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Gauge structure, anomalies, and mass generation in a three-dimensional Thirring model

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We consider a three-dimensional model of spinor fields with a Thirring-like, quadrilinear self-interaction. Using either two- or four-component Dirac spinors, we prove that the $1/N$ expansion for the model is renormalizable if a gauge structure to select physical quantities is introduced. For certain values of the coupling, the leading $1/N$ approximation exhibits bound-state poles. Dynamical breaking of parity or chiral symmetry is shown to occur as a cooperative effect of different orders of $1/N$, if N is smaller than the critical value $N_c=128/\pi^2 D$, where D is two or four depending on whether the fermion field has two or four components.

I. INTRODUCTION

An important characteristic of field theory in three space-time dimensions is the possibility for Dirac fields to have either a two- or a four-component representation. In the two-component representation a variety of interesting effects occur. Among these is the fact that a mass term in the Lagrangian violates parity, and then, if the Dirac field is coupled to an external electromagnetic field, a Chern-Simons term is induced. The breaking of parity may have a dynamical origin as it happens in the three-dimensional analogue of the Gross-Neveu model or may be present from the beginning in the Lagrangian.[1] In any case, the induced Chern-Simons term is the source of intriguing peculiarities as exotic statistics, fractional spin,[2,3] and a mass for the gauge field.[4,5] These features may be relevant to the quantized Hall effect[6] and to high-T_c superconductivity.[7] The presence of a Chern-Simons term seems also to be essential to a recent conjecture on bosonization of fermions in three dimensions.[8]

Other classes of effects may be present if four-component spinors are used. Indeed, for massless theories a continuous chiral symmetry can be implemented, and mechanisms for its spontaneous breaking may be investigated. To some extent, this has been done in the context of three-dimensional QED (QED_3), where an adequate use of the Schwinger-Dyson equations and the $1/N$ expansion has revealed the existence of a massive phase.[9]

As is well known, the Feynman amplitudes of the $1/N$ expansion have a better ultraviolet behavior than those of the usual perturbative scheme. This makes it possible to consider more general interactions than those allowed by the power-counting criterion of the perturbative approach. Within this extended class, quadrilinear self-interactions of fermionic[10] fields are of primary interest not only for methodological reasons, but also because they are the basic interactions in fermionic formulations of bosonic Chern-Simons models.[11]

In this work we investigate the theory of N Dirac fields interacting via a quadrilinear, Thirring-like interaction, specified by the Lagrangian

$$\mathcal{L}=i\bar{\psi}\partial\!\!\!/\psi-\frac{g}{2N}(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi)\ . \tag{1}$$

We will study two versions of the theory associated with (1), ψ having either two or four Dirac components. For large N, in the general case where a mass term $M\bar{\psi}\psi$ is added, we found vectorial bound states with a mass^2 m^2 in the region $0<m^2<4M^2$. This happens for g positive in the four-component version, whereas g must be greater than $-2\pi/M$ if two-component fermions are used. If g is outside these values, complex poles signaling instabilities occur.

In analyzing the renormalization of the $1/N$ expansion for this model, we will show the natural emergence of a gauge structure providing a principle to select the physical content of the theory. Using four-component fermions, we will prove that, to any finite order of $1/N$, the model does not present anomalies in the conservation of vector or the axial-vector currents. These conservation laws correspond to U(2) symmetry which arises as a result of the reducibility of the representation used for the Dirac matrices. For large N the absence of anomalies prevents the generation of a mass for the fermion field. Mass generation may occur only at not very large values of N, as a cooperation of different $1/N$ orders, and we discuss this possibility for both two- and four-component versions of the model.

The paper is organized as follows. In Sec. II the properties of the three-dimensional Thirring model employing two-component Dirac fermions are discussed. A version using four-component spinors is considered in Sec. III. There we prove the absence of anomalies as mentioned before. The possible occurrence of mass generation is analyzed in Sec. IV, using the Schwinger-Dyson equations as a basic tool. After some reasonable simplifications, a solution violating either chiral or parity symmetry is found.

II. TWO-COMPONENT REPRESENTATION

The most efficient way to derive the $1/N$ expansion for the model (1) is to use the equivalent Lagrangian

$$\mathcal{L}=i\bar{\psi}\partial\!\!\!/\psi-M\bar{\psi}\psi-\frac{A_\mu}{\sqrt{N}}(\bar{\psi}\gamma^\mu\psi)+\frac{1}{2g}A^2\ , \tag{2}$$

where A_μ plays the role of an auxiliary vector field [classically, $A_\mu=(g/\sqrt{N})\bar{\psi}\gamma_\mu\psi$ and $\partial_\mu A^\mu=0$] and a mass term for the fermion field has been added.

