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Continuum limit of exactly solvable Hamiltonian in lattice gauge theory

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We investigate the modified lattice gauge Hamiltonian proposed by Guo, Zheng, and Liu, whose vacuum wave function is known exactly. The result strongly suggests that the Hamiltonian does not have the correct continuum limit in either (2+1)-dimensional or (3+1)-dimensional space-time. We also show that the modified Hamiltonian cannot be improved so as to make its difference from the standard Kogut-Susskind Hamiltonian negligible in the continuum limit under the condition that its exact vacuum wave function is the product of the functions of a single plaquette variable.

One of the important problems in understanding the low-energy properties of non-Abelian gauge theory is to know the structure of its vacuum. Recently, a modified Hamiltonian was proposed by Guo, Zheng, and Liu,[1] whose vacuum wave function is known exactly to be the product of the functions of a single plaquette variable. They claimed that the difference ΔH between their Hamiltonian and the standard Kogut-Susskind Hamiltonian H_0 (Ref. 2) is irrelevant in the continuum limit. This would be, if correct, a significant advance in this region.

The operator ΔH contains the square of the difference of neighboring plaquette variables and the proof of their claim assumes that it can be replaced by the square of the derivative of the field strength in the continuum limit. It was pointed out, however, by Roskies[3] that, in the case of (2+1)-dimensional space-time, the difference cannot be replaced by the derivative of the field strength because the exact vacuum wave function of the modified Hamiltonian describes a completely independent fluctuation of the color-magnetic field at each plaquette. In fact, Roskies evaluated the vacuum expectation value of ΔH to be divergent in the continuum limit in 2+1 dimensions.

Guo *et al.*[4] later gave a counterexample to Roskie's Comment, in which the addition of an operator to the Hamiltonian does not affect the low-energy behavior of the system in the continuum limit, even if the vacuum expectation value of the added operator is divergent. But it should be noted that, in the example, the vacuum expectation value (or the zero-point fluctuation) of the added operator is less singular in the power behavior with respect to the lattice spacing than that of the original Hamiltonian.

In 3+1 dimensions, on the other hand, at first glance there seem to be no problems pointed out by Roskies since their exact vacuum wave function implies that color-magnetic fields in neighboring plaquettes are strongly correlated in the continuum limit due to the Bianchi identity in this dimension and it seems that the square of the difference of neighboring plaquette variables may be replaced by the derivative of the field strength. Then, after naive power counting, ΔH tends to zero in the continuum limit. In fact, Duncan and Roskies[5] agree with this argument, although they suppose that the effect of the term ΔH would be enhanced nonperturbatively.

In this paper we analyze the modified Hamiltonian proposed by Guo *et al.* and the result indicates that it does not have the correct continuum limit in either 2+1 or in 3+1 dimensions. We first point out that, in 2+1 dimensions, the expectation value $\langle \Delta H \rangle_0$ with the exact vacuum wave function of the modified Hamiltonian has the same order of magnitude with respect to the lattice spacing as the divergent vacuum expectation value $\langle H_0 \rangle_0$ of the Kogut-Susskind Hamiltonian in the continuum limit. We next evaluate $\langle \Delta H \rangle_0$ in 3+1 dimensions and show that it also has the same order of magnitude as $\langle H_0 \rangle_0$ up to a logarithmic factor. Here, each of the vacuum expectation values $\langle H_0 \rangle_0$ and $\langle \Delta H \rangle_0$ comes wholly from the quantum fluctuation of the gauge field. From these it is implausible that the operator ΔH is an irrelevant operator in both 2+1 and 3+1 dimensions; i.e., the operator ΔH would affect the low-energy behavior of the system in the same order of magnitude as the Kogut-Susskind Hamiltonian H_0 does. We further show that one cannot improve the modified Hamiltonian in either 2+1 or 3+1 dimensions so as to make ΔH less divergent in the continuum limit as far as the exact vacuum wave function is the product of the functions of a single plaquette variable.

The modified Hamiltonian proposed by Guo, Zheng, and Liu for the SU(N) lattice gauge theory has the form of

$$H=\frac{g^2}{2a}\sum_l \exp(-R)E_l^a\exp(2R)E_l^a\exp(-R) + \frac{1}{ag^2}\sum_p \mathrm{Tr}(2) , \qquad (1)$$

where R is a gauge-invariant function of plaquette variables. We have added the constant term $(1/ag^2)\sum_p \mathrm{Tr}(2)$, which was omitted in Ref. 1. It does not affect the property of the system at all. The exact ground state $|\Psi_0\rangle$ for the Hamiltonian is

$$|\Psi_0\rangle = \exp(R)|0\rangle \ , \qquad (2)$$

where $|0\rangle$ is defined by

$$E_l^a|0\rangle = 0 \ . \qquad (3)$$

It has the exact ground-state energy

$$H|\Psi_0\rangle = \frac{1}{ag^2}\sum_p \mathrm{Tr}(2)|\Psi_0\rangle \ . \qquad (4)$$

