Local expressions for one-loop calculations

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We develop local expressions for the contributions of the short-wavelength vacuum modes to the one-loop vacuum energy. These expressions significantly improve the convergence properties of various "brute-force" calculational methods. They also provide a continuous series of approximations that interpolate between the brute-force calculations and the derivative expansion.

I. INTRODUCTION

There has been much effort to develop techniques for performing one-loop calculations in static, nonuniform background fields. This is because such techniques are needed to perform relativistic bound-state calculations in the semiclassical limit.[1] At the one-loop level, the vacuum consists of occupied negative-energy fermion states and the zero-point fluctuations of the meson field. The presence of a background field, for instance, that generated by valence particles, perturbs the vacuum modes and generates a nonzero energy density in the vacuum. The induced energy in the vacuum ($E_{\rm vac}$) is known as the Casimir energy; its calculation is the subject of this paper.[2]

Schematically, calculating the Casimir energy means performing a sum of the form

$$E_{\rm vac} \sim \lim_{\Lambda\to\infty} \sum_{\epsilon_i^0<\Lambda} (\epsilon_i - \epsilon_i^0) + E_{\rm ct}(\Lambda) , \tag{1.1}$$

where ϵ_i are the energies of the vacuum modes in the presence of the background field and ϵ_i^0 are the energies in the absence of the field. This bare sum, which is equivalent to the sum of one-loop diagrams with external legs corresponding to the external field, is divergent due to the associated divergences in the one-loop diagrams. The finite physical energy is determined by combining the bare sum with the energy contribution of the one-loop counterterms $E_{\rm ct}(\Lambda)$.

Many techniques have been developed to evaluate the induced energy density. One is to use a derivative expansion of the effective action.[3] When this series converges it provides a rapid means of calculating the vacuum energy. When it diverges, many have resorted to various "brute-force" computational techniques that involve calculating explicitly either Eq. (1.1) or some equivalent equation.[4–8] Typically, a rather large value of Λ is required to achieve convergence. At this value of Λ, the energy sum and the counterterm can be individually much larger than the final answer, meaning that high numerical precision is required.

In this paper, we develop techniques that greatly accelerate the convergence of the brute-force calculations. The basic idea is to use various approximations, which we generically call WKB approximations,[9] to describe the short-wavelength vacuum modes with energy above some cutoff $\tilde{\Lambda}$. This results in the decomposition

$$E_{\rm vac} \sim \lim_{\tilde{\Lambda}\to\infty} \sum_{\epsilon_i^0<\tilde{\Lambda}} (\epsilon_i - \epsilon_i^0) + E_{\rm WKB}(\tilde{\Lambda}) , \tag{1.2}$$

where

$$E_{\rm WKB}(\tilde{\Lambda}) \sim \lim_{\Lambda\to\infty} \sum_{\tilde{\Lambda}<\epsilon_i^0<\Lambda} (\epsilon_i^{\rm WKB} - \epsilon_i^0) + E_{\rm ct}(\Lambda) . \tag{1.3}$$

A comparison of Eqs. (1.2) and (1.1) shows that we have effectively introduced a WKB-improved counterterm. As a finite part of this improved counterterm contains much of the finite part of the final answer, the WKB-improved sum converges at a significantly lower energy than does the "brute-force" sum. This reduces the computational burden required and provides us with an analytical expression for the contribution of the high-energy states to the vacuum energy.

We apply this idea to two types of brute-force calculations. The first uses the standard mode sum given in Eq. (1.1). After expressing this sum as an integral over the phase shifts of the continuum states, we use the WKB approximation to these phase shifts, thereby generating a local approximation for the short-wavelength modes. The second type, which is known as the Green's-function method,[7] uses an expression obtained by performing the energy integration of the one-loop effective action along the imaginary energy axis. In this case, we develop two different techniques for generating local approximations. The first is based on the WKB approximation to the Green's function for the short-wavelength modes; the second uses a derivative expansion of this same Green's function.

Our presentation is organized as follows. Section II reviews the one-loop formalism and the various computational techniques. Section III applies the WKB approximation to the mode sum (first type) calculation and Sec.

IV applies the WKB approximation to the Green's-function (second type) calculation. In Sec. V, we develop the derivative expansion approach for the Green's-function calculation. In all these sections, we illustrate our methods for (1+1)-dimensional boson-loop calculations and indicate how they work for other types of one-loop calculations. Section VI summarizes the paper.

II. FORMALISM

In this section, we review the one-loop formalism and various computational techniques. In particular, we will illustrate our methods by concentrating on a self-interacting scalar field in 1+1 dimensions as described by the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi-V(\phi)+\mathcal{L}_{\rm ct}(\phi)\ , \tag{2.1}$$

where V is the classical potential and $\mathcal{L}_{\rm ct}(\phi)$ represents the counterterms. We assume that the potential V has a minimum at $\phi=\phi_v$, which is chosen as the classical vacuum state for the theory, and define $M^2=V''(\phi_v)$. The quantum theory is renormalized so as to maintain the vacuum expectation value of ϕ to ϕ_v. Our primary interest is to calculate the one-loop effective action for spatially varying fields $\phi_{\rm cl}(x)$ that have the property $V''(\phi_{\rm cl}(\pm\infty))=M^2$ so that the background field causes a finite perturbation to the total energy.