Whenever convenient, we could adopt

$$\gamma^0=\sigma^3,\quad \gamma^1=i\sigma^1,\quad \text{and}\quad \gamma^2=i\sigma^2\ , \tag{3}$$

as an explicit realization for the Dirac matrices. Note that the dimension of ψ is one so that the Thirring interaction has dimension four, being consequently perturbatively nonrenormalizable. To generate the $1/N$ expansion, one either integrates over the ψ field or, equivalently, sums an infinite chain of fermion bubble graphs. In particular, the two-point proper vertex function of the auxiliary field is equal to

$$\Gamma_{\mu\nu}(p)=\frac{1}{g}g_{\mu\nu}+\rho_{\mu\nu}(p)\ , \tag{4}$$

where the polarization tensor $\rho_{\mu\nu}$ is given by

$$\rho^{\mu\nu}(p)=-i\int\frac{d^3k}{(2\pi)^3}\mathrm{Tr}\left[\gamma^\mu\frac{i}{k\!\!\!/-M}\gamma^\nu\frac{i}{p\!\!\!/+k\!\!\!/-M}\right]\ . \tag{5}$$

Taking into account that $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho)=-2i\epsilon^{\mu\nu\rho}$, we obtain

$$\rho^{\mu\nu}(p)=2M\epsilon^{\mu\rho\nu}p_\rho F(p^2)+\pi^{\mu\nu}\ , \tag{6}$$

where $F(p^2)$ is the integral

$$F(p^2)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2-M^2)[(k+p)^2-M^2]} \tag{7}$$

and

$$\pi_{\mu\nu}=2i\int\frac{d^3k}{(2\pi)^3}\frac{(k+p)_\mu k_\nu+(\mu\leftrightarrow\nu)-g_{\mu\nu}[k\cdot(k+p)-M^2]}{(k^2-M^2)[(k+p)^2-M^2]}\ . \tag{8}$$

The first contribution to the right-hand side of (6) is a nonlocal Chern-Simons term. This term is essential to the large distance physics, causing transmutation of the spin of ψ field. It breaks parity and time reversal, and being proportional to M, it indicates that the cause for this breaking is the mass term in the Lagrangian (2); actually, that is a well-known result.[4,5]

The second term on the right-hand side of (6) is (linearly) divergent. Now, by its very definition, $\rho_{\mu\nu}$ agrees with the lowest-order contribution to the two-point function of the current $\bar{\psi}\gamma^\mu\psi$. It is therefore natural to enforce its conservation by requiring that the renormalized $\rho_{\mu\nu}$ be transversal. This imposes the same restriction on $\pi_{\mu\nu}$. Clearly, it is convenient to use a regularization scheme furnishing a transversal tensor. For example, one could use, alternatively, the dimensional or Pauli-Villars regularization. In any case, the final result is

$$\pi_{\mu\nu}=\frac{1}{4\pi}[M+i2\pi(4M^2+p^2)F(p^2)]\left[g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right]\ . \tag{9}$$

As happens in massive QED,[12] the propagator obtained by inverting $\Gamma_{\mu\nu}$ has a longitudinal piece which behaves as a constant when the momentum p is scaled to infinity. However, as the auxiliary field A_μ interacts with a conserved current, this bad behavior, although affecting Green's functions in general, does not affect S-matrix elements and observables constructed as gauge-invariant (a gauge transformation on A_μ and ψ can be defined similarly to what is made in conventional QED, although our Lagrangian is not invariant) combinations of the basic fields and their derivatives.

If, alternatively, we do not want to worry about what quantities should be required to be renormalizable, we can improve the ultraviolet behavior of all the Green's functions by adding to the Lagrangian the term $(\lambda/2)(\partial_\mu A^\mu)^2$ and postulating that the observables are those quantities which are independent of λ. (This new theory has a restricted gauge symmetry. It is easy to see that this new Lagrangian is invariant under gauge transformations whose parameter Λ satisfies $[\lambda\,\partial^\mu\partial_\mu-(1/g)]\Lambda=0$.) As in the former case, these physical quantities coincide with gauge-invariant combinations of ψ, $\bar{\psi}$, and A_μ.

The propagator, after the introduction of the gauge-fixing term, is

$$\Delta^{\mu\nu}(p)=\frac{i(G+1/g)}{(1/g+G)^2-4M^2p^2F^2}\left[g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right]-\frac{i}{\lambda p^2-1/g}\frac{p^\mu p^\nu}{p^2}+\frac{2MF}{(1/g+G)^2-4M^2p^2F}e^{\mu\alpha\nu}p_\alpha\ , \tag{10}$$

where $F(p)$ is given in (7), and

$$G(p)=\frac{1}{4\pi}[M+i2\pi(4M^2+p^2)F(p)]\ . \tag{11}$$

The last term in the denominator of the transversal part of $\Delta^{\mu\nu}$, namely, $4M^2p^2F^2$, arises as a result of the induced Chern-Simons term. It is absent if four-component spinors are employed. In that case the propagator has a simpler form

$$\Delta^{\mu\nu}(p)=\frac{i}{1/g+2G}\left[g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right]-\frac{i}{\lambda p^2-1/g}\frac{p^\mu p^\nu}{p^2}\ . \tag{12}$$

For each positive g this propagator shows a bound-state pole in the region $0<p^2<4M^2$. However, for negative g, tachyons are present, indicating the breakdown of the $1/N$ approximation. These conclusions are drawn from a close examination of the denominator of the transversal part of the propagator given above. The function $F(p)$ is given by

$$F(p)=\frac{i}{4\pi(p^2)^{1/2}}\operatorname{arctanh}\frac{(p^2)^{1/2}}{2M}\ , \tag{13}$$

for $0<p^2<4M^2$. Outside the region, $F(p)$ is obtained by an analytic continuation of this formula.

