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Hadron momentum spectra in ultrarelativistic heavy-ion collisions

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Deviations from local thermal equilibrium in ultrarelativistic heavy-ion collisions are shown to arise when the hadronic collision rate cannot keep up with the rapid expansion. A method is proposed for computing hadron momentum spectra that incorporates the effect of finite collision rates. The method predicts more hadrons with high momentum than expected from the equilibrium approach.

I. INTRODUCTION

The most prominent feature of ultrarelativistic heavy-ion collisions is the exponential falloff of the hadronic cross sections with momentum $p_\perp$ transverse to the beam axis:[1,2]

$$E\frac{dN}{d^3p}\approx\exp(-p_\perp/T_{\rm eff})\ . \tag{1.1}$$

Normally, $T_{\rm eff}$ is considered a measure of the lowest, i.e., final, temperature of the rapidly cooling fireball. This is correct if the temperature decrease is gradual in space and time. But large thermal gradients can invalidate the approximation of local thermal equilibrium.[3] This paper will show that realistic thermal gradients and collision rates can seriously modify not just the coefficient of (1.1) but the value of $T_{\rm eff}$.

The necessary condition for local thermal equilibrium is that the hadronic collision rate Γ be large compared to the thermal gradients:

$$R\equiv\frac{\Gamma T^2}{|p^\mu\partial_\mu T|}\gg 1\ . \tag{1.2}$$

This condition comes directly from the relativistic Boltzman transport equation[4] for the one-particle distribution function $f(x,p)$:

$$p^\mu\partial_\mu f=C[f]\ , \tag{1.3}$$

where the collision integral $C[f]$ is an integral over the momenta of the other particles in the collision.[5] The distribution function which describes local equilibrium,

$$f_{\rm eq}(x,p)=\frac{1}{\exp(p\cdot u/T)\pm 1}\ , \tag{1.4}$$

makes $c[f_{\rm eq}]=0$. It fails to make the left-hand side of (1.3) vanish because of the space-time dependence of the fluid velocity $u^\mu(x)$ and of the temperature $T(x)$. To find the corrections to (1.4) one puts $f=f_{\rm eq}+\delta f$. Then $c[f]\approx-(p\cdot u)\Gamma\,\delta f$, where Γ is the collision rate. The transport equation becomes

$$\frac{\delta f}{f_{\rm eq}}\approx p^\mu\partial_\mu\left[\frac{p\cdot u}{T}\right]\Big/(p\cdot u)\Gamma\ . \tag{1.5}$$

Since derivatives of $p\cdot u$ are comparable to derivatives of temperature, this relation is roughly

$$\left|\frac{\delta f}{f_{\rm eq}}\right|\approx\frac{1}{R}\ , \tag{1.6}$$

with R given by (1.2). Thus δf is a small correction to the local equilibrium form provided $R\gg 1$.

The hadronic matter produced in an ultrarelativistic heavy-ion collision may not satisfy $R\gg 1$. Typically, $\partial_\mu T\approx 50$ MeV/fm $\approx 10^4$ MeV^2, which is about the same as T^2. Consequently $R\approx\Gamma/E$. Collision rates are in the range $\Gamma\approx 100-300$ MeV and decrease with E, so that at high energies R cannot be large. Therefore high-energy particles cannot stay in local equilibrium as the temperature falls.

The experimental implications of this are simple. Hadrons that emerge with high energy are likely to come from deep within the fireball, where T is large, not from the surface, where T is smaller. This will produce an excess of high-energy hadrons so that the parameter $T_{\rm eff}$ in (1.1) will be larger than expected from local equilibrium.

A. Standard model of hadron production at T_f

The standard model for hadron production proposed by Landau was put into a modern context by Cooper and Frye.[6] The model assumes that all hadrons are kept in thermal contact by continual rescatterings until some final decoupling temperature T_f, after which the hadrons are effectively free. Since different hadron species have different collision rates they will consequently have different freezeout temperatures. The space-time points of the fluid that lie on isotherm T_f define a three-dimensional hypersurface with surface element $d\sigma_\alpha$. The flux of particles through this surface gives a Lorentz-invariant hadron spectrum

$$E\frac{dN}{d^3p}=g\int_{[T_f]}\frac{d\sigma_\alpha p^\alpha}{(2\pi)^3}\frac{1}{\exp(p\cdot u/T_f)\pm 1}\ , \tag{1.7}$$

where g is the degeneracy factor of the hadron species. Figure 1 shows a typical isotherm in the r-t plane for a spherically symmetric expansion ($r=|\mathbf{r}|$). If there is no fluid velocity, $u^\mu=(1,0,0,0)$, then the exponential can be