The one-loop effective action for a background field $\phi_{\rm cl}$ is given by[10]

$$\Gamma_{\rm eff}(\phi_{\rm cl})=\int dx\,dt\left[\frac{1}{2}\partial^{\mu}\phi_{\rm cl}\partial_{\mu}\phi_{\rm cl}-V(\phi_{\rm cl})+V(\phi_v)+\frac{i}{2}\left\langle x,t\left|\ln[-\partial^2-V''(\phi_{\rm cl})]-\ln[-\partial^2-V''(\phi_v)]\right|x,t\right\rangle+\mathcal{L}_{\rm ct}(\phi_{\rm cl})\right]\ . \tag{2.2}$$

where the logarithms arise from the boson loops and $\mathcal{L}_{\rm ct}$ represents all the one-loop counterterms of the theory. In 1+1 dimensions the only divergent one-loop graphs are those with one vertex. A specific form of the counterterm is then

$$\mathcal{L}_{\rm ct}(\phi)=\frac{1}{2}\left\langle x,t\left|\frac{i}{-\partial^2-V''(\phi_v)+i\epsilon}\right|x,t\right\rangle[V''(\phi)-V''(\phi_v)]\ . \tag{2.3}$$

In this expression, we have chosen the finite part of the one-loop diagram with one vertex to vanish at zero external momentum.

In this paper we will study only the effective action for time-independent fields, in which case the effective action can be written as

$$\Gamma_{\rm eff}(\phi_{\rm cl})=-E(\phi_{\rm cl})\int dt\ . \tag{2.4}$$

It is convenient to isolate the vacuum contribution to the energy by writing

$$E(\phi_{\rm cl})=E_{\rm cl}(\phi_{\rm cl})+E_{\rm vac}(\phi_{\rm cl})\ , \tag{2.5}$$

where

$$E_{\rm cl}(\phi_{\rm cl})=\int_{-\infty}^{\infty}dx\,[\tfrac{1}{2}(\nabla\phi_{\rm cl})^2+V(\phi_{\rm cl})-V(\phi_v)] \tag{2.6}$$

is the classical energy of the background field. After some standard manipulations of Eq. (2.2) the vacuum energy can be written as[7,8]

$$E_{\rm vac}(\phi_{\rm cl})=i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2\int_{-\infty}^{\infty}dx\left\langle x\left|D(\omega,\phi_{\rm cl})-D(\omega,\phi_v)\right|x\right\rangle-\int_{-\infty}^{\infty}dx\,\mathcal{L}_{\rm ct}(\phi_{\rm cl})\ , \tag{2.7}$$

where

$$D(\omega,\phi(x))=\frac{1}{\omega^2+\nabla^2-V''(\phi)+i\epsilon}\ . \tag{2.8}$$

Note that the counterterm can be written in terms of D as

$$\int_{-\infty}^{\infty}dx\,\mathcal{L}_{\rm ct}(\phi_{\rm cl})=i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2\int_{-\infty}^{\infty}dx\left\langle x\left|D^2(\omega,\phi_v)[V''(\phi_{\rm cl})-V''(\phi_v)]\right|x\right\rangle\ . \tag{2.9}$$

All of the above expressions are purely formal until a means of regularization is specified.

The mode sum expression for the boson-loop energy is derived by placing the system in a large box, thereby discretizing the spectrum, and expressing the spatial traces of D in terms of a sum over a complete set of eigenstates of the spatial part of the inverse propagator:

$$E_{\rm vac}(\phi_{\rm cl})=i\int\frac{d\omega}{2\pi}\sum_n\left[\frac{\omega^2}{\omega^2-\omega_n^2+i\epsilon}-\frac{\omega^2}{\omega^2-\omega_n^{02}+i\epsilon}\right]-\int dx\,\mathcal{L}_{\rm ct}(\phi_{\rm cl})$$

$$=\frac{1}{2}\sum_{\arg(\omega_n)<\pi}(\omega_n-\omega_n^0)-\int dx\,\mathcal{L}_{\rm ct}(\phi_{\rm cl})\ . \tag{2.10}$$

Here ω_i are the square roots of the eigenvalues of the spatial part of the mean-field and vacuum propagators,

$$[-\nabla^2+V''(\phi_v)]\eta_i^0(x)=\omega_i^{02}\eta_i^0(x)\ , \tag{2.11a}$$

$$[-\nabla^2+V''(\phi_{\rm cl})]\eta_i(x)=\omega_i^2\eta_i(x)\ , \tag{2.11b}$$

and the energy integration has been performed by closing the contour and using the residue theorem. The taking of the box size to infinity is implicit.

Physically, this expression shows that, in the one-loop approximation, the energy of the boson vacuum is the sum of the zero-mode fluctuations of the boson fields. The presence of an external source that moves the expectation value of the field away from ϕ_v disturbs these oscillations and thereby alters the energy of the vacuum. There may be eigenstates with $\omega_i^2<0$, which contribute an imaginary amount to the energy and result in a complex total one-boson-loop energy. In such cases, the one-loop approximation has broken down.