The fact that the model is unstable for g negative can be understood by a variant of Dyson's argument.[13] For g positive the interaction among fermions through A_μ has the same form as in QED. We have then that particles with unlike charges are attracted, whereas those with charges of the same sign are repelled. For g negative, instead, particles with charges of the same sign are attracted and those with charges of different signs are repelled. Clustering of fermions in one region of space and antifermions in another is favored, and the vacuum is unstable.

The addition of the Chern-Simons term, which, in the two-component case, is dynamically generated, stabilizes the model even at some values of g that are forbidden in the four-component version. The propagator (10) presents bound-state poles in the region

$$\frac{M}{4\pi}+\frac{1}{g}>0\ , \tag{14}$$

and complex poles are found if this relation is violated.

The Thirring-like four-fermion interaction is perturbatively nonrenormalizable. In the $1/N$ expansion, however, the quadrilinear interaction is replaced by the trilinear interaction between the auxiliary field A_μ and the current $\bar{\psi}\gamma^\mu\psi$. Now, for large p^2, the A_μ propagator behaves as

$$\Delta_{\mu\nu}(p)\underset{p\to\infty}{\sim}\left[g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right]\frac{1}{(p^2)^{1/2}}\ , \tag{15}$$

and this provides additional decaying factors, which, as we will see shortly, turn the expansion renormalizable.

At any finite order of the $1/N$ expansion, Feynman amplitudes can be constructed using the following rules: Fermion propagator: $i/(\not p-M)$. A_μ propagator: $\Delta_{\mu\nu}$, given above. Trilinear vertex: the vertex associated with the term

$$-\frac{A_\mu}{\sqrt{N}}(\bar{\psi}\gamma^\mu\psi)\ . \tag{16}$$

Graphs containing as subgraphs the one-loop contribution to the A_μ propagator should be omitted since it has been explicitly taken into consideration. With these rules we obtain that the degree of superficial divergence associated with a proper graph γ is given by

$$d(\gamma)=3-N_F-N_{A_\mu}\ , \tag{17}$$

where N_F and N_{A_μ} are the number of external fermion and A_μ lines, respectively. From this we see that the $1/N$ expansion defines a renormalizable theory. Graphs with three external A_μ lines are logarithmically divergent, but as can be rapidly checked, the divergent contributions always involve an odd number of loop momenta factors, and a symmetric regularization is enough to eliminate them. Graphs having $N_{A_\mu}=2$ and $N_F=0$ are linearly divergent, but, again, because of the fact that A_μ couples to a conserved current, the resulting expression must be transversal. This imposition effectively reduces the degree of divergence by 2 so that no counterterm is needed. Differently, in four dimensions the same type of diagram is quadratically divergent and needs a counterterm of the type $F^{\mu\nu}F_{\mu\nu}$, making the $1/N$ expansion unrenormalizable.

The discussion of the observable content of the theory is the same as in massive QED_4.[12] Observable fields are those fields $\mathcal{O}_i(x_i)$ satisfying the following two conditions.

(1) Each $\mathcal{O}_i$ commutes with $\partial_\mu A^\mu$. This implies that the covariantized time-ordered function of those fields should obey

$$\begin{aligned}\left\langle 0\left|T\partial_\mu A^\mu(x)\prod_i \mathcal{O}_i X\right|0\right\rangle &=\frac{1}{\lambda}\sum_{j=1}^{l}\partial_{\nu_j}\Delta_F\left[x-x_j;\frac{1}{\lambda g}\right]\left\langle 0\left|T\prod_i \mathcal{O}_i X_j\right|0\right\rangle\\ &+\frac{1}{\lambda}\sum_{j=1}^{N}\left[\Delta_F\left[x-w_j;\frac{1}{\lambda g}\right]-\Delta_F\left[x-z_j;\frac{1}{\lambda g}\right]\right]\\ &\times\left\langle 0\left|T\prod_i \mathcal{O}_i X\right|0\right\rangle\ ,\end{aligned} \tag{18}$$

where X is an arbitrary product of the fields,

$$X=\prod_{i=1}^{l}A_{\nu_i}(x_i)\prod_{j=1}^{N}\psi(w_j)\prod_{k=1}^{N}\bar{\psi}(z_k)\ . \tag{19}$$

and X_i is equal to X with the field $A_{\nu_i}(x_i)$ deleted.

