Chiral-symmetry breaking in the disordered QCD vacuum

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(Received 14 May 1990)

The spectrum of quark eigenvalues $\nu(\Lambda)$ is found for an ensemble of vacuum fields, consisting of statistically independent pieces ("elementary fields"). The spectrum $\nu(\Lambda)$ is the Wigner semicircle curve, centered around poles of the "elementary" S matrix. Spontaneous chiral-symmetry breaking occurs when overlap integrals of different pieces are larger than the displacement of these "elementary" poles from the zero point. For the instanton-gas model the corresponding spectrum density is exactly reproduced, and a discussion of numerical studies of the instanton liquid is given. A general mechanism of chiral-symmetry restoration is briefly outlined.

I. INTRODUCTION

Chiral-symmetry breaking (CSB) defines many features of low-energy QCD and is now well understood on the phenomenological side; e.g., we know that CSB occurs spontaneously,[1] and the effective chiral Lagrangians are useful tools in describing not only pions,[2] but also nucleons.[3]

Much less is known about the mechanism of CSB itself, i.e., about the configurations of the QCD vacuum responsible for CSB. The instantons have been considered as a possible source of CSB already at the first stage of the instanton-gas model of the QCD vacuum.[4,5] A detailed picture of CSB inside that model has been developed in a series of recent papers[6-10] following the original idea of Dyakonov and Petrov[7] that a mixing and collectivization of zero (anti-)instanton modes can lead to CSB. This idea has been also checked numerically, using an ensemble of (anti-)instantons to simulate the QCD vacuum.[11]

There are three arguments, however, which make the instantonic gas (or liquid) model not a good candidate for a realistic QCD vacuum.

First, in the instanton model confinement is absent.[12] Recently this property has been understood as a consequence of an integer topological charge of instantons, which makes the string tension vanish.[13] Therefore, additional configurations are needed to ensure confinement, or instantons should be strongly distorted, so that all higher cumulants would vanish.[14]

Second, Monte Carlo calculations on relatively small lattices[2] give evidence for CSB,[15] while the size of lattices (≤ 1 fm) is too small to give room to an ensemble of instantons (in the instanton-liquid model instantons have a size of ~ 0.3 fm and the mean distance between them is ~ 1 fm).

Third, the lattice calculations at nonzero temperature show that the deconfinement and the restoration of chiral symmetry seem to occur at the same temperature T_c.[16] This would indicate that vacuum configurations responsible for confinement and CSB are either the same, or strongly interdependent, again ruling out instantons as the only configurations in the QCD vacuum.

Thus one can expect that CSB is probably due to a more universal (and therefore more crude) mechanism, which involves confinement and is connected to more common vacuum configurations, than fragile and sophisticated (from the lattice point of view) instantons.

Here we note for completeness that other CSB mechanisms have been suggested which try to directly connect confinement and CSB.[17-19] However, it is still difficult to draw any conclusions about them, since most of them usually are not explicitly gauge invariant. Also, the use of $1/q^4$ as a modified gluon propagator is questionable from the point of view of analyticity since a two-gluon glueball correlator would have the wrong analytic properties (for more discussion of these approaches see Ref. 20).

The purpose of this paper is to describe a universal mechanism of CSB, which is common to any stochastic vacuum configuration and can be applied also to the instantonic vacuum. The proposed mechanism is closely connected to the zero-mode collectivization mechanism of Dyakonov and Petrov,[7] but is more general and has a much wider scope of applications. For example, it applies to any pieces of (anti-) self-dual fields, such as torons,[21] or any randomly distributed lumps of field, capable of supporting nearly zero fermion modes.

The paper is organized as follows. In Sec. II we introduce the general formalism, in which we split the vacuum field into "elementary" configurations, centered around some points in four-dimensional (4D) Euclidean space. In full analogy with the scattering theory on many scattering centers, the full quark Green's function S is expressed through a Green's function (t matrices) of "elementary" fields, via a linear equation with elementary t matrices entering the kernel of the equation.

The phenomenon of spontaneous chiral-symmetry breaking (SCSB) is connected to the spectrum of S, which in turn depends on the "elementary" spectrum, i.e., on the spectral representation of t.

The latter is investigated in Sec. III. A new phenomenon of the "condensation of zero modes" is de-

scribed, which consists of accumulating zero modes in the spectrum of an "elementary" field t matrix when the strength and/or range of that field increases. It is shown that self-dual fields are preferable both because of their larger stability and because they create more zero modes for the same amount of the action integral.

In Sec. IV we connect properties of elementary fields and spectral properties of the global Green's function. The key element is the observation that the global spectrum is defined by a random ensemble of matrices, with the elements equal to overlap integrals of fermion modes from different centers.