The Hamiltonian can be rewritten as

$$H = H_0 + \Delta H \ , \qquad (5)$$

where

$$H_0 = \frac{g^2}{2a}(E_l^aE_l^a - [E_l^a,[E_l^a,R]]) \ , \qquad (6)$$

$$\Delta H = -\frac{g^2}{2a}[E_l^a,R][E_l^a,R] \ . \qquad (7)$$

If one takes the following form of R as

$$R = \sum_p \alpha\, \mathrm{Tr}(U_p + U_p^\dagger) \quad \left[\alpha = \frac{1}{2C_N g^4}\right] \ , \qquad (8)$$

then H_0 is exactly the standard Kogut-Susskind Hamiltonian:

$$H_0 = \frac{g^2}{2a}\sum_l E_l^aE_l^a + \frac{1}{ag^2}\sum_p \mathrm{Tr}(2 - U_p - U_p^\dagger) \ . \qquad (9)$$

If the effect of the remaining term

$$\Delta H = -\frac{1}{8C_N^2 g^6 a}\sum_l \chi_l \ ,$$
$$\chi_l = \sum_{p,p'\supset l} \mathrm{Tr}[\Lambda^a(U_p - U_p^\dagger)]\mathrm{Tr}[\Lambda^a(U_{p'} - U_{p'}^\dagger)] \qquad (10)$$

would vanish in the continuum limit, at least on the low-energy behavior of the system, one could say that the modified Hamiltonian is the correct one.

We first concentrate on the case of 2+1 dimensions. The vacuum expectation value of ΔH is evaluated by Roskies to be divergent like $-V/a^4$ with respect to the lattice spacing a in the continuum limit.[3] On the other hand, the vacuum energy

$$\frac{1}{ag^2}\sum_p \mathrm{Tr}(2) = \langle H_0\rangle_0 + \langle \Delta H\rangle_0$$

is also divergent like V/a^4, using the relation $g^2 = ae^2$, where e is the invariant gauge coupling. This means that $\langle \Delta H\rangle_0$ has the same order of magnitude as $\langle H_0\rangle_0$ in the continuum limit. Note that each of the vacuum expectation values $\langle H_0\rangle_0$ and $\langle \Delta H\rangle_0$ comes exclusively from the quantum fluctuation of the gauge field, because the classical ground state, in which $E_l^a=0$ and $U_p=1$, gives $H_0=0$ and $\Delta H=0$. (To make $H_0=0$ in the classical ground state is the reason why we have added the constant term to the original modified Hamiltonian.) From these it is implausible that the operator ΔH is an irrelevant operator, since there is no reason to expect that the operator ΔH does not give any effect on the property of the low-lying states such as mass gap and string tension while the operator H_0 does give various effects, in spite of the fact that the quantum vacuum fluctuation of the gauge field gives contributions to both of the operators in the same order of magnitude.

Next we consider the case of 3+1 dimensions. We estimate the continuum limit of the vacuum expectation value of ΔH, for which we evaluate the vacuum expectation value of the dimensionless quantity χ_l:

$$\langle \chi_l\rangle_0 = \frac{\int [dU]\chi_l \exp(2R)}{\int [dU]\exp(2R)} \ , \qquad (11)$$

with respect to g^2. Note that it is exactly the same as the vacuum expectation value of χ_l in the Euclidean lattice gauge theory in three-dimensional space-time with the coupling constant $\tilde{g}^2 = C_N g^4/2$ as

$$\langle \chi_l\rangle_0 = \frac{\int [dU]\chi_l \exp\left[\sum_p (1/2\tilde{g}^2)\mathrm{Tr}(U_p + U_p^\dagger)\right]}{\int [dU]\exp\left[\sum_p (1/2\tilde{g}^2)\mathrm{Tr}(U_p + U_p^\dagger)\right]} \ . \qquad (12)$$