(2) Independence of λ. This means that

$$\frac{\partial}{\partial\lambda}\left\langle 0\left|T\prod_i \mathcal{O}_i X\right|0\right\rangle=\text{terms vanishing on shell}\ . \tag{20}$$

It must be stressed that our construction is solely motivated by the bad high-momentum behavior of the longitudinal part of the propagator of the auxiliary field. In two dimensions the imposition of a gauge structure as in (1) and (2) would be too restrictive since the behavior at large momentum is highly improved and λ can be put equal to zero from the very beginning.

III. FOUR-COMPONENT REPRESENTATION

Theories using a two-component fermion field have the property that the fermionic mass term produces a violation of parity. A parity-conservating Lagrangian can be constructed by doubling the number of fermion fields. This leads to a four-component representation which uses four-by-four Dirac matrices. These three Dirac matrices can be taken as the first three Dirac matrices used in four-dimensional calculations. For definiteness we choose the representation

$$\gamma^0=\begin{bmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{bmatrix},\quad \gamma^1=\begin{bmatrix}i\sigma^1 & 0\\ 0 & -i\sigma^1\end{bmatrix},\quad \gamma^2=\begin{bmatrix}i\sigma^2 & 0\\ 0 & -i\sigma^2\end{bmatrix}. \tag{21}$$

In the free-field situation, the use of the above matrices leads to Dirac equations for two-component spinors of masses M and $-M$. In addition to those matrices, we will use

$$\gamma^3=i\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}$$

and (22)

$$\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}.$$

Because the Lagrangian uses only three Dirac matrices, the parity transformation, corresponding to $x_1\to -x_1$,

$$\psi(x^0,x^1,x^2)\to P_\epsilon\psi(x^0,-x^1,x^2)\ , \tag{23}$$

may be implemented by any of the operators

$$P_\epsilon=\frac{1+\epsilon}{2}P_1+\frac{1-\epsilon}{2}P_2\ , \tag{24}$$

$$P_1=-i\gamma_1\gamma_3,\quad P_2=-\gamma_1\gamma_5\ , \tag{25}$$

depending on the parameter ϵ, $|\epsilon|=1$. Other discrete symmetries, such as charge conjugation and time reversal, may also depend on free parameters. We have

$$\mathcal{C}\psi\mathcal{C}^{-1}=\bar\psi C_\eta \quad \text{for charge conjugation}\ , \tag{26}$$

$$\mathcal{T}\psi\mathcal{T}^{-1}=B_\rho\psi(-x_0,\mathbf{x}) \quad \text{for time reversal}\ , \tag{27}$$

where $\mathcal{C}$ is unitary and $\mathcal{T}$ is antiunitary. B_ρ and C_η are four-by-four matrices given by

$$B_\rho=-\left[\frac{1+\rho}{2}\right]\gamma^2\gamma^3-i\left[\frac{1-\rho}{2}\right]\gamma^2\gamma^5\ , \tag{28}$$

$$C_\eta=-i\left[\frac{1+\eta}{2}\right]\gamma^0\gamma^1+\left[\frac{1-\eta}{2}\right]\gamma^2\ , \tag{29}$$

where ρ and η are unitary complex numbers. Observe that both B and C are unitary matrices. Bilinears in $\psi,\bar\psi$ or their derivatives, regardless of the values of the parameters ϵ, η, and ρ, have simpler transformation properties if they involve only the γ^μ matrices. This happens, for example, with the bilinears present into the Lagrangian. Some of these bilinears are considered in Table I. There, for notational simplicity, we introduced the matrix $\tilde\gamma^\mu$, defined by $\tilde\gamma^0=\gamma^0$, $\tilde\gamma^1=-\gamma^1$, and $\tilde\gamma^2=\gamma^2$. The arguments of the transformed fields are $\tilde x=(x^0,-x^1,x^2)$, in the case of parity, and $\hat x=(-x^0,x^1,x^2)$, in the case of time reversal.

TABLE I. P, C, and T transformation properties of some scalar bilinears.

	P	C	T
$\bar\psi\psi(x)$	+	+	+
$\bar\psi\gamma^\mu\psi(x)$	$\bar\psi\tilde\gamma^\mu\psi(\tilde x)$	$-\bar\psi\gamma^\mu\psi(x)$	$\bar\psi\gamma_\mu\psi(\hat x)$
$\bar\psi\gamma^3\gamma^5\psi(x)$	−	+	−

As A_μ couples to the current $\bar\psi\gamma^\mu\psi$, the invariance of the Lagrangian under P, C, and T implies that

$$\begin{aligned} &A_\mu(x)\to \tilde A_\mu(\tilde x) \quad \text{under } P\ ,\\ &A_\mu(x)\to -A_\mu(x) \quad \text{under } C\ ,\\ &A_\mu(x)\to A^\mu(\hat x) \quad \text{under } T\ ,\end{aligned} \tag{30}$$

irrespective of the values of the parameters ϵ, η, and ρ. The transformed field $\tilde A_\mu$ is defined by $\tilde A_0=A_0$, $\tilde A_1=-A_1$, and $\tilde A_2=A_2$.

