Renormalization of a (2+1)-dimensional supersymmetric nonlinear σ model in $1/N$ expansion

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We explicitly carry out the renormalization of the O(N)-invariant supersymmetric nonlinear σ model up to next-to-leading order in $1/N$ and show that the wave-function renormalization removes all divergences. Cancellations between fermion and boson loops prevent counterterms from being induced which would otherwise spoil the nonlinear constraint. This aspect leads to the renormalization of the model by the conventional Bogoliubov-Parasiuk-Hepp-Zimmermann scheme in contrast with the ordinary nonlinear σ model. Supersymmetry constrains the anomalous dimensions of the Lagrange multiplier fields to be exactly zero, enabling one to prove renormalizability to all orders in $1/N$. Furthermore, the β function for this model vanishes.

I. INTRODUCTION

Recently there has been considerable interest in quantum field theories in spacetime dimension $d \equiv 2+1$. For physicists, the two main motivations for investigating planar [i.e., (2+1)-dimensional] field theories are that these theories have possible physical applications and they may provide us with deeper insight into quantum field theories in general. Among the possible physical applications, it appears that the quantum Hall effect and high-T_c superconductivity involve planar gauge dynamics.[1] Furthermore, motion in the presence of long cosmic strings is adequately described by planar gravity.[2] As for deeper insight into quantum field theory, massive planar QED has been investigated with a view to better understand the relationship between infrared (IR) singularities and confinement.[3] Also, since planar quantum gravity is renormalizable by power counting[4] it may serve as a useful "laboratory" for investigating the difficulties associated with quantum gravity in four dimensions.[4,5] Indeed, it is possible that quantum gravity in $d=3+1$, though not perturbatively renormalizable, may be a physically sensible theory when dealt with by nonperturbative means.[6]

It has been argued that, in $d=2+1$, even though the four-fermion interaction[7] and the nonlinear σ model[8,9] (NSM) are not renormalizable in weak-coupling perturbation theory, they are renormalizable order by order in the $1/N$ expansion. In this context, the planar supersymmetric nonlinear σ model (SSNSM) is particularly interesting because it contains within it a four-fermion interaction sector as well as an ordinary NSM sector.[10]

In this paper we examine in detail the renormalization of the SSNSM by calculating all the divergences in the O(N)-symmetric phase up to next-to-leading order in $1/N$. In addition, we deal with all the primitive divergences and this enables us to discuss the question of renormalizability to all orders in $1/N$. This model also exhibits a phase in which O(N) symmetry is broken to O($N-1$) symmetry and an O(N)-symmetric critical phase. Since the particular realization of a phase does not affect ultraviolet behavior, our demonstration of renormalizability in the O(N) phase automatically carries over to the critical and broken phases as well.[11]

The presence of supersymmetry (SUSY) greatly improves the divergence structure of the four-fermion and NSM sectors of the model. For instance, the four-fermion sector does not require a fine-tuning of the coupling constant, and this ultimately means that the corresponding auxiliary field and its induced propagator do not need to be renormalized. In the NSM sector, all the logarithmically divergent corrections to the gap equation, which arise beyond leading order, sum to zero because of cancellations between fermion and boson loops. Thus, there is no renormalization of the coupling constant other than the fine-tuning which determines the phase of the SSNSM; this also means that the β function vanishes. Furthermore, we see that the problems associated with the NSM which have been dealt with using unconventional renormalization procedures disappear in the supersymmetric model, making the latter renormalizable using the conventional Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) scheme.[12]

The organization of this paper is as follows. In Sec. II we briefly review the essential features in the $1/N$ expansion of the NSM and the four-fermion model; we see that the four-fermion model may be renormalized at least up to next-to-leading order in $1/N$ by conventional BPHZ techniques. But since the effective propagator for the auxiliary field in the model must also be renormalized, the renormalization proof to all orders in $1/N$ is less clear-cut. We then detail the problems encountered in renormalizing the NSM; we see that the way these problems have been dealt with in the literature[8,9] requires one to leave some Green's functions unrenormalized and to use these as subdiagrams to cancel other divergences; thus the NSM has not been renormalized using the conventional BPHZ scheme. In Sec. III we review the SSNSM[10,13] and its dynamics. Taking advantage of the O(N) symmetry we generate a $1/N$ expansion and derive its Feynman rules. In Sec. IV we compute all the primitive divergences and *explicitly* carry out the renormalization up to next-to-leading order in $1/N$. Because of the

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presence of SUSY, we show that cancellations between fermion and boson loops allow the model (and hence the NSM sector within it) to be renormalized by the conventional BPHZ scheme. Moreover, we show that cancellations due to SUSY between fermion and boson loops ensure that counterterms, which would otherwise spoil the nonlinear constraint, are not induced at higher orders in the SSNSM; it is these counterterms that persist in the absence of SUSY and are at the root of the difficulties encountered in the ordinary NSM. So, in the SSNSM all the divergent one-particle-irreducible (1PI) diagrams are made finite by counterterms which are of the same form as those in the bare Lagrangian and thus the model is renormalizable order by order by the conventional BPHZ scheme. This is in contrast with the ordinary NSM where, if all the counterterms demanded by the conventional BPHZ scheme are taken into account, the resulting counterterm Lagrangian is not of the same form as the bare Lagrangian. Finally, we discuss the role of SUSY in the renormalization to all orders of $1/N$ and in Sec. V we conclude with a few summarizing remarks.

II. $1/N$ RENORMALIZABILITY OF THE NSM AND THE FOUR-FERMION MODEL

A. The four-fermion model

The four-fermion model is described by the Lagrangian[7]

$$\mathcal{L}=\frac{i}{2}\bar{\psi}_j\partial\!\!\!/\psi_j+\frac{g^2}{8N}(\bar{\psi}_j\psi_j)^2\ , \tag{2.1}$$

where the sum of the the flavor index j runs from 1 to N and we require that g^2 remain constant as N goes to infinity. The fields ψ_j are two-component Majorana spinors and the γ matrices are $\gamma_0=\sigma_2$, $\gamma_1=i\sigma_3$, and $\gamma_2=i\sigma_1$, where the σ's are the Pauli spin matrices. This Lagrangian is invariant under parity transformation.[10] By introducing the scalar auxiliary field σ we may rewrite (2.1) as

$$\mathcal{L}=\frac{i}{2}\bar{\psi}_j\partial\!\!\!/\psi_j+\frac{1}{2}\sigma\bar{\psi}_j\psi_j-\frac{N\sigma^2}{2g^2}\ . \tag{2.2}$$

The corresponding generating functional in *Euclidean* space is[14]

$$\mathcal{Z}[\eta_j]=\int\mathcal{D}\psi_j\mathcal{D}\sigma\exp\left[\int d^3x_E\left[\frac{1}{2}\bar{\psi}_j(i\partial\!\!\!/_E+\sigma)\psi_j-\frac{N}{2g^2}\sigma^2+\bar{\eta}_j\psi_j\right]\right]\ . \tag{2.3}$$

Integrating over the fields ψ_j we get a nonlocal effective action for the field σ:

$$\mathcal{Z}[\eta_j]=\int\mathcal{D}\sigma\exp(-S_{\rm eff}[\sigma,\eta_j])\ , \tag{2.4}$$

where

$$S_{\rm eff}[\sigma,\eta_j]=\frac{N}{2g^2}\int d^3x_E\sigma^2-\frac{N}{2}{\rm Tr}\ln(i\partial\!\!\!/_E+\sigma)+\tfrac{1}{2}\int d^3x_E\bar{\eta}_j(i\partial\!\!\!/_E+\sigma)^{-1}\eta_j\ . \tag{2.5}$$

The N dependence is now explicit and we may evaluate $\mathcal{Z}$ by the large N saddle-point approximation. We thus impose the stationary condition which gives the gap equation

$$\frac{NM}{g^2}-\frac{N}{2}\int\frac{d^3k_E}{(2\pi)^3}{\rm tr}\frac{1}{-k\!\!\!/_E+M}=0\ , \tag{2.6}$$

where $M=\langle\sigma\rangle$ is the vacuum expectation value (VEV) of σ. The saddle point exists only within the branch

$$0<\frac{1}{g^2}\le\frac{1}{g^2_{\rm cr}}\ , \tag{2.7}$$

where

$$\frac{1}{g^2_{\rm cr}}\equiv\int\frac{d^3k_E}{(2\pi)^3}\frac{1}{k_E^2}\ . \tag{2.8}$$