This expression for the vacuum energy is purely formal since it involves two separately infinite terms: the mode sum and the counterterm. To regulate these expressions, we truncate the mode sum at some value $\omega_i^0=\Lambda$ and perform the counterterm integrations with the same cutoff to obtain

$$E_{\rm vac}(\phi_{\rm cl})=\lim_{\Lambda\to\infty}\tfrac{1}{2}\sum_{\omega_i^0<\Lambda}(\omega_i-\omega_i^0)+E_{\rm ct}(\Lambda,\phi_{\rm cl})\ , \tag{2.12}$$

where [see Eq. (2.3)]

$$E_{\rm ct}(\Lambda,\phi_{\rm cl})=-\tfrac{1}{2}\int_{-\Lambda_M}^{\Lambda_M}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{dk^0}{2\pi}\frac{i}{k^{02}-k^2-V''(\phi_v)+i\epsilon}[V''(\phi_{\rm cl})-V''(\phi_v)]$$

$$=\frac{1}{4\pi}\ln\left[\frac{\Lambda+\sqrt{\Lambda^2-V''(\phi_v)}}{\sqrt{V''(\phi_v)}}\right][V''(\phi_{\rm cl})-V''(\phi_v)] \tag{2.13}$$

and $\Lambda_M^2=\Lambda^2-M^2$.

In practice, calculations with the mode-sum formula are performed by placing the system in a large box, numerically solving for the lowest n eigenstates of Eq. (2.11b), and explicitly performing the eigenvalue sum in Eq. (2.12). Increasing eigenvalues are calculated until the mode sum converges. As the convergence can be very slow, various extrapolation techniques are generally used.[4,5] We will refer to this method as the brute-force mode-sum method.

An alternative to the mode-sum method is the Green's-function method, derived by performing the energy integration in Eq. (2.7) along a different contour.[7,8] As long as the boson-loop energy is well defined (in the sense that all the $\omega_n^2>0$), the contour of the energy integral in Eq. (2.7) can be deformed to run along the imaginary energy axis. This is equivalent to the replacement $\omega\to i\Omega$ in the integrand and allows the vacuum energy to be written as

$$E_{\rm vac}(\phi_{\rm cl})=\frac{1}{2\pi}\lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda}d\Omega\,\Omega^2\int_{-\infty}^{\infty}dx\{\langle x|D(i\Omega,\phi_{\rm cl})|x\rangle-\langle x|D(i\Omega,\phi_v)|x\rangle$$

$$-\langle x|D^2(i\Omega,\phi_v)|x\rangle[V''(\phi_{\rm cl})-V''(\phi_v)]\}\ , \tag{2.14}$$

where we have explicitly included the counterterm in the energy integral and the integrals are evaluated directly, not by using the residue theorem.

The matrix elements of $D(i\Omega,\phi_{\rm cl})$ in coordinate space are easily evaluated in terms of the solutions of

$$[\nabla^2-\Omega^2-V''(\phi_{\rm cl})]\eta(x)=0\ . \tag{2.15}$$

These solutions diverge exponentially at $x=\pm\infty$. If the solution that is regular at $x=-\infty$ is $u_\Omega(x)$ and the solution that is regular at $x=+\infty$ is $u_\Omega(x)$, then

$$\langle x'|D(i\Omega,\phi_{\rm cl})|x\rangle=[\theta(x'-x)u_\Omega(x)v_\Omega(x')+\theta(x-x')u_\Omega(x)v_\Omega(x')]/W_\Omega\ , \tag{2.16}$$

where W_Ω is the x-independent Wronskian:

$$W_\Omega=u_\Omega(x)v_\Omega'(x)-v_\Omega(x)u_\Omega'(x)\ . \tag{2.17}$$

The matrix elements of $D(i\Omega,\phi_v)$ can be evaluated analytically using Eq. (2.16) since the solutions of Eq. (2.15) are pure

exponentials when $\phi_{\rm cl}(x)=\phi_v$, yielding

$$E_{\rm vac}(\phi_{\rm cl})=\frac{1}{2\pi}\lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda}d\Omega\,\Omega^2\int_{-\infty}^{\infty}dx\left[\frac{u_\Omega(x)v_\Omega(x)}{W_\Omega}+\frac{1}{2[\Omega^2+V''(\phi_v)]^{1/2}}-\frac{1}{4[\Omega^2+V''(\phi_v)]^{3/2}}[V''(\phi_{\rm cl})-V''(\phi_v)]\right]. \tag{2.18}$$

An advantage of this expression is that it is simply an integral over a smooth function, as opposed to Eq. (2.12), which is a sum over discrete eigenvalues.

Green's-function method calculations are performed by determining the appropriate solutions of Eq. (2.15) on a sufficiently fine grid of energies, up to some maximum energy, that the integral in Eq. (2.18) can be evaluated accurately. The maximum energy is then increased until the boson-loop energy converges. As with the mode sum, the convergence is usually slow enough that extrapolation techniques are required.[7,8] We will refer to this computational method as the brute-force Green's-function method.

The preceding results are all exact expressions for the boson-loop energy for arbitrary fields. In general, they must be evaluated numerically, but in some cases an analytic solution is possible. For uniform fields, $\phi_{\rm cl}(x)=\phi$, the boson-loop energy density is easily evaluated to be the effective potential:

$$V_{\rm eff}(\phi)=\frac{1}{8\pi}\left[V''(\phi)-V''(\phi_v)-V''(\phi)\ln\left[\frac{V''(\phi)}{V''(\phi_v)}\right]\right]. \tag{2.19}$$

Observe that if $V''(\phi)<0$ then the effective potential is complex and the one-loop approximation has broken down.