We discuss the condition under which this random ensemble is a Gaussian unitary ensemble, and then the global spectrum is the famous "semicircle law" of Wigner. In Sec. V we then conclude that SCSB takes place and make comparisons with the instanton-liquid model, where the same "semicircle law" spectrum has been obtained by a more complicated method. Section VI is devoted to a discussion of results and possible generalizations. Throughout the paper we, for simplicity, consider the case of one flavor. A generalization to a larger N_f is straightforward and will be presented elsewhere.

II. GENERAL FORMALISM

The QCD partition function can be written as (regularization factors are irrelevant in what follows)

$$Z=\langle \det(-S^{-1})\rangle_A \ , \tag{1}$$

where $S(A)$ is the full quark Green's function in the vacuum field

$$S(x,y;A)=\langle \psi(x)\psi^\dagger(y)\rangle=(-i\hat{D}-im)^{-1} \tag{2}$$

and $\langle\ \rangle_A$ means averaging over all vacuum gluonic fields. Here $D_\mu=\partial_\mu-igA_\mu$, and we are always in the Euclidean space, $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$. $S(x,y)$ satisfies the equation

$$(-i\gamma_\mu D_\mu-im)S(x,y)=\delta^{(4)}(x-y) \tag{3}$$

with suitable boundary conditions on the sides of a big volume V_4.

In what follows we shall extensively use the spectral representation for S:

$$S(x,y)=\sum_n \frac{w_n(x)w_n^\dagger(y)}{\Lambda_n-im} \ , \tag{4}$$

where eigenfunctions w_n and eigenvalues Λ_n are to be found from the equation

$$[-i\gamma_\mu D_\mu(A)]w_n=\Lambda_n w_n \tag{5}$$

with periodic or scattering-type boundary conditions.

The distribution of eigenvalues Λ_n is the central issue in CSB, since the chiral condensate $\langle\bar{q}q\rangle$ can be expressed through them,[17,7]

$$-i\langle\bar{q}q\rangle_{\text{Mink}}=\langle\psi^\dagger(x)\psi(x)\rangle_{\text{Eucl}}=\text{tr}\langle S(x,x)\rangle_A$$
$$=\frac{1}{V_4}\left\langle\sum_n\frac{1}{\Lambda_n-im}\right\rangle_A \ , \tag{6}$$

where we have used the orthonormality condition for discrete states:

$$\int w_n^\dagger(x)w_n(x)d^4x=\delta_{nm} \ . \tag{7}$$

Since states with negative Λ_n also belong to the spectrum, namely, from (5),

$$\gamma_5 w(+\Lambda_n)=w(-\Lambda_n) \ , \tag{8}$$

the spectrum in (6) is symmetric around $\Lambda=0$, and we can proceed in (6), introducing the density of eigenvalues $\nu(\Lambda)$, which makes sense when the volume V_4 is very large, so that $\nu(\Lambda)$ is proportional to V_4. We have

$$\langle\bar{q}q\rangle=\frac{i}{\nu_4}\sum_{\Lambda_n\geq 0}\frac{2im}{\Lambda_n^2+m^2}\to\frac{-2m}{V_4}\int_0^\infty\frac{\nu(\Lambda)d\Lambda}{\Lambda^2+m^2} \ . \tag{9}$$

Now in the proper chiral limit $m\to 0$ one obtains the usual expression

$$\langle\bar{q}q\rangle=-\frac{\pi}{V_4}\nu(0) \ , \tag{10}$$

where it is assumed that $\nu(\Lambda)$ changes at $\Lambda\sim 0$ at values Λ much larger than m.

Nonzero values of $\langle\bar{q}q\rangle$ signal spontaneous CSB in the case of two or more flavors [for one flavor chiral symmetry actually reduces to $U_A(1)$ which is explicitly broken by anomaly].

Thus the problem of the mechanism of CSB can be formulated as follows: to find vacuum configurations $\{A_\mu(x)\}$, for which the averaged level density $\nu(\Lambda)$ is nonzero in the limit $\Lambda\to 0$, $V_4\to\infty$. We shall represent these configurations as a superposition of "elementary" configurations $A_\mu(i)$,

$$A_\mu=\sum_{i=1}^{N}A_\mu^{(i)}(x) \tag{11}$$

and we shall assume that N is large when V_4 is large, so that for $V_4\to\infty$, also $N\to\infty$. At the moment we do not make any assumptions about $A^{(i)}(x)$; it may be an instanton or anti-instanton at the point $R^{(i)}$, or a toron or regularized meron around that point, or simply a piece of field in the region around $R^{(i)}$.