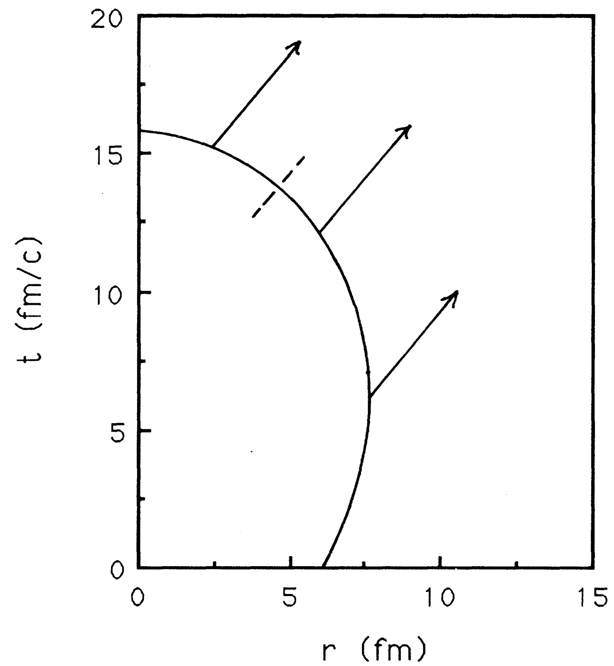

FIG. 1. Hadrons of a fixed momentum **p** emitted from a typical decoupling surface defined by the isotherm $T_f(r,t)=150$ MeV. The dashed line shows the separation between the timelike part of the surface and later t and the spacelike part of the surface at earlier t.

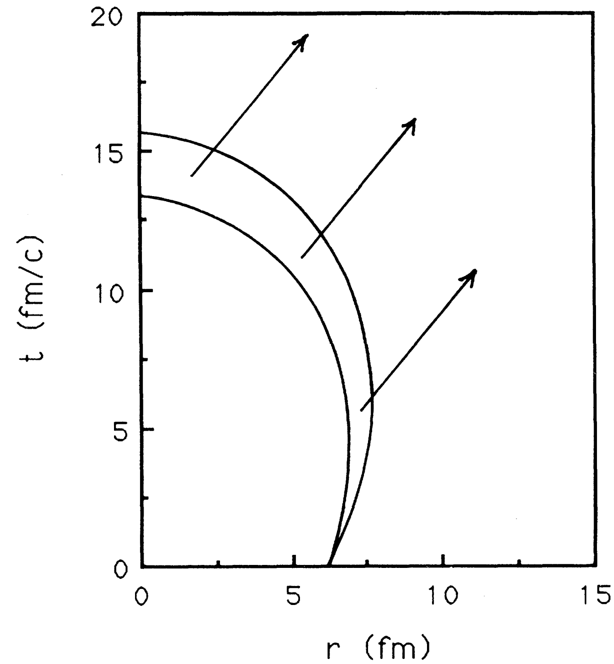

FIG. 2. Hadrons of a fixed momentum **p** emitted from the space-time volume between the inner isotherm of the hadron phase, $T_i=200$ MeV, and the outer isotherm $T_f=150$ MeV.

brought outside the integral and the spectrum becomes $\exp(-E/T_f)$. When there is a fluid velocity it is necessary to integrate before comparing to the data. The saddle-point approximation gives the behavior (1.1) with $T_{\rm eff}=T_f\exp(\eta_m)$, where η_m is the maximum transverse rapidity of the fluid on the decoupling isotherm.[7] The systematic application of (1.7) using the equation of state plus hydrodynamics has been developed in Refs. 8–10 and is reviewed in Refs. 11 and 12.

The neglect of particle emission from the interior of the region, where $T>T_f$, was checked numerically in Ref. 13. The cascade code of Bertsch *et al.*[14] was used to follow individual trajectories of pions. For a decoupling surface with a transverse radius of 4–5 fm, the last collision point of the pions typically occurred anywhere from 2–3 fm below the nominal decoupling surface to 2–3 fm above it.