The true vacuum is given by[7,10] $M\neq0$ so that we have dynamical mass generation which spontaneously breaks parity invariance. If we expand about the shifted field $\sigma'=\sigma-M$, then for $1/g^2$ obeying the gap equation (2.6), the quadratic term gives the inverse effective propagator

$$D^{-1}_{\sigma'}(p^2)=\frac{N}{g^2}-\frac{N}{2}\int\frac{d^3k_E}{(2\pi)^3}\times{\rm tr}\frac{1}{(-k\!\!\!/_E+M)[-(k\!\!\!/_E-p\!\!\!/_E)+M]}\ . \tag{2.9}$$

Using (2.6), we get

$$D_{\sigma'}(p^2)=\frac{8\pi}{N}\frac{1}{p^2+4M^2}\frac{\sqrt{p^2}}{\arctan(\sqrt{p^2}/|M|)}\ . \tag{2.10}$$

This propagator is cutoff independent and nonzero, a necessary condition for renormalizability. The Feynman rules for the $1/N$ expansion are given in Fig. 1; the graphs of Fig. 2 are illegal as subgraphs because they have already been accounted for by the gap equation (2.6)

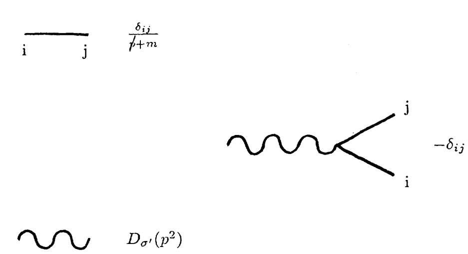

FIG. 1. Feynman rules for the $1/N$ expansion of the four-fermion model.

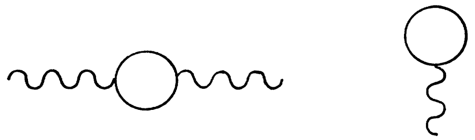

FIG. 2. Illegal subgraphs in the four-fermion model.

and the effective propagator (2.10).

It turns out that all of the infinities may be removed by counterterms which are of the same form as the bare Lagrangian (2.2). Although this has only been explicitly demonstrated up to next-to-leading order in $1/N$,[7] it has been argued that this model is indeed renormalizable to all orders in $1/N$ since the renormalization up to next-to-leading order may be performed by the conventional BPHZ method.[7]

Finally, note that outside the branch (2.7) we must have $M=0$ in order for (2.6) to be satisfied; thus, in this branch we have an O(N)-symmetric and parity-conserving phase. The effective propagator in this phase is

$$D_{\sigma'}(p^2)=\frac{2\pi}{N}\frac{1}{\sqrt{p^2+v^2}}\ , \tag{2.11}$$

where

$$\frac{1}{g^2}-\frac{1}{g_{\rm cr}^2}\equiv v^2\geq 0\ . \tag{2.12}$$

It is important to note that there are no tachyons in this phase because we have to take the positive branch of the square-root function in (2.11); this may be seen by taking the limit $M\to 0$ in (2.9) and using (2.12). However, since v^2 does not correspond to any condensate, the physical dynamics in this phase is obscure. On the other hand, as we see later, the four-fermion sector within the SSNSM does not suffer from such obscurities because v^2 corresponds to the nonzero VEV of one of the N bosons of the supermultiplet.

B. The nonlinear σ model (NSM)

The Lagrangian for the NSM[8] is defined by

$$\mathcal{L}=-\tfrac{1}{2}A_j\partial^2 A_j\ , \tag{2.13}$$

with the local nonlinear constraint

$$A_jA_j=\frac{N}{g^2}\ . \tag{2.14}$$

The sum over the flavor index j runs from 1 to N. The constraint (2.14) may be implemented by introducing a Lagrange multiplier field $\alpha(x)$. The corresponding *Euclidean* path integral then becomes

$$Z=\int\mathcal{D}A_j\mathcal{D}\alpha\exp\left[-\int d^3x_E\left[\frac{1}{2}A_j(-\partial_E^2+\alpha)A_j-\frac{N}{2g^2}\alpha-J_jA_j\right]\right]\ . \tag{2.15}$$

Integrating over the fields A_j, we get an effective action for the field α:

$$Z=\int\mathcal{D}\alpha\exp(-S_{\rm eff}[\alpha,J_j])\ , \tag{2.16}$$

where

$$S_{\rm eff}[\alpha,J_j]=-\frac{N}{2g^2}\int d^3x\,\alpha+\frac{N}{2}{\rm tr}\ln(-\partial^2+\alpha)+\tfrac{1}{2}\int d^3x\,J_j(-\partial^2+\alpha)^{-1}J_j\ . \tag{2.17}$$

The N dependence is now explicit and we may evaluate Z by the large N saddle-point approximation. From (2.17) and the usual Legendre transformation we see that the leading-order effective action is given by

$$\Gamma[A_j,\alpha]=\int d^3x_E\left[-\frac{1}{2}A_j(\partial_E^2+\alpha)A_j-\frac{N}{2g^2}\alpha\right]+\frac{N}{2}{\rm tr}\ln(-\partial_E^2+\alpha)\ . \tag{2.18}$$

Because of the O(N) symmetry, the VEV of $\mathbf{A}\equiv(A_1,\ldots,A_N)$ may be written as

$$\langle\mathbf{A}\rangle=(0,0,\ldots,\sqrt{N}v)\ , \tag{2.19}$$

so that the effective potential becomes

$$\frac{1}{N}V(v,\langle\alpha\rangle)=\left[v^2-\frac{1}{g^2}\right]\langle\alpha\rangle+\int\frac{d^3k_E}{(2\pi)^3}\ln(k_E^2+\langle\alpha\rangle)\ . \tag{2.20}$$

The stationary conditions of this potential are thus

$$\frac{1}{N}\frac{\partial V}{\partial v}=2v\langle\alpha\rangle=0\ , \tag{2.21}$$

$$\frac{1}{N}\frac{\partial V}{\partial\langle\alpha\rangle}=v^2-\frac{1}{g^2}+\int\frac{d^3k_E}{(2\pi)^3}\frac{1}{k_E^2+\langle\alpha\rangle}=0\ . \tag{2.22}$$

It is now apparent from (2.21) that this model exhibits two phases corresponding to broken ($v\neq 0,\langle\alpha\rangle=0$) and unbroken ($v=0,\langle\alpha\rangle\neq 0$), O($N$) symmetry. The second stationary condition (2.22) gives the relation between $\langle\alpha\rangle$ and v. The O(N)-symmetric critical point $g^2=g_{\rm cr}^2$ separating the two phases is determined from (2.22) by putting $\langle\alpha\rangle=v=0$:

$$\frac{1}{g_{\rm cr}^2}=\int\frac{d^3k_E}{(2\pi)^3}\frac{1}{k_E^2}\ . \tag{2.23}$$

Now, integrating over k_E in (2.22) we get

$$v^2-\frac{|\langle\alpha\rangle^{1/2}|}{4\pi}=\frac{1}{g^2}-\frac{1}{g_{\rm cr}^2}\equiv\frac{M}{4\pi}\ . \tag{2.24}$$

In the broken phase ($M>0$) we have $v^2=M/4\pi$, $\langle\alpha\rangle=0$ and the first $n-1$ fields A_j are massless Goldstone bosons but the Nth field A_N, which is now a composite, is unstable and decays into Goldstone bosons. In the symmetric phase ($M\leq 0$) we have $v=0$, $\langle\alpha\rangle=M^2$ so that all N of the bosons A_j acquire a mass $|M|$. More details on these results may be found in Refs. 8 and 9.

C. Renormalizability

If we can prove renormalizability for zero temperature then the renormalizability for all finite temperatures is assured because temperature acts as an infrared cutoff and does not affect the ultraviolet behavior.[11] Accordingly, we shall consider the renormalization in the symmetric phase where the coupling g^2 for zero temperature is given by (2.24) with $v^2=0$. Naturally, since the particular realization of a given phase is purely an "infrared effect," the renormalizability proof in the symmetric phase would automatically carry over to the critical and broken phases as well.[7,8,9,11] The Feynamn rules in the symmetric phase are given in Fig. 3 and the illegal subgraphs are shown in Fig. 4.

