, Phys. Rev. D **20**, 1191 (1979).

[13]V. I. Zakharov, L. A. Kondratyuk, and L. A. Ponomarev, Yad. Fiz. **8**, 783 (1969) [Sov. J. Nucl. Phys. **8**, 456 (1969)]; M. M. Giannini, in *Perspectives in Nuclear Physics at Intermediate Energies*, proceedings of the Workshop, Trieste, Italy, 1983, edited by S. Boffi, C. Ciofi degli Atti, and M. M. Giannini (World Scientific, Singapore, 1984), p. 466; D. Drechsel and M. M. Giannini (unpublished).

[14]B. M. K. Nefkens *et al.*, Phys. Rev. D **18**, 3911 (1978); P. Pascual and R. Tarrach, Nucl. Phys. **B134**, 133 (1978).

[15]D. W. Hertzog *et al.*, Phys. Rev. Lett. **51**, 1131 (1983).

[16]R. M. Sternheimer and M. Goldhaber, Phys. Rev. A **8**, 2207 (1973).