B. Contributions from higher T

This paper is an attempt to improve on (1.7) by including the effects of $\Gamma\neq\infty$. The new feature is to include hadrons that decouple at a higher temperature $T>T_f$ and subsequently stream through the fluid without changing their momentum. As shown in Fig. 2, such hadrons can come from anywhere within the space-time volume bounded by the isotherms T_i, the initial temperature of the hadron phase, and T_f, the final decoupling temperature of the hadron phase. The hadrons that propagate with unchanging momentum **p** do undergo forward scattering, which is accounted for as follows. The hadrons propagate through a media of density N with a complex index of refraction

$$n=1+\frac{2\pi N}{p^2}f(E,0)\ , \tag{1.8}$$

where $f(E,0)$ is the hadronic forward-scattering amplitude. The probability of propagating a distance ds is reduced by

$$|\exp(inp\,ds)|^2=\exp(-ds/\lambda)\ , \tag{1.9}$$

where $\lambda^{-1}=N4\pi\,{\rm Im}f(E,0)/p=N\sigma$ is the inverse mean free path and σ is the total cross section. This attenuation represents the fact that after several mean free paths most hadrons will have had their direction and/or their energy changed. it can also be expressed as

$$\exp(-ds/\lambda)=\exp(-\Gamma\,dt)\ , \tag{1.10}$$

where $\Gamma=vn\sigma$ is the collision rate.

It is easy to make a crude comparison of the hadronic production rates at a T slightly higher than T_f. Assume there is no fluid velocity. The contribution of high-energy hadrons that are produced at the decoupling isotherm T_f is proportional to

$$N(T_f)=\exp(-E/T_f)\ . \tag{1.11}$$

Hadrons of the same energy that come from an isotherm at higher temperature $T > T_f$ will be exponentially attenuated:

$$N(T)=\exp(-\Gamma dt)\exp(-E/T) , \tag{1.12}$$

where dt is the time required for the hadron to go from isotherm T to isotherm T_f. Despite the exponential attenuation, $N(T)$ can be larger than $N(T_f)$ when T is large, i.e., when there is a large temperature gradient. A hadron that moves from $(t,\mathbf{x})$ on isotherm T to $(t+dt,\mathbf{x}+\mathbf{v}\,dt)$ on isotherm T_f, experiences a temperature decrease of magnitude

$$|dT|=\left|\frac{\partial T}{\partial t}+\mathbf{v}\cdot\nabla T\right|dt=\left|p^\mu\frac{\partial T}{\partial x^\mu}\right|\frac{dt}{E} . \tag{1.13}$$

Thus

$$\Gamma dt=\Gamma E|dT|/|p^\mu\partial_\mu T| . \tag{1.14}$$

Using this, the ratio of hadron multiplicities becomes

$$\frac{N(T)}{N(T_f)}=\exp\left[(1-R)\frac{E|dT|}{T_f^2}\right] , \tag{1.15}$$

with R defined in (1.2). This ratio is small if $R \gg 1$ as required for local equilibrium and as assumed in (1.7). But the ratio is large when $R \not> 1$ and grows exponentially with E/T_f, thus invalidating (1.7).

This crude comparison of particle production from different isotherms contains the essential physics to be developed in this paper. In Sec. II the contribution from each isotherm is summed to obtain a general formula for the hadron momentum distribution that replaces (1.7). It can be written as

$$E\frac{dN}{d^3p}=g\int\frac{d^4x}{(2\pi)^3}\exp\left[-\int_t^{t_f}dt'\frac{\bar{E}}{E}\Gamma\right]\frac{\bar{E}\Gamma}{\exp(\bar{E}/T)-1} , \tag{1.16}$$

where $\bar{E}=p\cdot u$ and the integration is over the space-time volume of fluid between isotherms T_i and T_f as shown in Fig. 2. The collision rate Γ depends on space time through the temperature and the particle energy, $\bar{E}$, in the comoving frame. The result reduces to (1.7) in the limit $\Gamma\to\infty$. Since the new formula depends not just on the final isotherm $T_f(x)$ but on the earlier isotherms $T(x)$ as well, one should really solve the hydrodynamic equations for $T(x)$ and $u^\mu(x)$. Instead of this, Sec. III displays a pedagogical calculation of the hadron distributions using a simple analytic guess for the isotherms and no fluid velocity. Section IV concludes with some discussion of the functional dependence of the results on various parameters. For simplicity all chemical potentials have been omitted, but one can easily incorporate them.