To leading order, the only renormalization necessary is the coupling-constant renormalization which is determined by (2.24). However, difficulties arise at next-to-leading order. To begin with, a wave-function renormalization will take care of the logarithmic divergence that arises from the 1PI two-point function in Fig. 5 but there are linear and quadratic divergences as well which induce an A^2 counterterm, and this would spoil the nonlinear constraint (2.14). At even higher orders, the 1PI four-point and six-point functions shown in Figs. 6 and 7, respectively, also spoil the nonlinear constraint (2.14). Finally, the next-to-leading-order corrections to the effective propagator for the field α, shown in Fig. 8, induce α^2 counterterms, so α loses its role as a Lagrange multiplier, and this again spoils the nonlinear constraint (2.14). We may summarize these observations by saying that beyond leading order in $1/N$ counterterms are induced by 1PI diagrams that are not of the same form as the bare Lagrangian in (2.15), implying that the model is not renormalizable beyond leading order. The nonrenormalizability apparently stems from the fact that quantum corrections beyond leading order in $1/N$ spoil the nonlinear constraint (2.14).

The way this problem is corrected in the literature is to abandon the conventional BPHZ renormalization scheme altogether. That is, one leaves all diagrams with external α legs unrenormalized and uses these as subdiagrams to generate *one-particle-reducible* diagrams to cancel divergences of troublesome *primitive* 1PI graphs, with a particular prescription for integration over internal momenta.[8,9] This is not the conventional BPHZ procedure,[12] and it is not clear that this procedure can be generalized to all

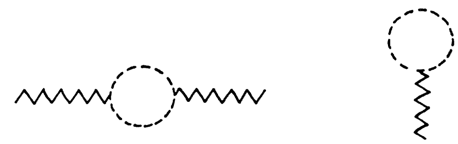

FIG. 4. Illegal subgraphs in the NSM.

orders in $1/N$. In a different attempt to solve the problem, it has been suggested that renormalization may be consistently carried out only via the dimensional-regularization technique.[15] In this scheme, one goes ahead and carries out the vertex renormalization and then, by rescaling the field α, the linear and quadratic divergences from the two-point function are absorbed into the vertex counterterm. This scheme, however, not only depends on the regularization procedure but assumes that there are no four-field counterterms induced if one uses dimensional regularization; but, in fact, a counterterm is induced because of the existence of a logarithmic divergence. In performing the vertex renormalization in this scheme, one is left with no way to eliminate the induced α^2, A^4, and A^6 counterterms, all of which spoil the nonlinear constraint and make the model inconsistent beyond leading order in $1/N$. Clearly, the NSM does not have conventional BPHZ renormalizability order by order in $1/N$, and the schemes that have been proposed to carry out the renormalization are very unconventional at best. However, these schemes have been justified by arguing that it is the nonlinear constraint that is the culprit and one has to abandon the standard renormalization techniques in order to deal with it at the quantum level. This has been emphasized in Ref. 9.

In the next two sections, we will see that these problems, which are blamed on the nonlinear constraint (2.14), go away in the supersymmetric model. We may thus argue that the natural place for the nonlinear σ model is in superspace.

III. THE SUPERSYMMETRIC NONLINEAR σ MODEL (SSNSM)

In $d=2+1$ the SSNSN is defined by the Lagrangian[16]

$$\mathcal{L}=\tfrac{1}{2}\int d^2\theta\,\Phi_J D^2\Phi_j \tag{3.1}$$

with the nonlinear constraint

$$\Phi_j\Phi_j=\frac{N}{g^2}\,, \tag{3.2}$$

where the sum of the flavor index j runs from 1 to N.

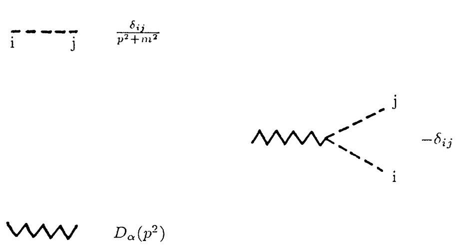

FIG. 3. Feynman rules in the O(N)-symmetric phase of the NSM.

FIG. 5. Next-to-leading-order correction to boson propagator.

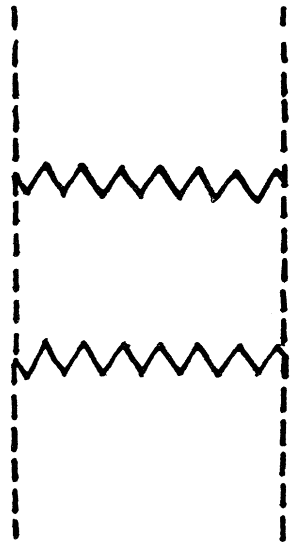

FIG. 6. Divergent A^4 counterterm induced at order $\sim 1/N^2$.

The superfields Φ_j may be expanded out in components,

$$\Phi_j = A_j + \bar{\theta}\psi_j + \tfrac{1}{2}\bar{\theta}\theta F_j \ , \tag{3.3}$$

and the supercovariant derivative is

$$D = \frac{\partial}{\partial\theta} - i\bar{\theta}\not{\partial} \ . \tag{3.4}$$

As usual, in order to express the constraint (3.2) as a δ functional we introduce a Lagrange multiplier superfield Σ:

$$\Sigma = \sigma + \bar{\theta}\xi + \tfrac{1}{2}\bar{\theta}\theta\alpha \ . \tag{3.5}$$

We thus arrive at the manifestly supersymmetric action for the SSNSM:

$$S = \int d^3x\, d^2\theta \left[\tfrac{1}{2}\Phi_j D^2 \Phi_j + \tfrac{1}{2}\Sigma \left[\Phi_j\Phi_j - \frac{N}{g^2} \right] \right] \ . \tag{3.6}$$

In component form, the Lagrangian from (3.6) is

$$\mathcal{L} = -\tfrac{1}{2}A_j\partial^2 A_j + \frac{i}{2}\bar{\psi}_j\not{\partial}\psi_j + \tfrac{1}{2}F_j^2 - \sigma A_j F_j - \tfrac{1}{2}\alpha A_j^2 + \tfrac{1}{2}\sigma\bar{\psi}_j\psi_j + \bar{\xi}\psi_j A_j + \frac{N}{2g^2}\alpha \ . \tag{3.7}$$

We now see that α, ξ, and σ are the respective Lagrange multipliers for the constraints

$$A_j A_j = \frac{N}{g^2} \ , \tag{3.8}$$

$$A_j\psi_j = 0 \ , \tag{3.9}$$

$$A_j F_j = \tfrac{1}{2}\bar{\psi}_j\psi_j \ , \tag{3.10}$$

which are the component version of (3.2). It is now straightforward to verify[10] that α accounts for the ordinary NSM sector, σ accounts for the four-fermion sector, and ξ accounts for the mixed sector of interaction between A and ψ.

Clearly, we may continue our treatment in superspace by using supergraph techniques,[16] but the relationship of the SSNSM to the models of Secs. II and III will not be readily apparent. By staying with the component treatment, we explicitly leave open the possibility that the renormalization is not supersymmetric and also see precisely how SUSY conspires to improve upon the renormalization properties of the models of Sec. II.

In component form, the SSNSM action functional in *Euclidean* space is

$$Z = \int \mathcal{D}A_j \mathcal{D}\psi_j \mathcal{D}\alpha\, \mathcal{D}\sigma\, \mathcal{D}\xi \exp\left[-\int d^3x_E [-\tfrac{1}{2}A_j\partial_E^2 A_j - (i/2)\bar{\psi}_j\not{\partial}_E\psi - (N/2g^2)\alpha + \tfrac{1}{2}\sigma^2 A_j^2 + \tfrac{1}{2}\alpha A_j^2 - \tfrac{1}{2}\sigma\bar{\psi}_j\psi_j - \bar{\xi}\psi_j A_j + J_j A_j - \bar{\eta}_j\psi_j] \right] \ , \tag{3.11}$$

where we have integrated over the fields F_j and set their source terms to zero since they are not true dynamical degrees of freedom. One should note, however, that the effect of F_j is still manifested through the four-field interaction $\tfrac{1}{2}\sigma^2 A_j^2$; thus this four-field interaction is an effective interaction brought about by contracting the nonpropagating F term to a point.