The stochastic ensemble of vacuum configurations $\{A_\mu\}$ consists of all fields of the type (11) and random copies thereof, made by taking all $\{R^{(i)}\}$ at random positions in V_4 and a random global but i-dependent transformation $A^{(i)}\to V^{(i)}A^{(i)}V^{+(i)}$.

This stochastic ensemble corresponds to the instanton-gas approximation, when $A^{(i)}$ is an instanton;[4,6] in Sec. IV we shall discuss the role of correlation of positions $R^{(i)}$.

The aim of this section is to express the global Green's function $S(x,y)$ in (2) in terms of an "elementary" Green's function $S^{(i)}$:

$$S^{(i)}=[-i(\hat{\partial}-ig\hat{A}^{(i)})-im]^{-1} \ . \tag{12}$$

This formal derivation is fully analogous to one usually done in the theory of scattering on many fixed centers[22] and one ends up with the Faddeev-type equations

$$S=S_0-\sum_{i,k=1}^{N} Q_{ik} \tag{13}$$

and

$$Q_{ik}=t_j\delta_{ik}-t_i S_0^{-1}\sum_{j\neq i} Q_{jk} , \tag{14}$$

where S_0 is the free Green's function

$$S_0=(-i\hat{\partial}-im)^{-1} , \tag{15}$$

and

$$t_i(x,y)=S_0(x,y)-S^{(i)}(x,y) . \tag{16}$$

The physical meaning of Q_{ik} is understood as that of a quark propagator with the first interaction at the center k and the last at center i.

The "elementary" Green's function has a spectral representation similar to (4):

$$S^{(i)}(x,y)=\sum_n \frac{u_n^{(i)}(x)u_n^{(i)\dagger}(y)}{\lambda_n^{(i)}-im} \tag{17}$$

with u_n, λ_n satisfying

$$-i\gamma_\mu D_\mu(A^{(i)})u_n^{(i)}(x)=\lambda_n^{(i)}u_n^{(i)}(x) . \tag{18}$$

Using (17), the explicit form of (14) is found to be

$$Q_{ik}(x,y)=+\delta_{ik}\left[S_0(x,y)-\sum_n \frac{u_n^{(i)}(x)u_n^{(i)\dagger}(y)}{\lambda_n^{(i)}-im}\right] + \int d^4z \sum_n \frac{u_n^{(i)}(x)u_n^{(i)\dagger}(z)}{\lambda_n^{(i)}-im}(-i\hat{\partial}-\lambda_n^{(i)}) \times\sum_{J\neq i} Q_{jk}(z,y) , \tag{19}$$

where we have used the completeness relation for $u_n^{(i)}$.

III. PROPERTIES OF ELEMENTARY SPECTRUM: CONDENSATION OF ZERO MODES

In this section we shall study the properties of $u_n^{(i)}(x),\{\lambda_n^{(i)}\}$ in their dependence on the form of the elementary field $A^{(i)}$. To this end we act on both sides of Eq. (18) with $-i\hat{D}$ to obtain a system of second-order equations [we omit everywhere in this section the superscript i, stressing that results will refer to any field $A(x)$]:

$$[-(\partial_\mu-igA_\mu)^2+g\boldsymbol{\sigma}\cdot(\mathbf{E}-\mathbf{B})](v_n-\bar{v}_n)=\lambda_n^2(v_n-\bar{v}_n) , \tag{20}$$

$$[-(\partial_\mu-igA_\mu)^2-g\boldsymbol{\sigma}\cdot(\mathbf{E}+\mathbf{B})](v_n+\bar{v}_n)=\lambda_n^2(v_n+\bar{v}_n) , \tag{21}$$

where we have used the representation $u_n=(v_n,\bar{v}_n)$. Since $-i\hat{D}$ is a Hermitian operator and $-\hat{D}^2$ a positive one, λ_n^2 on the right-hand side (RHS) of Eqs. (20) and (21) is always positive. Next, we note that for a spinless particle [when the σ terms in Eqs. (20) and (21) are absent] and nonzero A_μ (nonzero $F_{\mu\nu}$) the operator $-(\partial_\mu-igA_\mu)^2$ is positive and $\lambda_n^2\geq\lambda_{\min}^2$, $\lambda_{\min}$ depending on A_μ. In the case of the free quark $u_n(x)\sim e^{ipx}$ and $\lambda_n^2=p_\mu^2$, so that index n is continuous and coincides with p_μ. For quarks with spin $\frac{1}{2}$ we have an additional gain in attraction due to the σ terms in (20) and (21). Indeed for self-dual fields $\mathbf{E}=\mathbf{B}$, Eq. (20) tells us that a zero mode ($\lambda_n^2=0$) is possible when $v_n=\bar{v}_n$, i.e., the RHS chiral mode. But that would exist only if (21) has a nontrivial solution. One can see that the spin should be parallel to $\mathbf{E}=\mathbf{B}$ to get the lowest state.