Because of

$$P^{-1}\gamma^3P=-(\mathrm{Re}\epsilon)\gamma^3-(\mathrm{Im}\epsilon)\gamma^5\ , \tag{31}$$

and similar equations with P replaced by C and B, bilinears involving γ^3 and γ^5 will in general mix among themselves. However, there is a considerable simplification if the parameters are real. Table II illustrates this fact.

The classical massless Lagrangian is invariant under the U(2) transformations

$$\psi\to e^{iJ}\psi\ , \tag{32}$$

where J is a linear combination of the matrices $R=I$, γ^3,γ^5, and $\gamma^3\gamma^5$. These symmetries are generated by the currents

$$J_R^\mu=\bar\psi\gamma^\mu\psi,\quad \bar\psi\gamma^\mu\gamma^3\psi\ ,\quad \bar\psi\gamma^\mu\gamma^5\psi\ ,\quad \bar\psi\gamma^\mu\gamma^3\gamma^5\psi\ . \tag{33}$$

For the massive case the symmetries related to γ^3 and γ^5 are explicitly broken and the corresponding currents have divergencies $2iMJ_R$, where J_R is given by $\bar\psi\gamma^3\psi$ and $\bar\psi\gamma^5\psi$, respectively. At the quantum level we must yet look for possible anomalies in the conservation of the above four currents. As we shall see shortly, they are free from anomalies at any finite order of $1/N$.

Similarly to the two-component representation considered in the previous section, the $1/N$ expansion may be obtained by using the Lagrangian (2). In the present situation no Chern-Simons term is generated, of course. The two-point vertex function of the auxiliary field A_μ is

TABLE II. Transformation properties for special values of the free parameters.

	P		C		T	
	$\epsilon=1$	$\epsilon=-1$	$\eta=1$	$\eta=-1$	$\rho=1$	$\rho=-1$
$\bar\psi\gamma^3\psi(x)$	−	+	+	−	+	−
$\bar\psi\gamma^5\psi(x)$	+	−	+	−	−	+

equal to

$$\Gamma_{\mu\nu}=\frac{1}{g}g_{\mu\nu}+2\pi_{\mu\nu}\ , \tag{34}$$

where $\pi_{\mu\nu}$ is given by (8). It follows that the four-component theory has the same ultraviolet behavior as the two-component one. Thus renormalizability can be achieved by introducing the gauge structure specified in items (1) and (2) at the end of the preceding section.

The absence of anomalies in the conservation of J_R^μ can be proved by Fujikawa's method.[14] In that method these anomalies come from a possibly nontrivial Jacobian of the transformation of the measure of the functional integral induced by a change of the fields. Following Fujikawa's steps, we are led to

$$\partial_\mu\langle J_R^\mu\rangle=0\ , \tag{35}$$

for $R=I$ and $\gamma^3\gamma^5$, and

$$\partial_\mu\langle J_R^\mu\rangle=2iM\langle J_R\rangle+2\Lambda_R\ , \tag{36}$$

for $R=\gamma^3$ and γ^5, and where Λ_R is the anomaly

$$\Lambda_R\propto\lim_{\mathcal{M}\to\infty}\mathcal{M}^3\mathrm{Tr}\left[R\exp\left[\frac{i}{4\pi\mathcal{M}^2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right]\right]=0\ , \tag{37}$$

as consequence of the properties of the Dirac matrices.

More formally, another proof of the absence of anomalies can be obtained by considering the massive theory and using the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) procedure for subtracting divergent diagrams. The formal currents in (33) are quantized with a $\mathcal{N}_2$ normal product, and as a direct application of the BPHZ algorithm, we get[15]

$$\partial_\mu\langle 0|T\mathcal{N}_2(\bar\psi\gamma^\mu\psi)X|0\rangle=\sum_{j=1}^{N}[\delta(x-w_j)-\delta(x-z_j)]\langle 0|TX|0\rangle \tag{38}$$

and

$$\partial_\mu\langle 0|T\mathcal{N}_2(\bar\psi\gamma^\mu\gamma^3\gamma^5\psi)X|0\rangle=\sum_{j=1}^{N}[(\gamma^3\gamma^5)_{w_j}\delta(x-w_j)-(\gamma^3\gamma^5)^t_{z_j}\delta(x-z_j)]\langle 0|TX|0\rangle\ , \tag{39}$$

where X is given by (19) and the superscript t indices the transposed matrix. The fermionic mass term breaks the conservation of the other currents, giving

$$\partial_\mu\langle 0|T\mathcal{N}_2(\bar\psi\gamma^\mu\gamma^3\psi)X|0\rangle=2iM\langle 0|T\mathcal{N}_3(\bar\psi\gamma^3\psi)X|0\rangle+\sum_{j=1}^{N}[(\gamma^3)_{w_j}\delta(x-w_j)+(\gamma^3)^t_{z_j}\delta(x-z_j)]\langle 0|TX|0\rangle \tag{40}$$

and

$$\partial_\mu\langle 0|T\mathcal{N}_2(\bar\psi\gamma^\mu\gamma^5\psi)X|0\rangle=2iM\langle 0|T\mathcal{N}_3(\bar\psi\gamma^5\psi)X|0\rangle+\sum_{j=1}^{N}[(\gamma^5)_{w_j}\delta(x-w_j)+(\gamma^5)^t_{z_j}\delta(x-z_j)]\langle 0|TX|0\rangle\ . \tag{41}$$