Once again, to generate a $1/N$ expansion we integrate over the fields A_j and ψ_j to obtain an effective action for the fields α, σ, and ξ:

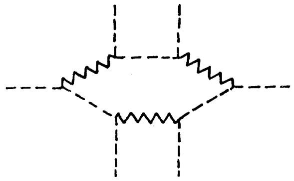

FIG. 7. Divergent A^6 counterterm induced at order $\sim 1/N^3$.

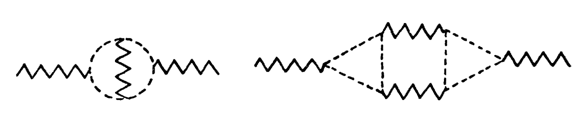

FIG. 8. Divergent next-to-leading-order corrections to the effective propagator for the field α.

$$Z[J_j,\eta_j]=\int \mathcal{D}\alpha\,\mathcal{D}\sigma\,\mathcal{D}\xi\,\exp(-S_{\text{eff}}[\alpha,\sigma,\xi,J_j,\eta_j])\;, \tag{3.12}$$

where

$$S_{\text{eff}}[\alpha,\sigma,\xi,J_j,\eta_j]=\frac{N}{2}\text{tr}\ln\left[-\partial_E^2+\alpha+\sigma^2+\bar{\xi}\frac{1}{i\partial\!\!\!/_E+\sigma}\xi\right]-\frac{N}{2}\text{Tr}\ln(i\partial\!\!\!/_E+\sigma)$$
$$-\frac{N}{2g^2}\int d^3x_E\,\alpha+\tfrac{1}{2}J_j(-\partial_E^2+\alpha+\sigma^2)^{-1}J_j\tfrac{1}{2}\bar{\eta}(i\partial\!\!\!/_E+\sigma)^{-1}\eta+\cdots\;, \tag{3.13}$$

where the ellipsis denotes source cross terms that will always vanish when differentiated and evaluated at the vacuum or at zero source, and hence they are irrelevant to our analysis. From (3.13) and the usual Legendre transformations we see that the leading-order effective action is

$$\Gamma[A_j,\psi_j,\alpha,\sigma,\xi]=\int d^3x_E[-\tfrac{1}{2}A_j(-\partial_E^2+\alpha+\sigma^2)A_j+\tfrac{1}{2}\bar{\psi}_j(i\partial\!\!\!/_E+\sigma)\psi_j$$
$$+(N/2)\text{tr}\ln(-\partial_E^2+\alpha+\sigma^2)-(N/2)\text{Tr}\ln(i\partial\!\!\!/_E+\sigma)]$$
$$+\bar{\xi}\psi_jA_j-\frac{N}{2g^2}\int d^3x_E\,\alpha+\cdots\;. \tag{3.14}$$

Once again, taking advantage of the O(N) symmetry we write the VEV of A_j as in (2.19) so that the effective potential for constant fields becomes

$$\frac{V}{N}=\frac{1}{2}\left[v^2-\frac{1}{g^2}\right]\langle\alpha\rangle+\frac{1}{2}v^2\langle\sigma\rangle^2+\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\ln(k_E^2+\langle\alpha\rangle+\langle\sigma\rangle^2)-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\text{tr}\ln(-k\!\!\!/_E+\langle\sigma\rangle)\;, \tag{3.15}$$

and the stationary conditions are

$$\frac{1}{N}\frac{\partial V}{\partial v}=(\langle\alpha\rangle+\langle\sigma\rangle^2)v=0\;, \tag{3.16}$$

$$\frac{1}{N}\frac{\partial V}{\partial\langle\alpha\rangle}=-\frac{1}{2g^2}+\frac{v^2}{2}+\frac{1}{2}\int\frac{d^3k_E}{(2\pi)^3}\frac{1}{k_E^2+\langle\alpha\rangle+\langle\sigma\rangle^2}=0\;, \tag{3.17}$$

$$\frac{1}{N}\frac{\partial V}{\partial\langle\sigma\rangle}=\langle\sigma\rangle v^2-\int\frac{d^3k_E}{(2\pi)^3}\frac{\langle\alpha\rangle\langle\sigma\rangle}{(k_E^2+\langle\sigma\rangle^2)(k_E^2+\langle\alpha\rangle+\langle\sigma\rangle^2)}=0\;. \tag{3.18}$$

Note that (3.18) is already finite and so it does not require a fine-tuning of the coupling as was the case in the four-fermion model of Sec. II. We see that the model allows two phases corresponding to broken ($v\neq0,\langle\alpha\rangle+\langle\sigma\rangle^2=0$) and unbroken ($v=0,\langle\alpha\rangle+\langle\sigma\rangle^2\neq0$) O($N$) symmetry. It is easy to show that, in both phases, $\langle\alpha\rangle\neq0$ leads to SUSY breaking and negative-norm states[17] and other instabilities, analogously to what happens in the $d=3+1$ model of Ref. 18. So we take $\langle\alpha\rangle=0$ in both phases, and SUSY is preserved.

In the broken phase $(A_1,A_2,\ldots,A_{N-1})$ and $(\psi_1,\psi_2,\ldots,\psi_{N-1})$ are massless. After performing the shift

$$A_N=A_N'+\sqrt{N}v\;, \tag{3.19}$$

with $\langle A_N'\rangle=0$ and expanding (3.13) and (3.14) about

$$(\langle A_1\rangle,\langle A_2\rangle,\ldots,\langle A_N\rangle)=(0,0,\ldots,\sqrt{N}v)\;,$$

and $\langle\alpha\rangle=\langle\sigma\rangle=0$ one may easily obtain the Feynman rules for this phase. In this phase, both A_N and ψ_N are unstable and decay into the remaining bosons and fermions and their effective propagators mix with the effective propagators of α' and ξ, respectively.

In the symmetric phase, all N bosons $(A_1,\ldots,A_N)$ and fermions $(\psi_1,\ldots,\psi_N)$ acquire the same dynamical mass $|M|$ so that SUSY is preserved. This time we perform the shift

$$\sigma=\sigma'+|M|\;, \tag{3.20}$$

with $\langle\sigma'\rangle=0$ and expand about $(\langle A_1\rangle,\langle A_2\rangle,\ldots,\langle A_N\rangle)=0$, $\alpha=0$, and $\sigma=|M|$ to obtain the Feynman rules listed in Fig. 9. To obtain these

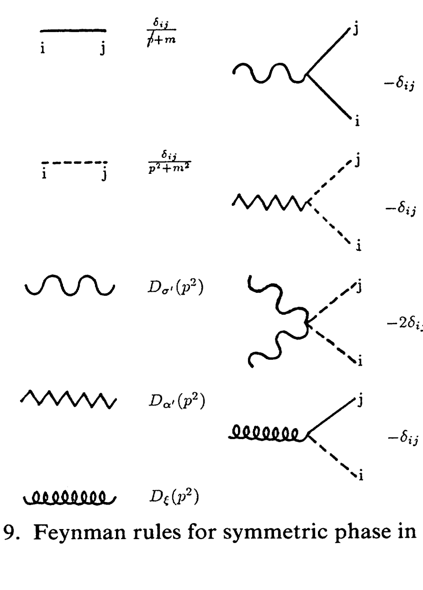

FIG. 9. Feynman rules for symmetric phase in SSNSM.

rules, we have also performed the transformation

$$\alpha'=\alpha+2m\sigma', \quad m\equiv|M| , \tag{3.21}$$

which diagonalizes the effective propagators for α' and σ' and makes the comparison between the NSM and four-fermion model with the SSNSM more transparent. This transformation was first introduced by Davis *et al.*[19] to study the SSNSM in $d=1+1$.