II. INCLUSION OF HADRONS PRODUCED AT $T > T_f$

A. General outline

It is convenient to first outline the argument leading to (1.16). The isotherm T_f defines the usual decoupling surface and it is assumed that outside that surface no hadronic interactions take place. A hadron trajectory that emerges from the decoupling surface with momentum $\mathbf{p}$ can be followed backward in time until the point where its momentum last changed (or to where the hadron was created with momentum $\mathbf{p}$). This last nonforward scattering occurred at a space-time point x, which lies on an isotherm $T(x)$. Let $\mathcal{P}$ be the inclusive probability of producing this hadron plus anything else. The probability will depend on the type of hadron, on its energy-momentum $(E,\mathbf{p})$ and on the local temperature $T(x)$ and fluid velocity $u^\mu(x)$. In the local equilibrium approximation

$$\mathcal{P}=\frac{g}{(2\pi)^3}\frac{1}{\exp(p\cdot u/T)\pm 1} . \tag{2.1}$$

The single-particle current due to that particular isotherm is

$$J^\alpha(x)=\int\frac{d^3p}{E}p^\alpha\mathcal{P} . \tag{2.2}$$

The integrand of (2.2) gives the current produced by hadrons of a particular momentum $\mathbf{p}$. Hadrons of momentum $\mathbf{p}$ that subsequently undergo only forward scattering will be attenuated exponentially:

$$A\exp(-B)\frac{d^3p}{E}p^\alpha\mathcal{P} , \tag{2.3}$$

where A and B are functions to be determined. The observed number of hadrons N results from integrating over the surface $d\sigma_\alpha$ of each isotherm T, and then integrating over all isotherms:

$$N=\int_{T_f}^{T_i}dT\int_{[T]}d\sigma_\alpha\int\frac{d^3p}{E}p^\alpha A\exp(-B)\mathcal{P} . \tag{2.4}$$

Here T_i is the initial temperature at which the hadron phase exists (≈ 200 MeV) and T_f is the final temperature at which they decouple (≈ 150 MeV). The differential multiplicity is

$$E\frac{dN}{d^3p}=\int_{T_f}^{T_i}dT\int_{[T]}d\sigma_\alpha p^\alpha A\exp(-B)\mathcal{P} . \tag{2.5}$$

In the limit that the collision rate $\Gamma\to\infty$, this calculation must agree with (1.7). Thus we require

$$\lim_{\Gamma\to\infty}A\exp(-B)=\delta(T-T_f) . \tag{2.6}$$

Before deducing the functions A and B, it is useful to first discuss the purely mathematical problem of integrating over the surface defined by an isotherm.

B. The isotherm surface element $d\sigma_\alpha$

Each isotherm $T(x)$ defines a three-dimensional hypersurface on which the temperature is constant.[15] One can parametrize the hypersurface by three new coordinates ξ^1,ξ^2,ξ^3. The space-time volume element in the new coordinates is

$$d^4x=J\,dT\,d^3\xi , \tag{2.7a}$$

$$J=\epsilon_{\alpha\beta\mu\nu}\frac{\partial x^\alpha}{\partial T}\frac{\partial x^\beta}{\partial\xi^1}\frac{\partial x^\mu}{\partial\xi^2}\frac{\partial x^\nu}{\partial\xi^3} . \tag{2.7b}$$

Any choice of coordinates is acceptable as long as J is finite and nonzero. (In simple models the hypersurface may have spherical or cylindrical symmetry and then it is natural to choose those coordinates.)

To integrate over the three-dimensional surface at constant T requires the integration element $d\sigma_\alpha$:

$$d\sigma_\alpha=\epsilon_{\alpha\beta\mu\nu}\frac{\partial x^\beta}{\partial\zeta^1}\frac{\partial x^\mu}{\partial\zeta^2}\frac{\partial x^\nu}{\partial\zeta^3}d\zeta^1\,d\zeta^2\,d\zeta^3\ . \tag{2.8}$$

For later use note that

$$d\sigma_\alpha\frac{\partial x^\alpha}{\partial T}=J\,d^3\zeta\ \text{ and }\ d\sigma_\alpha\frac{\partial x^\alpha}{\partial\zeta^j}=0\ . \tag{2.9}$$

The normal to the isotherms is the covariant vector $\partial T/\partial x^\alpha$. One can directly demonstrate that $d\sigma_\alpha$ is parallel to this normal. First note that the set of covariant vectors $\partial T/\partial x^\alpha$ and $\partial\zeta^j/\partial x^\alpha$ $(j=1,2,3)$ are linearly independent because

$$-\epsilon^{\alpha\beta\mu\nu}\frac{\partial T}{\partial x^\alpha}\frac{\partial\zeta^1}{\partial x^\beta}\frac{\partial\zeta^2}{\partial x^\mu}\frac{\partial\zeta^3}{\partial x^\nu}=\frac{1}{J}\neq 0\ . \tag{2.10}$$

The vector $d\sigma_\alpha$ can therefore be written as a linear combination of these vectors:

$$d\sigma_\alpha=C\frac{\partial T}{\partial x^\alpha}+\sum_{k=1}^{3}D_k\frac{\partial\zeta^k}{\partial x^\alpha}\ . \tag{2.11}$$