One can now check by the usual means[23] how deep the potential

$$V=g^2A_\mu^2-2g\boldsymbol{\sigma}\cdot\mathbf{E} \tag{22}$$

should be to support a bound state and come to the conclusion that it corresponds to the condition

$$J\equiv\frac{g^2}{32\pi^2}\int F_{\mu\nu}\tilde{F}_{\mu\nu}d^4x\geq 1 . \tag{23}$$

If we assume that $A_\mu^{(i)}(x)$ vanishes at large distances, then Eqs. (20) and (21) tells us that the spectrum of continuous states $\{\lambda_n^2\}$ always starts from $\lambda_n^2=0$, so that a bound state, if any, can exist only at $\lambda_n^2=0$. This is the case for instantons. At $J\neq 1$ we can have resonant states, which appear at complex λ_n^2.[24]

Another example of zero quark modes is provided by a constant self-dual field, $B_i^a=E_i^a$ with $a=3$. In the gauge where $A_\mu=-\frac{1}{2}F_{\mu\nu}x_\nu$ a zero mode centered at $x=z$ is given by[25]

$$\psi_z(x)=\frac{eB}{2\sqrt{2}\pi}\exp\left[\frac{ig}{2}z_\nu x_\mu F_{\mu\nu}\right] \times\exp\left[-\frac{gB}{4}(x-z)^2\right]\begin{bmatrix}\varphi_n^\dagger\\ \varphi_n^\dagger\end{bmatrix} , \tag{24}$$

where $F_{12}=F_{34}=B$.

The number of zero modes is infinite for the field constant everywhere in space-time.

Let us approach this limit of homogeneous field, starting with the configuration

$$A_\mu=-\tfrac{1}{2}F_{\mu\nu}x_\nu f(x^2),\quad f(x^2)\sim\theta(R^2-x^2) . \tag{25}$$

If one chooses $f(x^2)$ to be nonsingular, e.g.,

$$f(x^2)=\left[\exp\left[\frac{x^2-R^2}{a^2}\right]+1\right]^{-1} ,$$

then the topological charge J of the configuration (25) vanishes:

$$J=J_1+J_2\sim\int_0^\infty f^2x^2dx^2+\int fx^2\frac{df}{dx^2}dx^2=0 .$$

In this case J_1 and $|J_2|$ are increasing with $B\to\infty$ or $R\to\infty$, which leads to the appearance of resonance states with λ^2 close to zero with the support in the inner region $x^2<R^2$. The number of these states increases with J_1. At the same time resonance states appear in the surface region $x^2\sim R^2$, which have opposite chirality and their number is connected with J_2.

In the limit $R\to\infty$ (or $B\to\infty$) an infinite number of

poles (24), marked by their centers, coexist in space-time, with $\lambda_i^2=0$, $i=1,2,\ldots$.

One can also organize a self-dual field configuration with $J>0$. This can be done with A_μ having nontrivial boundary conditions at $x=0$ and/or at infinity, or else by prescribing twisted boundary conditions at the boundary of some volume. Increasing J we increase the number of zero bound and nearly zero resonant states.[24] These states again condense at $\lambda^2=0$.

Thus we observe a general *phenomenon of condensation of zero modes* for fields of large range R and/or strength B. The measure of this condensation n is given by the topological charge (23) for (anti-)self-dual fields and by the action integral for non-self-dual fields; a crude estimate is $n\sim\alpha_s B^2R^4$.

A special case is when fields are purely electric or magnetic; e.g., in the last case zero modes appear in pairs of opposite chirality; however, one can see in (21) that one needs roughly twice as big of a magnetic field to sustain zero modes as compared to the self-dual case.

The self-dual fields are singled out by the following properties: (i) as discussed above, in the compact lump of the field one needs larger values of the action $\frac{1}{4}\int d^4x\, F_{\mu\nu}F_{\mu\nu}$ to ensure nearly zero modes for purely magnetic or electric fields as compared to the self-dual or anti-self-dual case; (ii) classical lumps of (anti-)self-dual fields are stable with respect to quantum oscillations,[26] whereas magnetic (electric) fields are not. For this reason it would be natural to assume that "elementary" fields $A^{(i)}$ mostly consist of (anti-)self-dual lumps of fields, capable of supporting one (or more) nearly zero mode. This assumption is not crucial for the following and we shall discuss also other possible configurations.