IV. RENORMALIZATION OF THE SSNSM

As in the NSM, we detail the renormalization in the symmetric phase and the renormalizability would automatically carry over to the broken and critical phases as well. We only add that in the latter cases one deals with IR divergences by using the soft mass regularization introduced by Lowenstein and Zimmermann.[20] In the O(N)-symmetric phase, the dynamically generated mass regulates all IR divergences. From (3.11) and (3.21) we obtain the bare Lagrangian in the symmetric phase:

$$\mathcal{L}_0=\tfrac{1}{2}A_j(\partial_E^2-m^2)A_j+\tfrac{1}{2}\bar{\psi}_j(i\not{\partial}_E+m)\psi_j -\tfrac{1}{2}\alpha' A_jA_j-\tfrac{1}{2}\sigma'^2A_jA_j+\tfrac{1}{2}\sigma'\bar{\psi}_j\psi_j+\bar{\xi}\psi_jA_j +\frac{N}{2g^2}\alpha'-\frac{N}{2g^2}2m\sigma' . \tag{4.1}$$

For this model to be renormalizable, the counterterm Lagrangian has to be of the form

$$\mathcal{L}_{\rm CT}=\tfrac{1}{2}C_A A_j\partial_E^2A_j-\tfrac{1}{2}C'_A m^2A_jA_j+\tfrac{1}{2}C_\psi\bar{\psi}_j i\not{\partial}_E\psi_j +\tfrac{1}{2}C'_\psi m\bar{\psi}_j\psi_j-\tfrac{1}{2}C_\alpha\alpha' A_jA_j-\tfrac{1}{2}C_{\sigma^2}\sigma'^2A_jA_j +\tfrac{1}{2}C_\sigma\sigma'\bar{\psi}_j\psi_j+C_\xi\bar{\xi}\psi_jA_j+C'_\alpha\frac{N}{2g^2}\alpha' -C'_\sigma\frac{N}{2g^2}2m\sigma' , \tag{4.2}$$

and we will shortly see that it is so. We introduce the renormalized quantities

$$A=Z_A^{1/2}A_R, \quad \psi=Z_\psi^{1/2}\psi_R, \quad \sigma'=Z_{\sigma'}\sigma'_R , \quad \alpha'=Z_{\alpha'}\alpha'_R, \quad \xi=Z_\xi\xi_R, \quad m=Z_m m_R, \quad g^2=\mu^\epsilon Z_g g_R^2 , \tag{4.3}$$

where $d=3-\epsilon$ (as we will be using dimensional regularization). From (4.1), (4.2), and (4.3) we see that

$$\begin{aligned} &Z_A=1+C_A, \quad Z_\psi=1+C_\psi , \\ &Z_m^2Z_A=1+C'_A, \quad Z_mZ_\psi=1+C'_\psi , \\ &Z_AZ_{\alpha'}=1+C_\alpha, \quad Z_{\sigma'}Z_\psi=1+C_\sigma , \\ &Z_\xi(Z_AZ_\psi)^{1/2}=1+C_\xi, \quad Z_AZ_{\sigma'}^2=1+C_{\sigma^2} , \\ &Z_{\alpha'}Z_g^{-1}=1+C'_\alpha, \quad Z_{\sigma'}Z_g^{-1}=1+C'_\sigma . \end{aligned} \tag{4.4}$$

The leading-order tadpole graphs shown in Fig. 10 have already been accounted for by the gap equation (3.17) and so we need not discuss them here except to say that these tadpole graphs should be considered illegal as subgraphs. Likewise, the diagrams in Fig. 11 are also illegal subgraphs since they are taken into account by the

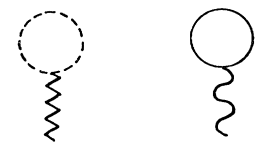

FIG. 10. Leading-order tadpole graphs.

effective propagators of the model:[10]

$$D_{\sigma'}(k^2)=\frac{8\pi}{N}\frac{1}{k^2+4m^2}I(k^2) , \tag{4.5}$$

$$D_{\alpha'}(k^2)=-\frac{8\pi}{N}I(k^2) , \tag{4.6}$$

$$D_\xi(k^2)=\frac{8\pi}{N}\frac{\not{k}-2m}{k^2+4m^2}I(k^2) , \tag{4.7}$$

where

$$I(k^2)=\frac{\sqrt{k^2}}{\arctan(\sqrt{k^2}/2m)} . \tag{4.8}$$

One should note, however, that the effective propagator in the four-fermion sector within the SSNSM, $D_{\sigma'}$, does not rely on the fine-tuning of the coupling constant unlike the effective propagator for the four-fermion model. This is because the linearly divergent terms exactly cancel between the fermion and boson loops because of SUSY; this should not be surprising because, as we saw from (3.18), fine-tuning of the coupling is not necessary in the four-fermion sector. We will shortly see that this improvement holds to higher order as well, and this means that the effective propagators do not need any renormalization; this is a consequence of SUSY. In fact, as we will see later, the corrections to the fine-tuning [see (3.17)] are not affected by logarithmic divergences since these cancel, in contrast to the NSM and four-fermion model.[7-9]

From the ultraviolet scaling of the induced effective propagators as well as the ψ and A propagators, we see that the superficial degree of divergence of a Feynman diagram is given by[10]

$$D=3-E_\sigma-E_\psi-\tfrac{1}{2}E_A-\tfrac{3}{2}E_\xi-2E_\alpha , \tag{4.9}$$

where E_f is the number of external legs of field f. It is now obvious from (4.9) that there is a finite number of primitive divergences as is necessary for the renormalizability of the model.

We now carry out the explicit renormalization of the SSNSM up to next-to-leading order in $1/N$. Note that in

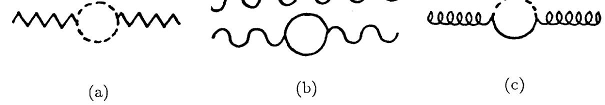

FIG. 11. Diagrams used to define the effective propagators. (a) The effective propagator for α. (b) The effective propagator for σ. (c) The effective propagator for ξ.

$d=2+1$ there is no analog of γ_5; furthermore, since we have no vector couplings, there are no Chern-Simons terms[1,4] and hence no Levi-Civita tensor $\epsilon^{\lambda\mu\nu}$ appears in our calculations. So, from here on we will use dimensional regularization,[12] but one may also use higher-derivative regularization.[21] In fact, at least up to next-to-leading order, the renormalization may even be carried out by a cutoff regularization. All the ensuing calculations will be carried out in Euclidean space.

A. Next-to-leading order corrections to A and ψ propagators

The next-to-leading-order corrections to the boson propagator are shown in Fig. 12. Here and elsewhere, the diagram with a small box denotes the counterterm. Adding the contributions from all of these graphs we get

$$-\frac{8\pi}{N}(k^2+m^2)\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+4m^2)[(l-k)^2+m^2]}I(l^2)\ , \tag{4.10}$$

where k is the external momentum, and, as usual, $d=3-\epsilon$ but we take the limit $\epsilon\to 0$ after we perform the integration. It is important to note that all of the linear and quadratic divergences cancel among these three diagrams and only a logarithmic divergence remains which may be taken care of by a wave-function renormalization. That is, upon expanding $I(l^2)$ in (4.10) and using the Feynman parametrization,[22] we isolate the divergent part to obtain

$$C_A=C'_A=\frac{4}{N\pi^2}\frac{2}{\epsilon}\ . \tag{4.11}$$

Thus, unlike the NSM, an independent A^2 counterterm is not induced and the nonlinear constraint is not broken.

The next-to-leading-order corrections to the fermion propagator are shown in Fig. 13. Once again, adding the contributions from these two graphs we get

$$-\frac{8\pi}{N}(\not{k}-m)\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+4m^2)[(l-k)^2+m^2]}I(l^2)\ , \tag{4.12}$$

and a similar calculation yields

$$C_\psi=C'_\psi=\frac{4}{N\pi^2}\frac{2}{\epsilon}\ . \tag{4.13}$$

The fermions thus undergo the same wave-function renormalization as the bosons and thus far it appears that the renormalization is going to be supersymmetric. We will soon see that not only does the renormalization preserve SUSY but this property is vital for the BPHZ scheme not to spoil the nonlinear constraint.

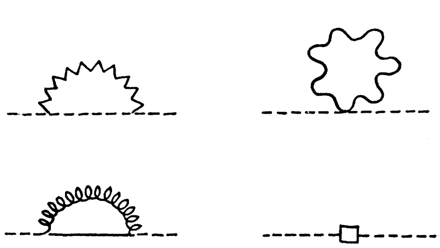

FIG. 12. Order-$(1/N)$ corrections to the boson (A_j) propagator.

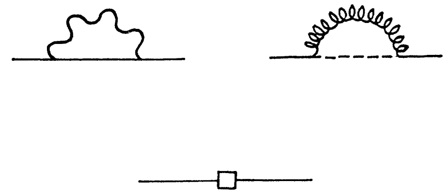

FIG. 13. Order-$(1/N)$ corrections to the fermion (ψ_j) propagator.