Contracting this with contravariant vectors and using the chain rule gives

$$C=d\sigma_\alpha\frac{\partial x^\alpha}{\partial T},\quad D_j=d\sigma_\alpha\frac{\partial x^\alpha}{\partial\zeta^j}\ . \tag{2.12}$$

By (2.9), $C=J\,d^3\zeta$ and $D_j=0$. Consequently

$$d\sigma_\alpha=\frac{\partial T}{\partial x^\alpha}J\,d^3\zeta\ . \tag{2.13}$$

This is a useful way of computing $d\sigma_\alpha$ even for the standard model (1.7). The normal to the isotherms will usually be timelike on some parts of the surface and spacelike on others as noted in Fig. 1. By contrast, the fluid four-velocity u_α is always timelike. (In simple models such as the Bjorken one-dimensional model[16] the normal is everywhere timelike and parallel to u_α, but this is not true generally.)

C. Attenuation in time

The attenuation exponent B that appears in (2.5) is to describe hadrons that are produced on an isotherm T and propagate through the fluid with constant four-momentum p^α until decoupling at isotherm T_f. Each space-time point in the fluid may be described either by its laboratory coordinate x^μ or by the coordinate in which the fluid is instantaneously at rest $\bar{x}^\mu$. The hadron four-momentum in the locally comoving frame is

$$\bar{p}^\mu=\frac{\partial\bar{x}^\mu}{\partial x^\alpha}p^\alpha\ . \tag{2.14}$$

In particular, the hadron energy in this frame is

$$\bar{E}=\frac{\partial\bar{t}}{\partial x^\alpha}p^\alpha=u_\alpha p^\alpha\ , \tag{2.15}$$

and the magnitude of the hadron velocity $\bar{v}=\bar{p}/\bar{E}$, where $\bar{p}=(\bar{E}^2-m^2)^{1/2}$.

In accordance with (1.10), a hadron that propagates through the fluid with only forward scattering will be attenuated by $\exp(-B)$ where

$$B=\int\Gamma\,d\bar{t}\ . \tag{2.16}$$

The collision rate Γ is a function of $T(x)$ and the local hadron momentum $\bar{p}(x)$. (Hadrons which happen to be moving with the fluid have $\bar{p}=0$ and suffer no damping because $\Gamma=0$.) Performing this integration directly is impractical because, although the hadron trajectory is straight in the laboratory coordinate,

$$dx^\alpha=p^\alpha\,dt/E\ , \tag{2.17}$$

it is curved in the locally comoving coordinate. Consequently it is simpler to transform (2.16) to the laboratory coordinates. The comoving time $\bar{t}$ is simply related to the laboratory time by

$$d\bar{t}=\frac{\partial\bar{t}}{\partial x^\alpha}dx^\alpha=u_\alpha\,dx^\alpha=\frac{\bar{E}}{E}dt\ . \tag{2.18}$$

Thus B can be expressed as an integral over laboratory time t:

$$B(t,\mathbf{x})=\int_t^{t_f}dt'\frac{\bar{E}}{E}\Gamma\ . \tag{2.19}$$

The path of integration is as follows. The hadron is produced at $(t,\mathbf{x})$ on an isotherm T. At a later time t' it will be at position $\mathbf{x}'=\mathbf{x}+\mathbf{v}(t'-t)$. It will decouple at a time t_f determined by $T_f=T(t_f,\mathbf{x}+\mathbf{v}(t_f-t))$. Both the integrand and the decoupling time t_f depend on the initial t. Consequently,

$$\left[\frac{\partial B}{\partial t}\right]_{\mathbf{x}}\neq-\frac{\bar{E}}{E}\Gamma\ . \tag{2.20}$$

D. Attenuation in temperature

In (2.19) B is written as a function of the space-time point $(\mathbf{x},t)$ where the hadron is produced. The same space-time point can also be labeled by the curvilinear coordinates T and ζ. As the time changes from t to $t+dt$, the hadron moves from $\mathbf{x}$ to $\mathbf{x}+\mathbf{v}dt$ and experiences a temperature change of

$$dT=\left[\frac{\partial T}{\partial t}+\mathbf{v}\cdot\nabla T\right]dt=\left[p^\mu\frac{\partial T}{\partial x^\mu}\right]\frac{dt}{E}\ . \tag{2.21}$$

Thus we can express B as an integral over isotherms:

$$\tilde{B}(T,\zeta)=\int_T^{T_f}dT'\frac{\bar{E}}{\psi}\Gamma,\quad \psi=p^\mu\frac{\partial T'}{\partial x^\mu}\ . \tag{2.22}$$