For finite systems, if the field is sufficiently smooth, then the energy can be evaluated adequately using the local density approximation

$$E_{\rm vac}^{\rm LDA}(\phi_{\rm cl})\simeq\int_{-\infty}^{\infty}dx\;V_{\rm eff}(\phi_{\rm cl}(x))\;. \tag{2.20}$$

Improved approximations can be obtained through the derivative expansion,[3] which yields

$$E_{\rm vac}(\phi_{\rm cl})=E_{\rm vac}^{\rm LDA}(\phi_{\rm cl})+\tfrac{1}{96}\int_{-\infty}^{\infty}dx\left[\frac{1}{V''(\phi_{\rm cl})}\frac{dV''(\phi_{\rm cl})}{dx}\right]^2+O((d/dx)^4) \tag{2.21}$$

for the Lagrangian density (2.1). When this expansion converges, it provides a convenient local expression for the vacuum energy. The convergence of this series has been studied by several authors[8] and although a simple convergence criterion does not exist, this expansion will typically converge if

$$\frac{1}{V''(\phi_{\rm cl})^{3/2}}\frac{dV''(\phi_{\rm cl}(x))}{dx}\ll 1\;. \tag{2.22}$$

Satisfying this criterion requires that the spatial variations of the background field be small compared to the local mass scale, $\sqrt{V''(\phi_{\rm cl})}$. These derivative expansion expressions are especially useful when many evaluations of the loop energy are required, as in self-consistent calculations, where the background field is adjusted to minimize the total energy of the system.[11,12]

III. LOCAL EXPRESSIONS FOR MODE-SUM CALCULATIONS

In this section, we derive our method for improving the mode-sum calculations. The method is based on the realization that the higher-energy vacuum modes are less perturbed by the background field than are the lower-energy vacuum modes. This allows us to develop a local formula for the vacuum energy contribution of the high-energy vacuum modes, so that only the low-energy modes need be treated explicitly.

Our method is derived by expressing the vacuum energy in terms of the phase shifts of the continuum states. The vacuum is a uniform state corresponding to $\phi=\phi_v$ where $V'(\phi_v)=0$ and $V''(\phi_v)=M^2>0$. We are interested in calculating the energy of a finite perturbation, $\phi_{\rm cl}(x)$, where $V''(\phi_{\rm cl}(\pm\infty))=M^2$. In the vacuum, the energy spectrum is given by Eq. (2.11a) and consists of a continuum of states with energy greater than M. The eigenstates are the plane waves $\sin(kx)$ and $\cos(kx)$ where the energy ω and momentum k are related via $\omega^2=M^2+k^2$. In the presence of $\phi_{\rm cl}(x)$, the energy spectrum is given by Eq. (2.11b) and consists of some discrete states with energy less than M and continuum states with energy greater than M. Although these continuum energy eigenstates are no longer momentum eigenstates (as are the vacuum states), it is nonetheless convenient to assign a momentum to these states via the energy-momentum relation $\omega^2=M^2+k^2$.

The difference in structure between the continuum states of Eqs. (2.11a) and (2.11b) is reflected in the difference between the density of states in the two systems. This difference is finite and can be calculated in terms of the relative phase shift $\delta(\omega)$ between respective continuum solutions. In all cases of interest the potential $V''(\phi_{\rm cl})$ is symmetric in x so that solutions of Eqs. (2.11a) and (2.11b) can be classified by their parities. If $\eta_{\rm even}(\omega,x)$ and $\eta_{\rm odd}(\omega,x)$ are the solutions of even and odd parity with energy ω, then their phase shifts are defined from the asymptotic behavior as $x\to\infty$:

$$\eta_{\rm even}(\omega,x)\to\cos[kx+\tfrac{1}{2}\delta_{\rm even}(\omega)]\;,\qquad \eta_{\rm odd}(\omega,x)\to\sin[kx+\tfrac{1}{2}\delta_{\rm odd}(\omega)]\;. \tag{3.1}$$

In general the antisymmetric and symmetric phase shifts at a given energy will be different, so we define $\delta=(\delta_{\rm even}+\delta_{\rm odd})/2$. In terms of these phase shifts, Dashen, Hasslacher, and Neveu have shown that the vacuum energy can be written as[1]

$$E_{\rm vac}(\phi_{\rm cl})=\lim_{\Lambda\to\infty}[E_{\rm MS}(\Lambda,\phi_{\rm cl})+E_{\rm ct}(\Lambda,\phi_{\rm cl})]\ , \tag{3.2}$$

where

$$E_{\rm MS}(\Lambda,\phi_{\rm cl})=\frac{1}{2}\sum_{\omega_i<M}(\omega_i-M)-\frac{1}{2\pi}\int_M^\Lambda\delta(\omega)d\omega \tag{3.3}$$

is the contribution of the bare mode sum and $E_{\rm ct}(\Lambda,\phi_{\rm cl})$ is the counterterm contribution defined in Eq. (2.13).