B. Corrections to the three- and four-point vertex functions

It is straightforward to verify either by dimensional arguments or by explicit expansion that all divergences for n-point functions with $n\geq 3$ do not depend on external momenta, so we may evaluate the renormalization constants for this class of diagrams by setting external momenta to zero.

The divergent contribution from each of the first two diagrams in Fig. 14 is

$$\frac{3(2^{10})m}{N^2\pi^3}\frac{2}{\epsilon}$$

and from the third one we get

$$-\frac{6(2^{10})m}{N^2\pi^3}\frac{2}{\epsilon}\ .$$

Because their sum is zero, no A^4 counterterm is induced. Likewise, no A^6 counterterm is induced because the divergent contributions from each of the diagrams shown in Fig. 15 exactly cancel. Clearly, such is not the case with the ordinary NSM because without SUSY there are no σ'^2A^2 and $\bar{\xi}\psi A$ interaction terms to bring about these miraculous cancellations.

All the next-to-leading-order corrections to the three-point vertex functions are shown in Figs. 16. The divergent parts of the first two diagrams of Fig. 16(a) are very easy to compute. We get

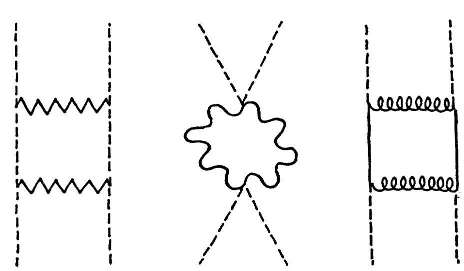

FIG. 14. Infinite parts from each of these four-point functions exactly cancel.

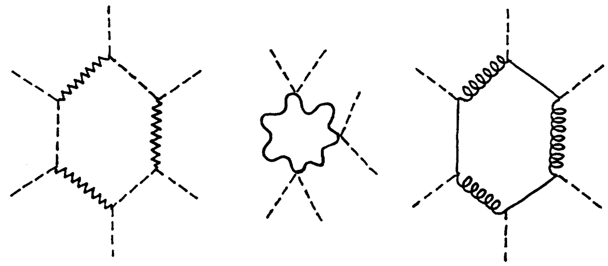

FIG. 15. Infinite parts from each of these six-point functions exactly cancel.

$$\frac{4}{N\pi^2}\frac{2}{\epsilon}$$

for the first diagram and

$$-\frac{64}{N\pi^4}\frac{2}{\epsilon}$$

for the second. For the third diagram, we have

$$-\frac{64\pi}{N}\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+m^2)^2[(l+k)^2+m^2]}\times\frac{1}{k^2+m^2}[I(k^2)]^2\ . \quad (4.14)$$

We first calculate the integral over l, which is the "triangle" subdiagram with three external α' legs:

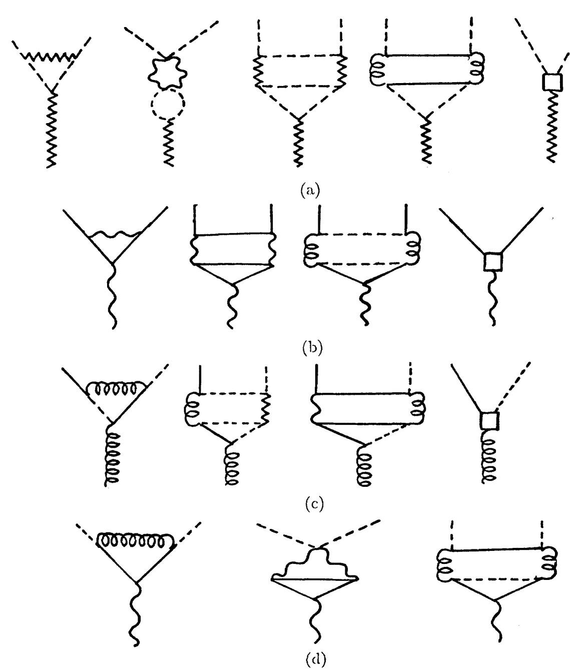

FIG. 16. Corrections to the three-point vertex functions: (a) $\alpha' A^2$, (b) $\sigma'\bar{\psi}\psi$, (c) $\bar{\xi}\psi A$, (d) $\sigma' A^2$.

$$-\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+m^2)^2[(l+k)^2+m^2]}=-\frac{1}{8\pi m}\frac{1}{k^2+4m^2}+O(\epsilon)\ , \quad (4.15)$$

where we have used Feynman parametrization. Putting (4.15) back into (4.14) and expanding $[I(k^2)]^2$, we may isolate the logarithmically divergent part. Using Feynman reparametrization once again, we get

$$-\frac{64}{N\pi^4}\frac{2}{\epsilon}\ .$$

The triangle subdiagram for the fourth graph of Fig. 16(a) is

$$-\int\frac{d^dl}{(2\pi)^d}\frac{\not{l}+\not{k}+m}{(l^2+m^2)^2[(l+k)^2+m^2]}=\frac{1}{16\pi m}\frac{\not{k}-2m}{k^2+4m^2}+O(\epsilon)\ , \quad (4.16)$$

and, by a calculation similar to that for the third graph, we get

$$-\frac{4}{N\pi^2}\left[2-\frac{32}{\pi^2}\right]\frac{2}{\epsilon}\ .$$

Adding all of these contributions together and multiplying by -1 gives us the value for the counterterm:

$$C_\alpha=\frac{4}{N\pi^2}\frac{2}{\epsilon}\ . \quad (4.17)$$

The divergent part of the first graph of Fig. 16(b) gives

$$\frac{4}{N\pi^2}\frac{2}{\epsilon}\ .$$

The triangle subdiagram for the second graph is

$$\int\frac{d^dl}{(2\pi)^d}\text{tr}\frac{(\not{l}+m)^2(\not{l}+\not{k}+m)}{(l^2+m^2)^2[(l+k)^2+m^2]}=-\frac{m}{\pi}\frac{\arctan(\sqrt{k^2}/2m)}{\sqrt{k^2}}+O(\epsilon)\ , \quad (4.18)$$

and one may readily verify that this graph is actually finite. The triangle subdiagram for the third graph is

$$-\int\frac{d^dl}{(2\pi)^d}\frac{(\not{l}+m)^2}{(l^2+m^2)^2[(l+k)^2+m^2]}=-\frac{1}{8\pi}\frac{\not{k}+2m}{k^2+4m^2}-\frac{1}{4\pi}\frac{\arctan(\sqrt{k^2}/2m)}{\sqrt{k^2}}-\frac{1}{\pi}\frac{\not{k}}{k^2}\left[1-\frac{2m}{\sqrt{k^2}}\arctan(\sqrt{k^2}/2m)\right]+O(\epsilon)\ , \quad (4.19)$$

and by a similar calculation as before, we isolate its divergent part, giving the value

$$-\frac{8}{N\pi^2}\frac{2}{\epsilon}\ .$$

Again, summing these divergent contributions and multiplying by -1 we get the value for the counterterm:

$$C_\sigma=\frac{4}{N\pi^2}\frac{2}{\epsilon}\ . \tag{4.20}$$

The divergent part of the first graph of Fig. 16(c) gives

$$\frac{4}{N\pi^2}\frac{2}{\epsilon}\ .$$

The triangle subdiagrams for the second and third graphs are

$$\begin{aligned}-\int\frac{d^dl}{(2\pi)^d}&\frac{(\not{l}+m)}{(l^2+m^2)^2[(l+k)^2+m^2]}\\&=\frac{1}{4\pi}\frac{m}{(k^2+4m^2)}\frac{\not{k}}{k^2}-\frac{1}{8\pi}\frac{1}{(k^2+4m^2)}\\&\quad-\frac{1}{8\pi}\frac{\not{k}}{k^2}\frac{\arctan(\sqrt{k^2}/2m)}{\sqrt{k^2}}+O(\epsilon)\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}-\int\frac{d^dl}{(2\pi)^d}&\frac{(\not{l}+m)(\not{l}+\not{k}+m)}{(l^2+m^2)^2[(l+k)^2+m^2]}\\&=\frac{1}{8\pi}\frac{\arctan(\sqrt{k^2}/2m)}{\sqrt{k^2}}\\&\quad+\frac{1}{8\pi}\frac{\not{k}}{k^2}\left[1-\frac{2m}{\sqrt{k^2}}\arctan(\sqrt{k^2}/2m)\right]+O(\epsilon)\ ,\end{aligned} \tag{4.22}$$

respectively. As before, we find the divergent part of each of the second and third graphs to be

$$-\frac{4}{N\pi^2}\frac{2}{\epsilon}\ .$$

Summing these divergent parts and multiplying by -1 we get

$$C_\xi=\frac{4}{N\pi^2}\frac{2}{\epsilon}\ . \tag{4.23}$$

The first graph of Fig. 16(d) is finite. Using (4.18) and (4.19) to compute the divergent parts of the second and third graphs, respectively, we find that they exactly cancel and thus there is no $\sigma' A^2$ counterterm induced. This is vital for BPHZ renormalizability because there is no term of this form in the bare Lagrangian (4.1). For the same reason, it is also vital that the four-point functions in Fig. 17 are finite, and the reader may quickly verify that they are. Of course, all other n-point functions for $n\geq 3$ not depicted here are also finite.