The function ψ was already encountered in (1.14)

Because the integrand does not depend on the initial temperature T,

$$\left[\frac{\partial\tilde{B}}{\partial T}\right]_\zeta=-\frac{\bar{E}}{\psi}\Gamma\ . \qquad (2.23)$$

The previously undetermined function A in (2.5) is $-\bar{E}\Gamma/\psi$ and satisfies

$$\lim_{\Gamma\to\infty}-\frac{\bar{E}}{\psi}\Gamma\exp(-\tilde{B})=\delta(T-T_f)\ . \qquad (2.24)$$

The explicit form of (2.5) is

$$E\frac{dN}{d^3p}=\int_{T_i}^{T_f}dT\int_{[T]}d\sigma_\alpha p^\alpha\frac{\bar{E}}{\psi}\Gamma\exp(-\tilde{B})\mathcal{P}\ . \qquad (2.25)$$

Even for finite Γ one can relate (2.25) to the usual result. Because of (2.23) one can integrate by parts on temperature to get

$$E\frac{dN}{d^3p}=\int_{[T_f]}d\sigma_\alpha p^\alpha\mathcal{P}+\mathrm{OT}\ , \qquad (2.26)$$

where the first term is the standard surface contribution (1.7), independent of Γ. OT denotes the two other terms resulting from the integration. One term is positive and one negative. Their sum OT can have either sign in general.

E. Space-time simplification

The complete four-dimensional integral (2.25) for the hadron spectra looks quite formidable. To actually compute it, the function ψ must be expressed in terms of T and ζ. Surprisingly, all these complications will disappear by using the mathematics of Sec. II B. The integration element is

$$dT\,d\sigma_\alpha p^\alpha=dT\left[\frac{\partial T}{\partial x^\alpha}p^\alpha\right]J\,d^3\zeta=\psi\,dT\,J\,d^3\zeta$$

because of (2.13). By (2.7) this can be changed to a space-time integration:

$$dT\,d\sigma_\alpha p^\alpha=\psi\,d^4x\ . \qquad (2.27)$$

When this is substituted into (2.25) the function ψ cancels and the result can be written as an integration over the space-time history of fluid:

$$E\frac{dN}{d^3p}=\int d^4x\,\bar{E}\Gamma\exp\left[-\int_t^{t_f}dt'\frac{\bar{E}}{E}\Gamma\right]\mathcal{P}\ . \qquad (2.28)$$

This is straightforward to compute given the space-time dependence of $T(x)$ and $u^\mu(x)$.

III. SAMPLE CALCULATION

To understand how (2.28) differs from the usual treatment it is helpful to discuss a specific calculation. For simplicity suppose there is no fluid velocity $u^\mu=(1,0,0,0)$, and that the isotherms depend only on t and the radial coordinate $r=|\mathbf{x}|$. Such isotherms are characteristic of a collision in which the heavy ions are completely stopped and the fireball expands spherically. Instead of attempting to solve the hydrodynamic equations, I will just adopt a simple analytic form that has reasonable qualitative properties:

$$T=\frac{bt}{t^2+a(r^2-r_0^2)}\ , \qquad (3.1)$$

where $a=1.6$, $b=9.0$, and $r_0=6.0$ fm. The extreme isotherms, $T_i=200$ MeV and $T_f=150$ MeV are shown in Fig. 2. One can also write (3.1) as

$$a(r_0^2-r^2)=t\left[t-\frac{b}{T}\right]\ . \qquad (3.2)$$

For definiteness the momentum distributions will be those of pions, although neither the mass nor the statistics is important when E is large. Different species would, of course, have different values for Γ, T_f, and possibly for T_i.

A. Standard calculation

Since there is no fluid velocity, the usual surface formula (1.7) becomes

$$E\frac{dN}{d^3p}=\frac{1}{(2\pi)^3}\frac{1}{\exp(E/T_f)-1}\int d\sigma_\alpha p^\alpha\ . \qquad (3.3)$$

Because of the spherical symmetry it is convenient to choose $(\zeta^1,\zeta^2,\zeta^3)=(r,\theta,\phi)$. The surface element normal to the isotherm is

$$d\sigma_\alpha=\left[\frac{\partial T}{\partial t},-\hat{r}\frac{\partial T}{\partial r}\right]J\,d^3\zeta$$
$$=(1,-\hat{r}X)r^2\sin\theta\,dr\,d\theta\,d\phi\ , \qquad (3.4)$$

where $X=(\partial T/\partial r)/(\partial T/\partial t)$. The region of the surface that has a spacelike normal, i.e., $|X|>1$, is indicated on the isotherm in Fig. 1. Here the θ integration eliminates dependence on X and gives

$$\frac{dN}{d^3p}=\frac{1}{(2\pi)^3}\frac{4\pi r_0^3/3}{\exp(E/T_f)-1}\ . \qquad (3.5)$$