Equation (3.3) involves an explicit sum over the energies of the bound states of Eq. (2.11b) and an integral over the phase shift of the continuum states. It is convenient to define $\delta_{\rm ct}(\omega)$ such that

$$E_{\rm ct}(\Lambda,\phi_{\rm cl})=\frac{1}{2\pi}\int_M^\Lambda\delta_{\rm ct}(\omega)d\omega\ , \tag{3.4}$$

in terms of which the vacuum energy is

$$E_{\rm vac}(\phi_{\rm cl})=\frac{1}{2}\sum_{\omega_i<M}(\omega_i-M)-\lim_{\Lambda\to\infty}\left[\frac{1}{2\pi}\int_M^\Lambda[\delta(\omega)-\delta_{\rm ct}(\omega)]d\omega\right]\ . \tag{3.5}$$

For the counterterm given by Eq. (2.13),

$$\delta_{\rm ct}(\omega)=-\frac{1}{2\sqrt{\omega^2-V''(\phi_v)}}\int_{-\infty}^{\infty}dx[V''(\phi_{\rm cl})-V''(\phi_v)]\ . \tag{3.6}$$

Equation (3.5) is an alternative formulation of the mode-sum technique that can be more efficient to use than the mode-sum formula, Eq. (2.12). Here one need calculate explicitly only the bound states and the phase shift at enough energies to enable an accurate evaluation of the phase-shift integral. Again, as in all brute-force calculations, the convergence of the answer with respect to increasing the upper limit of integration can be sufficiently slow that extrapolation procedures must be used.[6] We will refer to this technique as the brute-force phase-shift method.

We now introduce our method for improving the brute-force phase-shift calculations. For high-energies, the solutions to Eq. (2.11b) can be approximated by the WKB approximation[13]

$$\eta^{\rm WKB}(x)=\frac{1}{\sqrt{k(\omega,k)}}\exp\left[i\int_0^x k(\omega,z)dz\right]\ , \tag{3.7}$$

where

$$k(\omega,x)=\sqrt{\omega^2-V''(\phi_{\rm cl}(x))} \tag{3.8}$$

is the local wave number. These solutions are expected to be accurate if

$$\frac{1}{k^2}\frac{dk}{dx}\sim\frac{1}{\omega^3}\frac{dV''(\phi_{\rm cl}(x))}{dx}\ll 1\ . \tag{3.9}$$

A comparison of this criterion with that of the derivative expansion, Eq. (2.22), shows that here the variation in the fields must be small compared to a global energy scale, as opposed to the local mass scale.

The phase shift obtained from these solutions is

$$\delta^{\rm WKB}(\omega)=\int_{-\infty}^{\infty}[k(\omega,z)-k(\omega,\infty)]dz\ . \tag{3.10}$$

Using the exact phase shift for energies below $\tilde\Lambda$ and the WKB phase shifts for higher energies allows the vacuum energy to be written as

$$E_{\rm vac}(\phi_{\rm cl})=\lim_{\tilde\Lambda\to\infty}[E_{\rm MS}(\tilde\Lambda,\phi_{\rm cl})+E_{\rm MWKB}(\tilde\Lambda,\phi_{\rm cl})]\ , \tag{3.11}$$

where

$$E_{\rm MWKB}(\tilde\Lambda,\phi_{\rm cl})=\lim_{\Lambda\to\infty}\left[-\frac{1}{2\pi}\int_{\tilde\Lambda}^\Lambda\delta^{\rm WKB}(\omega)d\omega+E_{\rm ct}(\Lambda)\right]\ . \tag{3.12}$$

The energy integral of the WKB phase shift is

$$\int_{\tilde\Lambda}^\Lambda\delta^{\rm WKB}(\omega)=\int_{-\infty}^{\infty}dx\frac{1}{2}\left[\Lambda\sqrt{\Lambda^2-V''(\phi_{\rm cl})}-\Lambda\sqrt{\Lambda^2-V''(\phi_v)}-V''(\phi_{\rm cl})\ln\left[\frac{\Lambda+\sqrt{\Lambda^2-V''(\phi_{\rm cl})}}{\tilde\Lambda+\sqrt{\tilde\Lambda^2-V''(\phi_{\rm cl})}}\right]\right.$$
$$\left.+V''(\phi_v)\ln\left[\frac{\Lambda+\sqrt{\Lambda^2-V''(\phi_v)}}{\tilde\Lambda+\sqrt{\tilde\Lambda^2-V''(\phi_v)}}\right]-\tilde\Lambda\sqrt{\tilde\Lambda^2-V''(\phi_{\rm cl})}+\tilde\Lambda\sqrt{\tilde\Lambda^2-V''(\phi_v)}\right]\ . \tag{3.13}$$

Note that the logarithmic divergence in Λ is identical to that of the counterterm. Combining this phase-shift expression with the counterterm and taking the limit $\Lambda\to\infty$ gives

$$E_{\rm MWKB}(\tilde\Lambda,\phi_{\rm cl})=\frac{1}{4\pi}\int_{-\infty}^{\infty}dx\left[\frac{1}{2}[V''(\phi_{\rm cl})-V''(\phi_v)]+\tilde\Lambda\sqrt{\tilde\Lambda^2-V''(\phi_{\rm cl})}-\tilde\Lambda\sqrt{\tilde\Lambda^2-V''(\phi_v)}\right.$$
$$\left.+V''(\phi_{\rm cl})\ln\left[\frac{\sqrt{V''(\phi_v)}}{\Lambda+\sqrt{\tilde\Lambda^2-V''(\phi_{\rm cl})}}\right]-V''(\phi_v)\ln\left[\frac{\sqrt{V''(\phi_v)}}{\tilde\Lambda+\sqrt{\tilde\Lambda^2-V''(\phi_v)}}\right]\right]\ . \tag{3.14}$$

Equation (3.11) is of the same form as Eq. (3.2), implying that we have calculated a "WKB-improved" counterterm. For large $\tilde{\Lambda}$, $E_{\rm MWKB}(\tilde{\Lambda},\phi_{\rm cl})\to E_{\rm ct}(\tilde{\Lambda},\phi_{\rm cl})$, while, for small $\tilde{\Lambda}$, these energies differ; the bare counterterm is built out of plane waves, while the WKB term is built out of distorted plane waves. For the WKB term to be useful, it must allow calculations to converge more rapidly.