The only divergent four-point functions of order $1/N$ are shown in Fig. 18. The divergent parts of graphs (a) to (d) are the same as those of the graphs of Fig. 16(a) except for a factor of 2 due to the σ'^2A^2 vertex. The sum of the divergent contributions from these four graphs is

$$-\frac{8\pi}{N\pi^2}\frac{2}{\epsilon}\ .$$

Since graph (e) has a symmetry factor 1 and graph (f) a symmetry factor $\frac{1}{2}$, their sum is

$$\frac{64\pi^2}{N}\int\frac{d^dk}{(2\pi)^d}\left[\frac{\partial}{\partial m}\int\frac{d^dl}{(2\pi)^d}\mathrm{tr}\frac{-(\not{l}+m)^2(\not{l}+\not{k}+m)}{(l^2+m^2)^2[(1+k)^2+m^2]}\right]\frac{1}{(k^2+4m^2)^2}[I(k^2)]^2\ . \tag{4.24}$$

Recognizing the integral over l inside the large parentheses to be (4.18), we isolate the divergent part of (4.24) and it is

$$-\frac{32}{N\pi^2}\frac{2}{\epsilon}\ .$$

For graph (g), which has a symmetry factor 2, we have

$$\begin{aligned}-\frac{2(8\pi)^3}{N}\mathrm{tr}\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}&\frac{(\not{l}+m)^2}{(l^2+m^2)^2[(l+k)^2+m^2]}\\&\times\int\frac{d^dq}{(2\pi)^d}\frac{(\not{q}+m)^2}{(q^2+m^2)^2[(q+k)^2+m^2]}\frac{(\not{k}-2m)^3(\not{k}+m)}{(k^2+4m^2)^3(k^2+m^2)}[I(k^2)]^3\ .\end{aligned} \tag{4.25}$$

The l and q integrals are both given by (4.19). After some tedious algebra and integration, we finally isolate the divergent term

$$\frac{256}{N}\int\frac{d^dk}{(2\pi)^d}\frac{k^6}{(k^2+4m^2)^3(k^2+m^2)\sqrt{k^2}}\ , \tag{4.26}$$

which is evaluated using Feynman parametrization, giving

$$-\frac{64}{N\pi^2}\frac{2}{\epsilon}\ .$$

For graph (h), with symmetry factor 2, we have

$$-\frac{2(8\pi)^2}{N}\mathrm{tr}\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}\frac{(\not{l}+m)^3}{(l^2+m^2)^3[(l-k)^2+m^2]}\frac{(\not{k}-2m)^2(\not{k}+m)}{(k^4+4m^2)^2(k^2+m^2)}\ . \tag{4.27}$$

Now the integral over l is not given by any of the triangle subdiagrams obtained before but, after much tedious algebra and integration, we get

$$\int \frac{d^d l}{(2\pi)^d} \frac{(\not{l}+m)^3}{(l^2+m^2)^3[(l-k)^2+m^2]} = \frac{m}{2\pi} \frac{\not{k}(k^2+2m^2)}{k^2(k^2+4m^2)^2} - \frac{1}{4\pi} \frac{k^2+2m^2}{(k^2+4m^2)^2} - \frac{1}{8\pi} \frac{\not{k}}{k^2} \frac{\arctan(\sqrt{k^2}/2m)}{\sqrt{k^2}} + O(\epsilon) . \tag{4.28}$$

Substituting (4.28) into (4.27), we isolate the divergent piece

$$\frac{64}{N} \int \frac{d^d k}{(2\pi)^d} \frac{k^4}{(k^2+4m^2)^2(k^2+m^2)\sqrt{k^2}} , \tag{4.29}$$

which gives

$$\frac{16}{N\pi^2} \frac{2}{\epsilon} .$$

Similar but much easier calculations give the values for the divergent parts of the other eight diagrams of Fig. 18. The divergent piece for diagram (i) is

$$-\frac{16}{N\pi^2} \frac{2}{\epsilon} ,$$

but diagram (l) is finite. The divergent piece of each of the diagrams (j), (k), (m), (n), and (p) is

$$\frac{32}{N\pi^2} \frac{2}{\epsilon} ,$$

and for diagram (o) it is

$$-\frac{64}{N\pi^2} \frac{2}{\epsilon} .$$

Summing all the infinite contributions from the graphs of Fig. 18 and multiplying by $-\frac{1}{2}$, we get

$$C_{\sigma^2} = \frac{4}{N\pi^2} \frac{2}{\epsilon} . \tag{4.30}$$

C. Corrections to the effective propagators

The corrections to the effective propagators must be finite because terms of the form α'^2, σ'^2, and $\bar{\xi}\xi$ are not in the bare Lagrangian (4.1). Clearly, if such counterterms were induced at higher orders, then the fields α, σ, and ξ would lose their role as Lagrange multipliers, the nonlinear constraint would be violated, and we would have an inconsistent model. Again from the dimensional arguments or direct expansion it is easy to show that the divergent parts of these corrections do not depend on external momenta, so we evaluate the corresponding diagrams at zero external momenta.

The next-to-leading-order corrections to the effective propagator $D_{\alpha'}$ are shown in Fig. 19. Diagram (a) has a symmetry factor $\frac{1}{2}$ and we get

$$-4\pi \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2+m^2)^2[(k+l)^2+m^2]} I(k^2) . \tag{4.31}$$

We first integrate over l and then expand and integrate over k to isolate the divergent piece of this diagram, giving

$$-\frac{1}{2\pi^3 m} \frac{2}{\epsilon} .$$

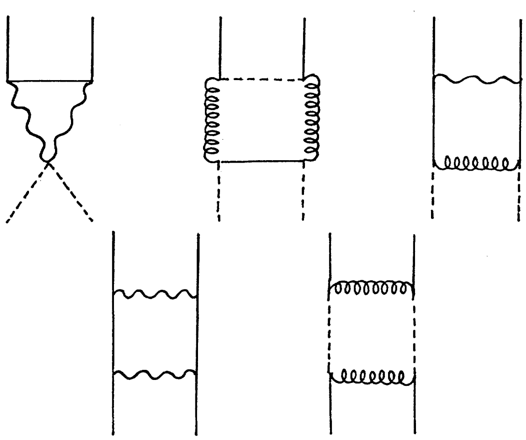

FIG. 17. Finite four-point functions.

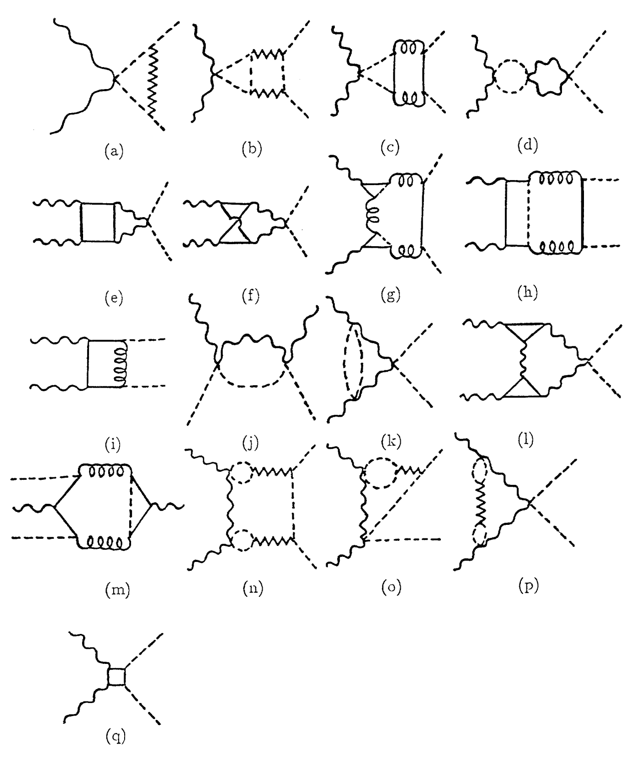

FIG. 18. Corrections to the four-point vertex functions.