B. Calculation of hadrons produced at $T>T_f$

Because the fluid velocity is zero, one can write (2.28) as

$$\frac{dN}{d^3p}=\int dt\int\frac{d^3r}{(2\pi)^3}\Gamma\exp(-B)\frac{1}{\exp(E/T)-1}\ . \qquad (3.6)$$

In this integration, r and t must lie within the extreme isotherms shown in Fig. 2. The collision rate Γ generally depends on p and T. For simplicity I will instead use a constant value. From $\pi\pi$ scattering[17] the momentum averaged rate is $\Gamma=T^5/12f_\pi^4$. For $T=200$ MeV this gives $\Gamma=350$ MeV$=1.8$ (fm/c)$^{-1}$. For $T=150$ MeV this gives $\Gamma=85$ MeV$=0.4$ (fm/c)$^{-1}$. A reasonable intermediate value is $\Gamma=1.0$ (fm/c)$^{-1}$, corresponding to a mean free path $\lambda=v/\Gamma\approx1.0$ fm. For constant Γ the damping integral is

$$B=\Gamma[t_f(r,\cos\theta,t)-t]\ , \qquad (3.7)$$

where t_f is determined by

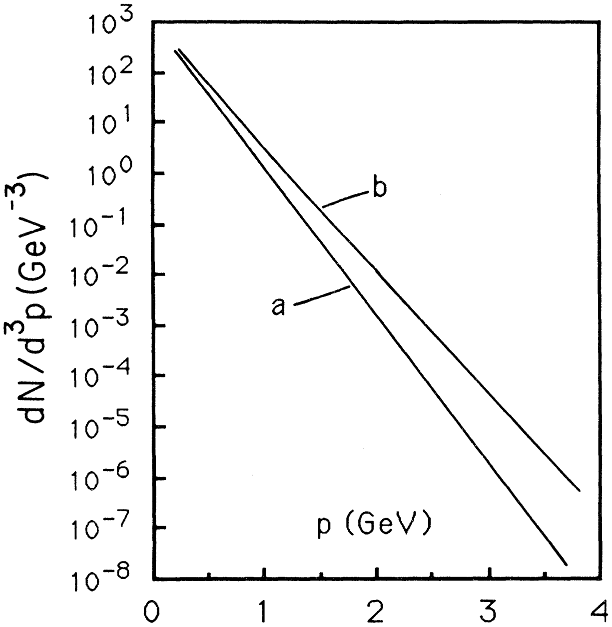

FIG. 3. Comparison of the hadron momentum spectra: (a) the standard emission from the decoupling surface shown in Fig. 1; (b) the result of including emission from the region between the isotherms shown in Fig. 2 with a collision rate $\Gamma=1.0$ (fm/c)$^{-1}$. Fitting to $\exp(-p/T_{\rm eff})$ gives $T_{\rm eff}=150$ MeV for (a) but $T_{\rm eff}=180$ MeV for (b).

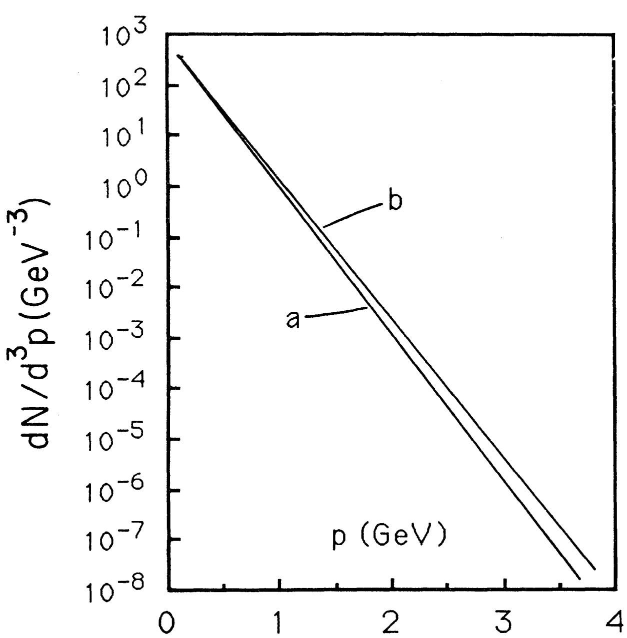

FIG. 4. The same as in Fig. 3 but with such a large collision rate, $\Gamma=10.0$ (fm/c)$^{-1}$ in (b), that the two models give very similar results: $T_{\rm eff}=150$ MeV for (a) and $T_{\rm eff}=158$ MeV for (b).

$$a[r_0^2-|\mathbf{x}+\mathbf{v}(t_f-t)|^2]=t_f\left[t_f-\frac{b}{T_f}\right] \tag{3.8}$$

and θ is the angle between $\mathbf{x}$ and the fixed velocity $\mathbf{v}$.