We now test this method for several boson-loop calculations in 1+1 dimensions. To illustrate our method, we choose the standard kink potential

$$V(\phi)=\tfrac{1}{4}\phi^4-\tfrac{1}{2}\phi^2 , \tag{3.15}$$

which is minimized at $\phi=\pm 1$. We choose $\phi_v=1$, implying $M^2=V''(\phi_v)=2$.

First consider the background field associated with the kink soliton

$$\phi_{\rm cl}(x)=\tanh(x/\sqrt{2}) , \tag{3.16}$$

which is a static solution of the classical equation of motion. For this background field, the boson-loop energy can be evaluated analytically.[1] Equation (2.11b) is found to have two bound states at $\omega^2=0$ and $\frac{3}{2}$ and a phase shift of

$$\delta_{\rm exact}(\omega)=2\pi-2\arctan[\sqrt{2(\omega^2-2)}] -2\arctan\left[\left[\frac{\omega^2-2}{2}\right]^{1/2}\right] , \tag{3.17}$$

resulting in a vacuum energy of

$$E_{\rm vac}^{\rm kink}=\left[\frac{1}{2\sqrt{6}}-\frac{3}{\pi\sqrt{2}}\right]=-0.471\ldots . \tag{3.18}$$

The bound states contribute -0.802 to the energy, and the continuum states and counterterm contribute 0.331 to this energy.

Figure 1 shows the WKB phase shift calculated using Eq. (3.10) plotted along with the exact phase shift. At $\omega=M$ the exact phase shift is 2π, corresponding to the existence of two bound states in this reflectionless potential, while the WKB phase shift is about 25% larger. The two phase shifts rapidly approach each other with increasing ω and by the time $\omega\sim 1.5M$ the two are nearly equal. The dotted line shows $-\delta_{\rm ct}$ plotted vs ω. The difference between this phase shift and δ corresponds to the finite contributions to the vacuum energy. The WKB phase shift converges to the exact phase shift much faster than $\delta_{\rm ct}$, since the WKB phase shift contains a significant amount of information about the finite part of the vacuum energy. Using the WKB counterterm should improve the convergence relative to using the bare counterterm.

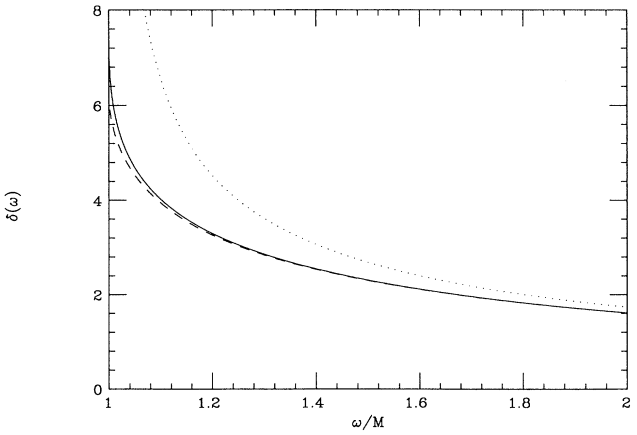

FIG. 1. Phase shifts as a function of ω for the kink soliton. The solid line is $\delta_{\rm exact}$, the dashed line $\delta_{\rm MWKB}$, and the dotted line is $-\delta_{\rm ct}$.

We now calculate $E_{\rm vac}(\phi_{\rm cl})$ using both the brute-force method, Eq. (3.2), and the WKB improved method, Eq. (3.11). This allows us to compare the convergence rates of the two methods. Table I shows both $E_{\rm MS}(\tilde{\Lambda},\phi_{\rm cl})+E_{\rm ct}(\tilde{\Lambda},\phi_{\rm cl})$ and $E_{\rm MS}(\tilde{\Lambda},\phi_{\rm cl})+E_{\rm MWKB}(\tilde{\Lambda},\phi_{\rm cl})$ for various values of $\tilde{\Lambda}$. The WKB-improved calculation converges more rapidly. In fact, using the WKB approximation for the entire continuum $[E_{\rm MS}(M,\phi_{\rm cl})+E_{\rm MWKB}(M,\phi_{\rm cl})]$ gives an answer already accurate to 2%. Typically the brute-force method requires an order of magnitude larger cutoff to achieve the same accuracy as the WKB-improved method. This improvement in convergence is typical for any background field.