IV. THE GLOBAL SPECTRUM AND RANDOM MATRIX ENSEMBLES

We study in this section the spectrum of eigenvalues Λ_n, Eqs. (4) and (5). Our goal is to obtain the eigenvalues density $\nu(\Lambda)$ and to discuss different averaging procedures, leading to different answers for $\nu(\Lambda)$.

As is easy to see from (4) and (14)–(19), an equation for eigenvalues Λ is obtained from the homogeneous part of Eq. (19), where im should be replaced by Λ:

$$Q_{ik}(x,y)=\int d^4z\sum_n\frac{u_n^{(i)}(x)u_n^{(i)\dagger}(z)}{\lambda_n^{(i)}-\Lambda}(-i\hat{\partial}-\lambda_n^{(i)})\times\sum_{j\neq i}Q_{jk}(z,y)\ . \tag{26}$$

Equation (26) implies that Q_{ik} has the structure

$$Q_{ik}(x,y)=\sum_n u_n^{(i)}(x)f_n^{(ik)}(y) \tag{27}$$

and for f_n we obtain a system of algebraic equations:

$$f_n^{(ik)}(y)=\frac{1}{\lambda_n^{(i)}-\Lambda}\sum_m V_{nm}^{(ij)}f_m^{(jk)}(y)\ , \tag{28}$$

where we have defined

$$V_{nm}^{(ij)}=\int d^4z\, u_n^{(i)\dagger}(z)(-i\hat{\partial}-\lambda_n^{(i)})u_m^{(j)}(z)\ . \tag{29}$$

The eigenvalues Λ_p, $p=1,2,\ldots$ are to be found from (28), which gives us an equation

$$\det[\delta_{nm}\delta^{ij}(\lambda_n^{(i)}-\Lambda)-V_{nm}^{(ij)}]=0\ . \tag{30}$$

The matrix in Eq. (30) is over indices (n,i) and (m,j), where $i,j=1,\ldots,N$, refer to elementary fields $A^{(i)}$, and $m,n=1,2,\ldots$ to eigenfunctions of elementary spectra (in principle, spectrum $\{\lambda_m^{(i)},u_m^{(i)}\}$ of the field $A^{(i)}$ depends on i and therefore we supply it with the superscript i). The matrix elements $V_{nm}^{(ij)}$ are subject to some randomization and the spectrum $\{\Lambda_p\}$ is the result of the averaging over a random ensemble. In this paper we assume that our ensemble of fields $A^{(i)}$ has uncorrelated centers $R^{(i)}$ and uncorrelated transformations $A^{(i)}\to\Omega^{(i)+}A^{(i)}\Omega^{(i)}$. This means for $V_{nm}^{(ij)}$ that

$$V_{nm}^{(ij)}\equiv V_{nm}^{(ij)}(R_i,R_j)=\int d^4z\, y_n^{(i)\dagger}(z-R_i)(-i\hat{\partial}-\lambda_n^{(i)})u_m^j(z-R_j) \tag{31}$$

and averaging over R_i,R_j has to be done independently; secondly, V transforms as

$$V^{(ij)}\to\int d^4z\, u_n^{(i)\dagger}\Omega^{(i)}(-i\hat{\partial}-\lambda_n^{(i)})\Omega^{(j)\dagger}u_m^{(j)}\ . \tag{32}$$

The randomization (31) and (32) implies the following averaging over R_i and $V^{(i)}$ with the Haar measure $d\Omega^{(i)}$:

$$\langle O\rangle=\int\prod_i\frac{dR_i}{V_4}d\Omega^{(i)}O(R_K,\Omega^{(k)})\ . \tag{33}$$

As a result we have

$$\langle V^{(ij)}\rangle=0\ , \tag{34}$$

$$\langle|V_{nm}^{(ij)}|^2\rangle=\frac{c}{N_cV_4}\int\frac{d^4p}{(2\pi)^4}|u_n^{(i)\dagger}(p)(\hat{p}-\lambda_n^{(i)})u_m^{(j)}(p)|^2\equiv\frac{(v_{nm}^{ij})^2}{N_cV_4}\ , \tag{35}$$

where c depends on the color structure of solutions $u_n(x)$.

Let us now discuss the general structure of our stochastic ensemble of matrices λ_m^i, $V_{nm}^{(ij)}$.