Note that the degree of the normal products on the right-hand sides of these equations has increased by 1. They can be related to minimally subtracted normal products through the Zimmermann identities. Technically, this is the cause for the existence of anomalies. More formally, the anomalies should have the same quantum numbers as the terms already present in the classical conservation laws. This puts a very strong restriction on the possible new terms. In fact, since A_μ itself transforms under P, C, and T independently of the values of the parameters ϵ, η, and ρ, it immediately follows that the anomalies cannot have terms depending only on the field A_μ. Moreover, the anomalies should be polynomials of canonical dimension 3, as follows from general considerations on the definition of composite fields. Since ψ and A_μ both have canonical dimensions equal to 1, it follows from Tables I and II that the possible anomalous terms must be independent of A_μ altogether. Thus only terms bilinear in ψ and $\bar\psi$ and having one derivative at most can contribute to the anomalies. Using Tables I and II, it is easily checked that only terms proportional to the divergence of the currents themselves can arise, i.e.,

$$\mathcal{N}_3(\bar\psi\gamma^3\psi)=\mathcal{N}_2(\bar\psi\gamma^3\psi)+s_1\mathcal{N}_3[\partial_\mu(\bar\psi\gamma^\mu\gamma^3\psi)] \tag{42}$$

and

$$\mathcal{N}_3(\bar\psi\gamma^5\psi)=\mathcal{N}_2(\bar\psi\gamma^5\psi)+s_2\mathcal{N}_3[\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi)]\ , \tag{43}$$

where the coefficients s_1 and s_2 can be computed order by order in $1/N$.

So, as claimed before, the anomalies are very mild, being possible to absorb them into the normalization of the currents.

IV. FERMION MASS GENERATION

Let us now consider the model (1) with $M=0$ and investigate if a mass can be dynamically generated, implicating either parity breaking or chiral-symmetry breaking in the two- or four-component versions. For simplicity we choose to work in Euclidean space. The Schwinger-Dyson equations are depicted in Fig. 1. The propagators represented by single lines are the ones read from (2), taking $N=\infty$. The propagators represented by

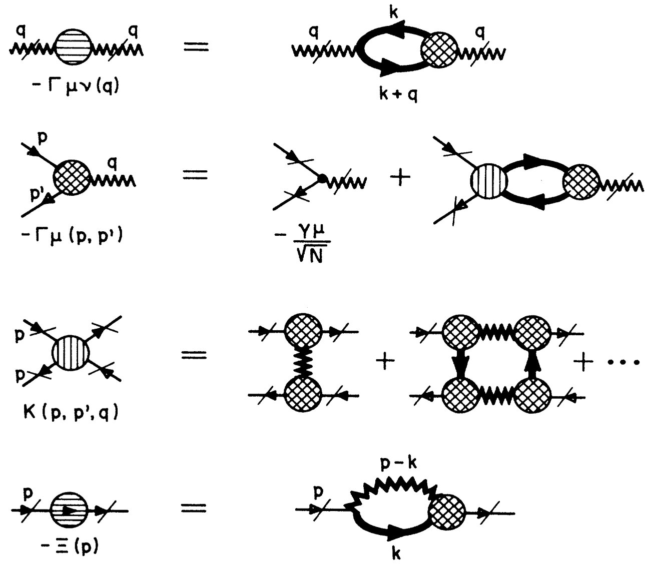

FIG. 1. Full set of self-coupled Schwinger-Dyson equations for the Thirring model, as described in (1).

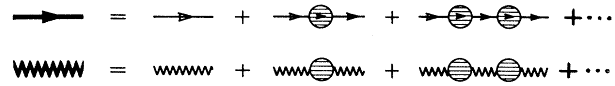

FIG. 2. Expansion of the full propagators of ψ and A_μ in terms of the one-particle-irreducible parts appearing in the Schwinger-Dyson equations.

double lines are the complete ones, with many self-energy insertions as indicated in Fig. 2. In the dominant order of $1/N$, Γ_μ is given by the trivial contribution $-\gamma_\mu/\sqrt{N}$ and the four-fermion kernel decouples from the system of equations. The relevant Schwinger-Dyson equations reduce to the photon and fermion self-energy parts as shown in Fig. 3. Writing the fermion self-energy as

$$\Xi(p)=i[A(p^2)\not{p}-\Sigma(p^2)]\ , \tag{44}$$

the full fermion propagator reads

$$S(p)=i\frac{\not{p}[1+A(p^2)]-\Sigma(p^2)}{p^2[1+A(p^2)]^2+\Sigma^2(p^2)}\ . \tag{45}$$

For the moment we will keep ourselves from doing an $1/N$ expansion for Σ. Let us instead consider the possibility of having, as the result of some cooperative effect among different orders of $1/N$, nonvanishing values for Σ and A.