Now consider a wide class of background fields parametrized as

$$\phi_{\rm cl}(x)=1-\frac{\phi_b}{1+\exp[(x^2-R^2)/T^2]} . \tag{3.19}$$

This functional form allows an arbitrary adjustment of the depth and surface shape of the background field. In particular, we can vary parameters from the regime where the local density approximation is valid to where the boson-loop approximation itself breaks down. Table II shows the results of boson-loop calculations for fields of varying depth at fixed R and T. We tabulate the exact one-boson-loop energy, calculated as above and with two approximate methods. The first uses the local density approximation defined by Eq. (2.20), which involves integration of a local formula in the fields over the spatial extent of the soliton. The second calculates the bound-state energies exactly, but uses the WKB formula, Eq. (3.11),

TABLE I. Comparison of the brute-force calculation $[E_{\rm MS}(\tilde{\Lambda},\phi_{\rm cl})+E_{\rm ct}(\tilde{\Lambda},\phi_{\rm cl})]$ with the WKB-improved calculation $[E_{\rm MS}(\tilde{\Lambda},\phi_{\rm cl})+E_{\rm MWKB}(\tilde{\Lambda},\phi_{\rm cl})]$ of the one-boson-loop correction to the kink soliton using the phase-shift technique for various values of Λ.

$\tilde{\Lambda}/\sqrt{2}$	$-E_{\rm MS}-E_{\rm MWKB}$	$-E_{\rm MS}-E_{\rm ct}$
1.0	0.479	0.802
1.1	0.474	0.607
2.0	0.471	0.495
4.0	0.471	0.477
8.0	0.471	0.472
16.0	0.471	0.471

TABLE II. Comparison of the exact boson-loop energy $[E_{\rm vac}(\phi_{\rm cl})]$ with that calculated by the WKB method for the entire continuum $[E_{\rm MS}(M,\phi_{\rm cl})+E_{\rm MWKB}(M,\phi_{\rm cl})]$ and with the local density approximation method $[E_{\rm LDA}(\phi_{\rm cl})]$ for the background field given by Eq. (3.19) with $R=1$ and $T=0.5$. The column labeled ω_{bs}^2 lists the values of ω^2 corresponding to the discrete eigenvalues of Eq. (2.12b) for each field configuration.

ϕ_b	$-E_{\rm vac}$	ω_{bs}^2	$-E_{\rm LDA}$	$-E_{\rm MS}(M)-E_{\rm MWKB}(M)$
0.2	0.021	1.52	0.024	0.030
0.4	0.083	0.90	0.106	0.092
0.6	0.188	1.97		0.186
		0.39		
0.8	0.354	1.89		0.352
		0.05		

with $\tilde{\Lambda}=M$ to approximate the entire contribution of the continuum states.

For small field strengths, the local density approximation provides an adequate evaluation of the boson-loop energy and approximates the exact boson-loop energy better than the WKB calculation with $\tilde{\Lambda}=M$. However, as the field strength increases, the local density approximation becomes worse, while the WKB calculation with $\tilde{\Lambda}=M$ becomes better. When the field becomes strong enough that $3\phi_{\rm cl}^2-1<0$ for some value of x, the local density approximation breaks down and can no longer be used. The $\tilde{\Lambda}=M$ WKB calculation has no such limitation and continues to provide an increasingly good approximation as the field strength increases. Eventually, the eigenvalue of the most bound state has $\omega^2<0$ and the one-loop approximation breaks down. Up to this point, however, using the WKB approximation for the entire continuum provides an accurate description of the full one-loop result.

The WKB method can be generalized straightforwardly to fermion loops in $1+1$ dimensions and to spherically symmetric background fields in $3+1$ dimensions where a partial wave decomposition allows the $(1+1)$-dimensional techniques to be applied.[12] In each of these cases, the increase in computational efficiency is comparable to that seen in the $(1+1)$-boson-loop calculation.

IV. LOCAL EXPRESSIONS FOR GREEN'S-FUNCTION CALCULATIONS

In this section, we use the WKB approximation to calculate local formulas for the Green's-function method. Recall that the Green's-function method requires solutions of Eq. (2.15) that are regular at $x=\pm\infty$, respectively. In the WKB approximation, these two solutions are of the form[13]

$$u_\Omega^{\rm WKB}(x)=\frac{1}{\sqrt{\kappa(x)}}\exp\left[\int_0^x\kappa(z)dz\right],$$
$$v_\Omega^{\rm WKB}(x)=\frac{1}{\sqrt{\kappa(x)}}\exp\left[-\int_0^x\kappa(z)dz\right], \tag{4.1}$$

where

$$\kappa(x)=\sqrt{\Omega^2+V''(\phi_{\rm cl})}\ . \tag{4.2}$$

The Wronskian for these solutions is -2, so the resulting Green's function in the WKB approximation is

$$\langle x'|D^{\rm WKB}(i\Omega,\phi_{\rm cl})|x\rangle=-\frac{1}{2\sqrt{\kappa(x)\kappa(x')}}\left[\theta(x'-x)\exp\left[\int_{x'}^x\kappa(z)dz\right]+\theta(x-x')\exp\left[\int_x^{x'}\kappa(z)dz\right]\right]. \tag{4.3}$$