If we make an identification

$$V_{nm}^{(ii)}\equiv-\lambda_n^{(i)} \tag{36}$$

we can see from (30) that the spectrum $\{\Lambda_n\}$ is the spectrum of a random ensemble of matrices $\hat{V}$. There are two types of disorder present in $\hat{V}$: (i) a diagonal disorder due to random variations of the spectrum $\{\lambda_n^{(i)}\}$ at each center i and from one center to another; (ii) an off-diagonal disorder due to the randomness of nondiagonal matrix elements of $\hat{V}$. Both types of disorder are well known in solid-state physics,[27] e.g., in the theory of amorphous semiconductors.[28] To say more about the stochastic ensemble $\hat{V}$, one needs to know the integration measure for averaging over the ensemble, which is actually proportional to the effective action expressed in terms of $\hat{V}$. The latter is readily given by

$$L_{\text{eff}}=\ln\det S^{-1}=\ln\det\left[S_0-\sum_{i,k=1}^{N}Q_{ij}\right]^{-1}\ . \tag{37}$$

Equations for Q_{ik} are given in (19) and if one keeps in the spectral representation of t_j, Eq. (16), several pole terms, then Q_{ik} can be written as

$$Q_{ik}(x,y)=-\sum_{n,m}\frac{u_n^{(i)}(x)W_{nm}^{(ik)}u_m^{(k)\dagger}(y)}{\lambda_n^{(i)}-im} . \tag{38}$$

The constant matrix W satisfies an algebraic equation

$$W_{nm}^{(ik)}=\delta_{ik}\delta_{nm}+\sum_{j\neq i}\frac{V_{np}^{(ij)}W_{pm}^{(jk)}}{\lambda_p^{(j)}-im} . \tag{39}$$

Substituting

$$\hat{W}=[1-\hat{V}(\hat{\lambda}-im)^{-1}]^{-1} \tag{40}$$

into (38) and (37) we obtain

$$\ln\det S^{-1}=\ln\det\{[S_0+u(\hat{\lambda}-im-\hat{V})^{-1}u^{\dagger}]^{-1}\} , \tag{41}$$

where we have used matrix notation for the diagonal matrix $\hat{\lambda}\equiv\lambda_n^i\delta_{nm}\delta_{ij}$ and $\hat{V}\equiv V_{nm}^{(ij)}$, and vectors $u\equiv u_n^{(i)}(x)$.

The effective action (41) without a λ term is invariant under the following unitary transformation:

$$u\to uU,\quad u^{\dagger}\to U^{\dagger}u^{\dagger},\quad \hat{V}\to U^{\dagger}\hat{V}U . \tag{42}$$

The same property holds true when $\hat{\lambda}$ is proportional to a unit matrix. In what follows we assume for simplicity that this is indeed the case, which means that we take the same value $\lambda=\lambda_0$ for all centers $j=1,\ldots,N$.

We are now in the position to recall the definition of the Gaussian unitary ensemble,[29,30] as an ensemble of Hermitian $N\times N$ matrices $\hat{H}$ with elements H_{ij} satisfying two requirements: (i) the integration measure $P(H)dH$ is invariant under automorphisms

$$H\to U^{-1}HU , \tag{43}$$

where U is any unitary matrix; (ii) various linearly independent components of H_{ij} are also statistically independent.

In this case the spectrum density of eigenvalues of matrices $\hat{H}$ is given by the well-known Wigner semicircle law.[31,29,30] Namely, when

$$\langle H_{ij}\rangle=0,\quad \langle H_{ij}^2\rangle\equiv V^2\ \text{(independent of } i,j) , \tag{44}$$

the density of eigenvalues $\sigma(\epsilon,V^2)$, normalized as

$$\int_{-\infty}^{\infty}d\epsilon\,\sigma(\epsilon,V^2)=N , \tag{45}$$

is equal to (see Ref. 30, Appendix 29)

$$\sigma(\epsilon,V^2)=\begin{cases}\frac{1}{\pi V^2}(2NV^2-\epsilon^2)^{1/2}, & \epsilon^2<2NV^2 ,\\ 0, & \epsilon>2NV^2 .\end{cases} \tag{46}$$

We now apply this theorem to our situation and to this end consider three cases of increasing complexity.

(a) We keep only one term with $\lambda_m^{(i)}\equiv\lambda_0$ the same for all matrices $t^{(i)}$. Then the matrix $\hat{\lambda}$ both in (41) and (30) is $\hat{\lambda}=\lambda_0\hat{1}$. Moreover, $(v_{nm}^{ij})^2\equiv v^2$ in (35) does not depend on indices and is a number.

Now the property (i) is satisfied, because the probability density $P(H)$ is proportional to (41) and is invariant under transformation (42).