So we consider Euclidean lattice gauge theory in three-dimensional space-time for a moment. The continuum limit of U_p can be expanded as

$$U_p = 1 + i\tilde{e}\,\tilde{a}^2\tilde{F}_{lj} + O(\tilde{a}^4) \ , \qquad (13)$$

where $\tilde{a}$, $\tilde{e}$, and $\tilde{F}$ are the lattice spacing, the invariant gauge coupling, and the field strength, respectively, in three-dimensional Euclidean lattice gauge theory with gauge coupling $\tilde{g}$ ($\tilde{g}^2 = \tilde{a}\,\tilde{e}^2$). Then the quantity χ_l becomes

$$\chi_l = (i\tilde{e}\,\tilde{a}^3\partial_j\tilde{F}_{lj})^2 \ , \qquad (14)$$

and the naive dimensional counting gives

$$\langle(\partial_j\tilde{F}_{lj})^2\rangle_0 \propto \tilde{e}^{10} \ . \qquad (15)$$

The reason why we can now use the naive dimensional count is that, in the expectation value (12), the correct vacuum wave function in (2+1)-dimensional gauge theory is generated automatically by the (path) integration of the link variables from the infinite past to the infinite future in the Euclidean time. Finally, we obtain

$$\langle \chi_l\rangle_0 \propto \tilde{g}^{12} \propto g^{24} \ , \qquad (16)$$

and, going back to the original (3+1)-dimensional theory, it gives

$$\langle \Delta H\rangle_0 \propto \frac{-Vg^{18}}{a^4} \ . \qquad (17)$$

Thus, the vacuum expectation value of ΔH is proportional to $-Vg^{18}/a^4$, which diverges like $-V/a^4(-\ln a)^9$ in the continuum limit, using the relation

$$a = \Lambda_L^{-1}\exp(-\mathrm{const}\times g^{-2})$$

in (3+1)-dimensional theory. On the other hand, the vacuum energy $(1/ag^2)\sum_p \mathrm{Tr}(2)$ diverges like $V(-\ln a)/a^4$. These imply that the vacuum expectation value of ΔH diverges in the same order of magnitude with respect to the lattice spacing a as that of the Kogut-Susskind Hamiltonian H_0 in the continuum limit up to the logarithmic factor and it strongly suggests that the operator ΔH is also relevant in 3+1 dimensions.

Next let us consider the modified Hamiltonian in 2+1 and 3+1 dimensions with the more complicated form of R as

$$R=\sum_p \sum_{n=1}^{\infty} \alpha_n [\mathrm{Tr}(U_p+U_p^\dagger)]^n \ . \tag{18}$$

This would give the most general vacuum wave function that is the product of the functions of a single plaquette variable as

$$\Psi_0(\{U\})=\prod_p \psi(U_p) \ , \tag{19}$$

taking into account the requirement that the vacuum wave function should be gauge invariant and translationally invariant. Now we will ask whether we can choose an appropriate set of parameters $\{\alpha_n\}$ so that the vacuum expectation value of ΔH be less divergent in the continuum limit. In fact, Guo *et al.* tried this issue in 2+1 dimensions because in this dimension, ΔH diverges even in the naive classical continuum limit.[1,6]

We assume that the continuum limit of U_p can be expanded as

$$U_p=1+iga^2F_{ij}-\frac{g^2a^4}{2}F_{ij}^2+O(a^6) \ , \tag{20}$$

where g should be replaced by the invariant gauge coupling e in 2+1 dimensions. Then the condition that H_0 [the right-hand side (rhs) of (6)] should approach the standard Hamiltonian in the continuum limit gives a constraint

$$\sum_{n=1}^{\infty} n[(n^2+1)n-2](2N)^{n-2}\alpha_n=\frac{1}{g^4} \ . \tag{21}$$

We further impose the condition that the leading term of ΔH [the rhs of (7)] with respect to a or g should vanish. It gives another constraint

$$\sum_{n=1}^{\infty} n(2N)^{n-1}\alpha_n=0 \ . \tag{22}$$

Equations (21) and (22) give

$$\sum_{n=2}^{\infty} n(n-1)(2N)^{n-2}\alpha_n=\frac{1}{g^4} \ . \tag{23}$$

The left-hand sides of Eqs. (22) and (23) are just the first and the second derivatives, respectively, of R with respect to $\mathrm{Tr}(U_p+U_p^\dagger)$ at the configuration $\{U_p=1\}$, and the two equations imply that the vacuum wave function $\exp(R)$ is not peaked at $\{U_p=1\}$, instead the configuration $\{U_p=1\}$ gives its local minimum. This is inconsistent with the expansion of U_p around the configuration $\{U_p=1\}$ as in Eq. (20).

The above consideration implies that one cannot improve the modified Hamiltonian so as to make ΔH less divergent in the continuum limit under the condition that its exact vacuum wave function is the product of the functions of a single plaquette variable as in Eq. (19). This would indicate that the exact vacuum wave function of lattice gauge theory in (2+1)- and (3+1)-dimensional space-time should involve more extended operators such as two plaquette variables.[7]

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