To proceed with the analysis, it is necessary to make some assumptions, the validity of which may be verified using consistency checks on the results. Specifically, we will assume that both $\Sigma(p)$ and $A(p)$ are small compared with the characteristic mass $\alpha=32/gD$ of the model and also that they tend rapidly to zero for values of $|p|$ above α. D is equal to either two or four for the two- or four-component versions of the model.

Let us first look at the photon self-energy. Adopting the aforementioned approximations and considering that most of the contributions to the fermion loop come from the region of integration $|k|>32/gD$, Σ and A can be taken as zero. This is the same kind of approximation used in QED_3.[9] The result is given by (4)–(11) with M put equal to zero. In Euclidean space we get

$$\Delta_{\mu\nu}(p)=\frac{1}{1/g+(D/32)|p|}\left[\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right]+\frac{1}{1/g+\lambda p^2}\frac{p_\mu p_\nu}{p^2}\ . \tag{46}$$

In more accurate calculations, where in the fermion loop Σ is not taken as zero, a nonlocal Chern-Simons term would also be induced if two-component spinors are used.

The simplified Schwinger-Dyson equation for the fermion self-energy,

$$\Xi(p)=\frac{1}{N}\int\frac{d^3k}{(2\pi)^3}\Delta_{\mu\nu}(p-k)\gamma_\mu S(k)\gamma_\nu\ , \tag{47}$$

with $\Gamma_\mu=-\gamma_\mu/\sqrt{N}$, after the substitution of (46) and after some traces are computed, gives

$$\Sigma(p)\simeq\frac{2}{N}\int\frac{d^3k}{(2\pi)^3}\frac{\Sigma(k)}{k^2[1+A(k)]^2+\Sigma^2(k)}\frac{1}{1/g+(D/32)|p-k|}+\frac{1}{N}\int\frac{d^3k}{(2\pi)^3}\frac{\Sigma(k)}{k^2[1+A(k)]^2+\Sigma^2(k)}\frac{1}{1/g+\lambda(p-k)^2} \tag{48}$$

and

$$p^2A(p)\simeq\frac{2}{N}\int\frac{d^3k}{(2\pi)^3}\frac{1+A}{k^2(1+A)^2+\Sigma^2}\frac{1}{1/g+(D/32)|p-k|}\frac{(p-k)\cdot p\,(p-k)\cdot k}{(p-k)^2}-\frac{2}{N}\int\frac{d^3k}{(2\pi)^3}\frac{1+A}{k^2(1+A)^2+\Sigma^2}\frac{1}{1/g+\lambda(p-k)^2}\left[\frac{(p-k)\cdot p\,(p-k)\cdot k}{(p-k)^2}-p\cdot k\right]\ . \tag{49}$$

Expanding A and Σ in powers of $1/N$,

$$A = a_0 + \frac{a_1}{N} + \frac{a_2}{N^2} + \frac{a_3}{N^3} + \cdots ,$$
$$\Sigma = \sigma_0 + \frac{\sigma_1}{N} + \frac{\sigma_2}{N^2} + \frac{\sigma_3}{N^3} + \cdots , \tag{50}$$

and equating the same powers of $1/N$ in each side of (48) and (49), we see that a_0 and all σ_i's are zero; that is, $A(p)$ can be possibly nonvanishing only at nonleading orders, whereas a mass is not generated at any finite order of $1/N$. To be fair, these results are strictly valid only to leading $1/N$ order. In computing subleading contributions, one should also consider corrections to the trilinear vertex and also to the four-fermion kernel, taking into account all four Schwinger-Dyson equations of Fig. 1. We must stress that the results are in accord with that of Sec. III concerning the absence of anomalies in the conservation of the currents. However, this does not preclude the possibility for Σ to be generated for small N because of cooperative effects of different orders of $1/N$. To explore this possibility, we put $A(p)=0$ in (48). After the angular integrations are done, we get

$$\Sigma(p) = \frac{16}{ND\pi^2}\frac{1}{p}\int_0^\infty dk \frac{k\Sigma(k)}{k^2+\Sigma^2(k)}\left[|p+k| - |p-k| + \alpha \ln\frac{|p-k|+\alpha}{|p+k|+\alpha}\right] + \frac{1}{8N\lambda\pi^2}\frac{1}{p}\int_0^\infty dk \frac{k\Sigma(k)}{k^2+\Sigma^2(k)} \ln\frac{1/g\lambda + (p-k)^2}{1/g\lambda+(p+k)^2} . \tag{51}$$