The property (ii) is satisfied due to the randomization (32), however only for large enough N_c, since the $SU(N_c)$ matrices $\Omega^{(i)}, i=1,\ldots,N$ contain $(N_c^2-1)N$ independent elements for $N(N-1)$ matrix elements $V^{(ij)}$.

Assuming N_c large, $N_c>N$, we can use (46) to obtain finally the spectral density of eigenvalues Λ:

$$\nu(\Lambda,\lambda_0)=\frac{1}{\pi V^2}[2NV^2-(\Lambda-\lambda_0)^2]^{1/2}\theta(2NV^2-(\Lambda-\lambda_0)^2) , \tag{47}$$

where according to (35) we have

$$V^2=\frac{v^2}{N_cV_4} . \tag{48}$$

(b) We again, as in case (a), keep only one and the same eigenvalue λ_0 in the matrix $\hat{\lambda}$, but assume that it is randomly distributed for our Gaussian unitary ensemble in the interval $(-\beta,+\beta)$ with a probability $p(\beta)$. If, in addition, the randomization of $\hat{\lambda}$ and $\hat{V}$ is statistically independent, then the resulting spectrum is

$$\nu_p(\Lambda)=\int_{-\beta}^{\beta}p(\beta)\nu(\Lambda,\beta)d\beta . \tag{49}$$

(c) A generalization of models (a) and (b) can be obtained in many ways, but an analytic solution is usually not available. For example, allowing for matrix $\hat{\lambda}$ depending on i or n, we immediately violate invariance (42) and the properties of the Gaussian unitary ensemble, together with the semicircle law (47). Numerical simulations are probably the best way in this case.

V. CHIRAL-SYMMETRY BREAKING FOR THE GAUSSIAN STOCHASTIC ENSEMBLE: INSTANTONS

The order parameter for CSB is the chiral condensate $\langle\bar{q}q\rangle$, which according to Eq. (10) is directly connected with $\nu(0)$.

For the model (a) we find, from Eq. (47),

$$\langle\bar{q}q\rangle=-\frac{1}{V^2V_4}(2NV^2-\lambda_0^2)^{1/2}\theta(2NV^2-\lambda_0^2) \tag{50}$$

so that CSB occurs for $\lambda_0^2<2NV^2$. For the model (b) we find from Eq. (49) that $\langle\bar{q}q\rangle$ is always nonzero, if $p>0$.

Physically the condition $\lambda_0^2<2NV^2=2Nv^2/N_cV_4$ means that the effective pole of the elementary field t matrix should be closer to zero than some critical value, depending on density N/V_4 of elementary fields and the strength v^2/N_c of the overlap interaction. When λ_0 is small, we have, from (50) and (48),

$$\langle\bar{q}q\rangle=-\left[\frac{2NN_c}{V_4v^2}\right]^{1/2} . \tag{51}$$

If we keep the size of elementary fields, then v^2 does not depend on N_c, while $N/V_4\sim\langle\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}\rangle\sim N_c$, and $\langle\bar{q}q\rangle=O(N_c)$.

We can now compare our result for the spectral density, Eq. (47), with that obtained in the instanton-liquid

model.[7] Two special features of the model are that (i) $\lambda_0=0$, since only exact zero modes are retained in the t matrix, and (ii) the interaction $V^{(ij)}$ exists only between instantons and anti-instantons (otherwise V vanishes). Therefore the interaction matrix $\hat{V}$ can be represented as a $N/2\times N/2$ matrix, and every matrix element is a 2×2 matrix:

$$V^{(ij)}\to\begin{pmatrix} 0 & V_{II}^{(ij)} \\ V_{\bar{I}\bar{I}}^{(ij)} & 0 \end{pmatrix} . \tag{52}$$

Thus a random ensemble is in this case a simplectic Gaussian ensemble[29,30] which has the same level density, as the unitary Gaussian ensemble, considered above.

One can calculate v^2 in (35) using the known zero-mode wave function φ',[7] and one obtains

$$v^2=2\int\frac{d^4pp^2}{(2\pi)^4}[\varphi'(p)]^4 , \tag{53}$$

where φ' (in notations of Ref. 7) is

$$\varphi'(p)=\pi\rho^2\frac{d}{dz}[I_0(z)K_0(z)-I_1(z)K_1(z)]\Big|_{z=p\rho/2} \tag{54}$$

and ρ is the average instanton radius. Substituting v^2 from (53) into (47) and (48) we obtain the form of spectrum $\nu(\Lambda,0)$, which exactly coincides with that found for the instanton liquid model:[7]

$$\nu(\Lambda)=\frac{N}{\pi\kappa}\left[1-\frac{\Lambda^2}{4\kappa^2}\right]^{1/2} , \tag{55}$$

where

$$\kappa^2=\frac{Nv^2}{2V_4N_c}=\frac{6.6\rho^2}{N_c}\frac{N}{V_4} . \tag{56}$$

For standard values of ρ, N/V_4 in that model one has numerically $\kappa\approx 100$ MeV.[7] From (51) we obtain for this value of $v^2(\kappa^2)$ a very reasonable chiral condensate: $\langle\bar{q}q\rangle\simeq-(250\text{ MeV})^3$.