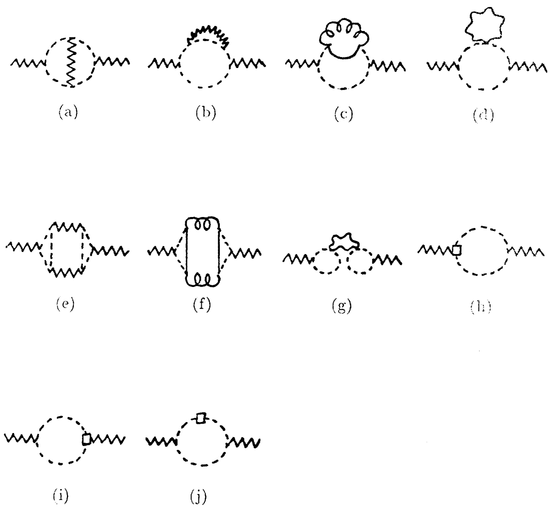

FIG. 19. Corrections to the effective propagator $D_{\alpha'}$.

The diagrams (b), (c), and (d) contain the corrections to the boson propagator whose sum is given by (4.10). The sum of these three diagrams is thus

$$-8\pi\int\frac{d^dl}{(2\pi)^d}\int\frac{d^dk}{(2\pi)^d}\frac{k^2+m^2}{(l^2+m^2)[(l-k)^2+m^2]}\times\frac{1}{(k^2+m^2)^3}I(l^2)\ , \qquad (4.32)$$

and their divergent contribution is

$$-\frac{1}{2\pi^3 m}\frac{2}{\epsilon}\ .$$

The diagrams (e) and (f) each contain two triangle subdiagrams, and their divergent contributions are easy to compute. We get

$$\frac{4}{\pi^5 m}\frac{2}{\epsilon}$$

for (e) and

$$\frac{1}{\pi^3 m}\left[1-\frac{8}{\pi^2}\right]\frac{2}{\epsilon}$$

for (f). The divergent contribution of graph (g) is easier to obtain, giving

$$\frac{4}{\pi^5 m}\frac{2}{\epsilon}\ .$$

The sum of these divergent contributions from the diagrams (a) to (g) is zero. Finally, it is easy to show that the sum of the divergent contributions for the graphs (h), (i), and (j), which contain counterterms for the divergent subdiagrams, exactly cancel. Hence no α'^2 counterterm is induced.

A similar calculation shows that all the divergent parts of the corrections to the σ' propagator sum to zero and there is no induced σ'^2 counterterm. Likewise, there is no induced $\bar{\xi}\xi$ counterterm because all the divergent parts of the corrections to the ξ propagator exactly cancel. Lastly, at next-to-leading order there are α'-σ' mixed propagator corrections, but they also do not give a net divergent contribution.

D. Corrections to the tadpole graphs

The next-to-leading order corrections to the VEV of α' are shown in Fig. 20, and the computation of their divergent parts is straightforward. For diagrams (a), (b), and (c) we get

$$\frac{m}{\pi^3}\left[1-\frac{4}{\pi^2}\right]\frac{2}{\epsilon},\ \frac{m}{\pi^3}\left[-2+\frac{8}{\pi^2}\right]\frac{2}{\epsilon}\ ,$$

and

$$\frac{m}{\pi^3}\left[1-\frac{4}{\pi^2}\right]\frac{2}{\epsilon}\ ,$$

respectively, and their sum is zero. For diagrams (d) and (e) we get

$$\frac{m}{2\pi^3}\frac{2}{\epsilon}\quad\text{and}\quad -\frac{m}{2\pi^3}\frac{2}{\epsilon}\ ,$$

respectively, and their sum is zero. So we get, surprisingly,

$$C'_\alpha=0\ . \qquad (4.33)$$

It is also straightforward to verify that the divergent parts of the diagrams of Fig. 21, for the VEV of σ', also sum to zero and we have

$$C'_\sigma=0\ . \qquad (4.34)$$

Equations (4.33) and (4.34) show that the gap equation remains unaffected by logarithmic divergences. This is certainly not true for the ordinary NSM but it is also not true for the SSNSM in $1+1$ dimensions.[19]

E. Renormalizability

We have computed all counterterms of (4.2) up to next-to-leading order in $1/N$. It is evident from (4.11),

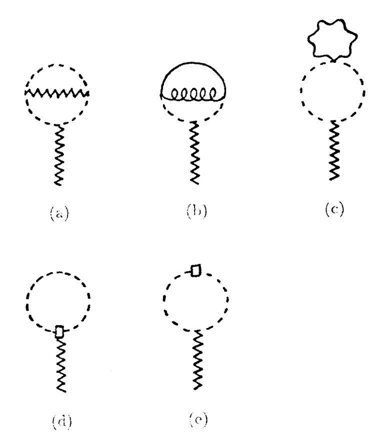

FIG. 20. Corrections to the VEV of α'.

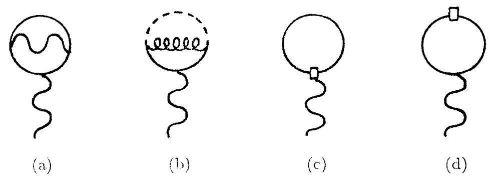

FIG. 21. Corrections to the VEV of σ'.

(4.13), (4.17), (4.20), (4.23), (4.30), (4.33), and (4.34) that a wave-function renormalization will remove all divergences. That is, recalling (4.3) and (4.4), we have

$$Z_A = Z_\psi = 1 + \frac{4}{N\pi^2}\frac{2}{\epsilon} \quad (4.35)$$

and

$$Z_m = Z_{\alpha'} = Z_{\sigma'} = Z_\xi = Z_g^{-1} = 1 \ . \quad (4.36)$$

Significantly, no renormalization is necessary for any of the vertices. *Thus the renormalization is supersymmetric.* As $Z_m = 1$, m does not get renormalized, and as $Z_g^{-1} = 1$, the coupling does not get renormalized and thus the β function is zero.

We have seen that, by virtue of SUSY, the auxiliary fields α', σ', and ξ do not get renormalized and this means that their anomalous dimensions are zero. This in turn establishes that α', σ', and ξ retain their respective "engineering dimensions" of 2, 1, and $\frac{3}{2}$. This is not the case for the auxiliary field in the NSM or four-fermion model.

Assuming that supersymmetry is preserved to all orders, it follows that the anomalous dimensions of the auxiliary fields are zero to all orders in $1/N$. Only A and ψ acquire anomalous dimensions but we can show that these are perturbatively small in the $1/N$ expansion by adapting the arguments of Polchinski[23] to the SSNSM. It then follows that the power counting of (4.9) is valid and supersymmetry forbids any counterterms other than the ones that we obtained up to next-to-leading order. This proves renormalizability to all orders in $1/N$. Details will be presented elsewhere.

V. CONCLUSION

We have explicitly carried out, up to next-to-leading order in $1/N$, the renormalization of an $O(N)$-invariant supersymmetric nonlinear σ model in 2+1 dimensions. We have shown that, because of supersymmetry, there are cancellations of divergences between fermion and boson loops which make it possible to renormalize all Green's functions by the conventional BPHZ method. Moreover, the renormalization itself is supersymmetric and a wave-function renormalization removes all divergences. Taking advantage of the constraints imposed by supersymmetry, it is then straightforward to prove renormalizability to all orders in $1/N$.

Quantum field theories in 2+1 dimensions have provided us with the first clear demonstration that the models which are not renormalizable in weak-coupling expansion may actually be quantitatively sensible when dealt with by nonperturbative methods. Moreover planar theories may exhibit phase transitions due to the breakdown of global symmetries, unlike theories in $d=1+1$. These features make planar theories much more realistic toy models for understanding more complicated features of models in $d=3+1$.

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