$\Sigma(p)$ is a gauge-dependent quantity. However, the fact that it is not identically zero has physical consequences (parity- or chiral-symmetry breaking), and it is therefore a gauge-invariant statement. For simplicity, we chose to work in the unitary ($\lambda \to 0$) and the Landau gauge ($\lambda \to \infty$). Moreover, we will restrict the study to the region $\Sigma(p) < p < \alpha$. Expanding the logarithms and keeping only the dominant terms in p/α, we get

$$\Sigma(p) \simeq \frac{16}{ND\pi^2}\left[\int_0^p dk \frac{k^2\Sigma(k)}{k^2+\Sigma^2(k)}\frac{2}{p+\alpha} + \int_p^\alpha dk \frac{k^2\Sigma(k)}{k^2+\Sigma^2(k)}\frac{2}{k+\alpha} + \frac{\xi}{\alpha}\int_0^\alpha dk \frac{k^2\Sigma(k)}{k^2+\Sigma^2(k)}\right] . \tag{52}$$

In light of the above approximations, we disregard the contributions of $k > \alpha$ to the integrals on the right-hand side of (52). ξ is one or zero, respectively, for the unitary and Landau gauge. The above integral equation is equivalent to the differential equation

$$\frac{d}{dp}\left[(p+\alpha)^2\frac{d\Sigma}{dp}\right] = -\frac{32}{ND\pi^2}\frac{p^2\Sigma}{p^2+\Sigma^2} , \tag{53}$$

subject to two boundary conditions[9,16] that we choose to be

$$2\alpha(1+\xi)\frac{d\Sigma}{dp}\bigg|_{p=\alpha} + \Sigma|_{p=\alpha} = 0 , \tag{54}$$

$$0 < \Sigma|_{p=0} < \infty . \tag{55}$$

As we already know, mass generation does not occur for N big enough. Thus, if it occurs for N small, there should exist a critical value N_c. For N smaller than but

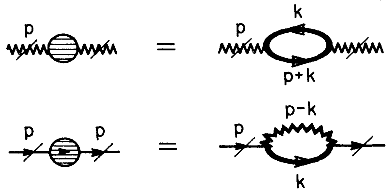

FIG. 3. System of self-coupled Schwinger-Dyson equations in light of the approximations of Sec. IV.

near N_c, we must have a region in which $\Sigma(p) \ll p < \alpha$, and there, the linearized equation

$$\frac{d}{dp}\left[(p+\alpha)^2\frac{d\Sigma}{dp}\right] = -\frac{32}{ND\pi^2}\Sigma \tag{56}$$

is a good approximation to (53).

Similarly to what happens in QED_4,[17] (56) has the solutions

$$\Sigma_\pm = 1/(p+\alpha)^{1/2 \pm 1/2(1-128/ND\pi^2)^{1/2}} . \tag{57}$$

Nevertheless, in our case they are real only for $N > N_c = 128/D\pi^2$ and so do not satisfy the requirement that N be small. Moreover, they do not satisfy (54) and are not solutions of the integral equation (52).

For $N < N_c$, (56) has the oscillatory solutions

$$\Sigma_n(p) = \frac{1}{(p+\alpha)^{1/2}} \times \sin\left[\frac{1}{2}\left[\frac{128}{ND\pi^2} - 1\right]^{1/2} \ln\frac{p+\alpha}{2\alpha} + n\pi + \delta\right] , \quad n = 0,1,2,\ldots , \tag{58}$$

where $\delta = -\pi/2$ for the unitary gauge and $\delta = -(128/ND\pi^2 - 1)^{1/2}$, for the Landau gauge. Σ_n satisfy (54) and so are solutions of (52). The oscillatory character of these solutions is essential to the compatibility of the assumption that we have made before, namely, that Σ tends to zero above a certain value of p. In fact, in the unitary gauge, the last term in (52) is a constant, independent of p. To be consistent with our assumption, this constant must vanish, which can be true for an oscillatory $\Sigma(p)$.

Which of the solutions Σ_n is energetically preferred should be inferred from an analysis of the effective action.

It is interesting to observe that the mechanism of mass generation in this model is more similar to the one found in QED_3 (Ref. 9) than that working on the Jona-Lasinio–Gross–Neveu model.[1,10] The fact that mass generation occurs as a result of contributions of terms of different orders of $1/N$ could be inferred from an examination of the identities among quartic fermionic self-couplings listed at the end of Ref. 1. In the case $D=2$, as far as $\Sigma(p)$ is nonvanishing, a Chern-Simons term will be induced, but it will be highly nonlocal.

To sum up, we have obtained a nonvanishing $\Sigma(p)$ both in the unitary and in the Landau gauges. The difference in the result for the two gauges is only due to phase δ in (58). The unitary gauge is known to be an ultraviolet problematic gauge, at least peturbatively. In the Schwinger-Dyson self-consistent approach, the main trouble comes from the last term in (52). But, as we saw, a finite solution is possible because of its two characteristics: (a) a rapid decay of Σ and A in the ultraviolet region and (b) oscillatory behavior.

The position $p^2=-m^2$ of the pole of the propagator is a physical quantity that must be independent of the gauge. The restriction of the validity of the solutions (58) to the region $\Sigma(p) \ll |p|$ does not allow one to investigate this possibility, although it seems greatly plausible. In any case, since $\Sigma(p)$ is not identically zero, either parity or chiral symmetry is dynamically broken.

Our analysis is still a bit crude, and a numerical verification would be welcome. That is in progress.

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