VI. SUMMARY AND DISCUSSION

We have obtained the density of eigenvalues $\nu(\Lambda)$, Eqs. (49) and (47), and now we comment on the possible shapes of $\nu(\Lambda)$ and mechanisms of SBCS, which is signaled by a nonzero value of $\nu(0)$.

Our result (49) and (47) shows that SBCS occurs whenever elementary spectra contain (nearly) zero modes λ_0 and conditions (i) and (ii) of Sec. IV are satisfied.

The last of these conditions, namely, the statistical independence of various V_{ij}, is most difficult to ensure. First of all, our randomization procedure, involving averaging over matrices $\Omega^{(i)}$, leads to the statistical independence only in the limit $N_c\to\infty$. This is in agreement with the proof of SBCS in this limit, which has been obtained in a general framework.[32] Second, correlations between positions of different "elementary" fields $R^{(i)}$ at finite N_c may occur due to the formation of "molecules."[11] In the case of instantons, each molecule consists of an instanton and anti-instanton, and this produces two effects. First, self-dual and antiself-dual fields in the molecule are partly compensated and zero modes disappear, shifting to some higher values of λ_0. Second, the interaction V_{ij} inside the molecule is highly correlated, while the interaction between molecules is still random (but with renormalized values of $|V_{ij}|^2$). Therefore the spectrum will have the form of Eq. (47) and for a small density N/V_4 the peak of $\nu(\Lambda,\lambda_0)$ is shifted off the origin, which means that SBCS does not occur.

However, some remnant dissociated elementary fields (instantons) can still be present in the ensemble, creating a separate semicircle law curve of $\nu(\Lambda)$ around zero, and this may ensure SBCS but with a much smaller magnitude of $\nu(0)$, proportional to the number of dissociated centers. Thus, we expect $\nu(\Lambda)$ to have in general the form of a superposition of Wigner's curves:

$$\nu(\Lambda)=\sum p_s\nu_s(\Lambda,\lambda_s) \tag{57}$$

with $p_s, s=1,2,3,\ldots$ denoting the number of clusters, and λ_s a typical or effective eigenvalue of the sth cluster.

Numerical studies of the instanton–anti-instanton ensemble[11] seem to support the form (57), and one can notice in the numerically computed $\nu(\Lambda)$ two bumps corresponding to dissociated instantons and molecules, respectively. The formation of molecules is a part of a more general mechanism of CS restoration, by which nearly zero modes of elementary field disappear.

Finally we make several remarks about a possible role of the number of flavors. Our consideration has been everywhere for the case of only one flavor where CS is broken explicitly by anomaly. A generalization of the method to more flavors formally is straightforward as we mentioned in the Introduction, since eigenfunctions and eigenvalues, both local and global, Eqs. (5) and (18), do not depend on flavors and diagonal in flavor indices. In the averaging procedure, however, which is necessary to calculate the physical observables, e.g., the quark condensate, one should average over all vacuum fields with the weight, which contains the product of fermionic determinants with all flavors. This weight tends to stimulate correlations between (anti-)instantons and the formation of instantonic molecules, and this tendency becomes stronger for larger N_f. Recently this effect has been found numerically in Ref. 33. To account for the correlations in the framework of the present approach, one should allow in the general and still correct equations (26) and (30) different spectra λ_n for different centers $i=1,\ldots,N$. In our discussion of the Gaussian unitary ensemble instead we have fixed λ to be proportional to the unit matrix, i.e., the same for all centers. A more detailed discussion of the case $N_f>1$ will be given in a subsequent publication.

ACKNOWLEDGMENTS

This work has been done while the author was visiting the Theoretical Physics Institute of the University of Minnesota. It is a pleasure to thank the staff of the Theoretical Physics Institute for their kind hospitality. Useful discussions with D. I. Dyakonov and L. McLerran are gratefully acknowledged. I am grateful to M. Hamermesh and B. Shklovskii for references, comments, and discussions. I also acknowledge support under Contact No. DE-FG02-87ER40328.

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