Figure 3 shows a comparison of (3.5) and (3.6) for the case $\Gamma=1.0$ (fm/c)$^{-1}$. The main feature, anticipated in Sec. I, is that the high-energy particles are more abundant than predicted by the surface model. The slope of the surface model gives $T_{\rm eff}=T_f=150$ MeV, whereas the volume integration gives $T_{\rm eff}=180$ MeV.

As argued in Sec. I, the two calculations must agree when $\Gamma\rightarrow\infty$. It is interesting to see in practice how large the collision rate must be for agreement. Figure 4 shows a comparison of (3.5) and (3.6) for $\Gamma=10.0$ (fm/c)$^{-1}$ corresponding to a mean free path $\lambda=v/\Gamma\approx0.1$ fm. The agreement here is already quite good; $T_{\rm eff}=150$ MeV vs 158 MeV. For still larger $\Gamma=50.0$ (fm/c)$^{-1}$, the two approaches are indistinguishable.

IV. CONCLUSIONS

The sample calculation demonstrates that the slope parameter $T_{\rm eff}$ is not determined just by the behavior of the hadronic fluid on the decoupling isotherm. It can depend crucially on the hydrodynamic history of the fluid and on the hadronic collision rates. In this example, the isotherms $T(x)$ as well as T_i and T_f are presumed to be known. Then there are two extreme approaches to understand a slope $T_{\rm eff}>T_f$: The conventional approach is to include only surface emission. Then a large fluid velocity is required to explain $T_{\rm eff}>T_f$. This is indeed the correct explanation if Γ is large [$\approx10-50$ (fm/c)$^{-1}$ for the Sec. III calculation]. The new possibility is that Γ is smaller, ≈1 (fm/c)$^{-1}$, and the fluid velocity is not so important.

Of course the simplicity of the sample calculation is misleading. In reality, the shape of the isotherms $T(x)$ and the value of T_i and T_f are not so certain and may have to be deduced from the data.[10] The flow velocity $u(x)$ is certainly not zero and should be determined from hydrodynamics to the extent possible. Fortunately, the momentum spectrum (2.28) depends rather sensitively on the various inputs.

Thermal gradients. When gradients are large, the new formula supersedes the surface emission. This fact is hidden in the space-time form (2.28), but the temperature formulation (2.22) shows that the attentuation factor is small when $\psi=p^\mu\partial T/\partial x^\mu$ is large. This is also apparent in (1.14). The integrand of (2.22) agrees with the estimate (1.15) since $\Gamma E/\psi=RE/T^2$.

Initial temperature T_i. Increasing T_i will increase the multiplicity because it adds hadrons that are produced deep within the fireball without changing anything else.

Final temperature T_f. The multiplicity is not monotonic in T_f. It has a maximum at some value T_f^*. For smaller T_f the multiplicity is less because the hadrons suffer more exponential attenuation. For larger T_f the multiplicity is also less because the production volume decreases.

Collision rate Γ. The multiplicity is also quite sensitive to Γ, having a maximum at some value Γ^*. A Γ larger than this produces more attenuation and decreases the multiplicity (as in going from Fig. 3 to Fig. 4). However a Γ smaller than Γ^* also decreases the multiplicity because of the explicit Γ factor in the integrand of (2.28). The above remarks hold at each value of p. Actual collision rates are functions of momentum and decrease at large p. This can produce a momentum spectrum that is concave up on a semi-log plot [as if Fig. 4(b) were used for small momenta and Fig. 3(b) for large momenta].

Fluid velocity u. The dependence on the fluid velocity has not been investigated at all. The geometry of the collision (e.g., spherical versus cylindrical expansion) requires examination and can affect the hadronic rapidity distributions.

The fundamental problem of how to systematically treat the hadronic fluid when it is not in local equilibrium remains unsolved. This paper focuses on a simple, though important, consequence of $R \gg 1$. The sensitive dependence on the various inputs may make the momentum spectra a more stringent test of the underlying theory than expected.

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