We now use this WKB Green's function to approximate the contribution to the boson-loop energy integral, Eq. (2.14), for energies greater than some value, Λ. Explicitly, we write

$$E_{\rm vac}(\phi_{\rm cl})=\lim_{\Lambda\to\infty}\left[E_{\rm GF}(\Lambda,\phi_{\rm cl})+E_{\rm GWKB}(\Lambda,\phi_{\rm cl})\right], \tag{4.4}$$

where

$$E_{\rm GF}(\Lambda,\phi_{\rm cl})=\frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}d\Omega\,\Omega^2\int_{-\infty}^{\infty}dx\left[\frac{u_\Omega(x)v_\Omega(x)}{W_\Omega}+\frac{1}{2[\Omega^2+V''(\phi_v)]^{1/2}}-\frac{1}{4[\Omega^2+V''(\phi_v)]^{3/2}}[V''(\phi_{\rm cl})-V''(\phi_v)]\right] \tag{4.5}$$

is calculated using the exact solutions and

$$E_{\rm GWKB}(\Lambda,\phi_{\rm cl})=\frac{1}{2\pi}\int_{\Lambda}^{\infty}d\Omega\,\Omega^2\int_{-\infty}^{\infty}dx\left[-\frac{1}{[\Omega^2+V''(\phi_{\rm cl})]^{1/2}}+\frac{1}{[\Omega^2+V''(\phi_v)]^{1/2}}-\frac{1}{2[\Omega^2+V''(\phi_v)]^{3/2}}[V''(\phi_{\rm cl})-V''(\phi_v)]\right] \tag{4.6}$$

is calculated using the WKB solutions. Note that, in contrast with the strategy followed in the preceding section, we have split the counterterm between the exact and WKB expressions. As a consequence $E_{\rm GWKB}(\Lambda,\phi_{\rm cl})\to 0$ as $\Lambda\to\infty$.

A brute-force Green's-function calculation consists of simply using $E_{\rm GF}(\Lambda,\phi_{\rm cl})$ to approximate the boson-loop ener-

gy, which is exactly Eq. (2.19), while the WKB-improved calculation consists of using Eq. (4.4). Explicitly,

$$E_{\mathrm{GWKB}}(\Lambda,\phi_{\mathrm{cl}})=\frac{1}{4\pi}\int_{-\infty}^{\infty}dx\,\tfrac{1}{2}[V''(\phi_{\mathrm{cl}})-V''(\phi_v)]+\Lambda[\sqrt{\Lambda^2+V''(\phi_{\mathrm{cl}})}-\sqrt{\Lambda^2+V''(\phi_v)}]$$
$$-\frac{\Lambda[V''(\phi_{\mathrm{cl}})-V''(\phi_v)]}{\sqrt{\Lambda^2+V''(\phi_v)}}-V''(\phi_{\mathrm{cl}})\ln\left[\frac{\Lambda+\sqrt{\Lambda^2+V''(\phi_{\mathrm{cl}})}}{\Lambda+\sqrt{\Lambda^2+V''(\phi_v)}}\right]. \tag{4.7}$$

Note that, for $\Lambda=0$,

$$E_{\mathrm{GWKB}}(0,\phi_{\mathrm{cl}})=\frac{1}{8\pi}\int_{-\infty}^{\infty}dx\;V''(\phi_{\mathrm{cl}})-V''(\phi_v)-V''(\phi_{\mathrm{cl}})\ln\left[\frac{V''(\phi_{\mathrm{cl}})}{V''(\phi_v)}\right]$$
$$=E_{\mathrm{LDA}}(\phi_{\mathrm{cl}})\,. \tag{4.8}$$

Thus, at zero cutoff the WKB method exactly duplicates the local density approximation. Hence as the cutoff is increased, the exact answer is approached from an initial value corresponding to the local density approximation. The WKB method thus provides a smooth interpolation between the brute-force calculation and the local density approximation. Note, however, that just as the local density approximation can only be used if $V''(\phi_{\mathrm{cl}})>0$, the WKB method can only be used if $\Lambda^2+V''(\phi_{\mathrm{cl}})>0$.

We now illustrate this method by applying it to the kink soliton background field, Eq. (3.15). Since $3\phi^2-1\geq-1$ for the kink soliton, the WKB formula given in Eq. (4.6) can only be used for $\Lambda\geq 1$. To compare the exact and WKB wave functions, we compare the various contributions to the integrands in Eqs. (4.5) and (4.6) at $x=0$. Figure 2 plots

$$\Omega^2\frac{u_\Omega(0)v_\Omega(0)}{W_\Omega}\,,\quad -\Omega^2\frac{1}{2\sqrt{\Omega^2+V''(0)}}\,,$$

and

$$\Omega^2\left[\frac{1}{2[\Omega^2+V''(\phi_v)]^{1/2}}-\frac{1}{4[\Omega^2+V''(\phi_v)]^{3/2}}[V''(0)-V''(\phi_v)]\right]$$

as functions of Ω. The first quantity is the contribution of the exact wave functions at $x=0$, the second quantity is the contribution of the WKB wave functions at $x=0$, and the last quantity is the contribution of the counterterm at $x=0$. Any difference between the first quantity and the last quantity corresponds to a finite contribution to the boson-loop energy. As with the mode-sum method, the WKB wave functions converge to the true ones long before the true wave functions converge to the counterterm, showing that the WKB wave functions contain much information about the finite part of the boson-loop energy. Similar results are seen at all other values of x.

A comparison of the WKB and brute-force methods for the entire energy at various values of Λ is shown in the second and fourth columns of Table III, respectively. As expected, the WKB method converges much more rapidly than does the brute-force method. This advantage persists for a wide range of background fields.

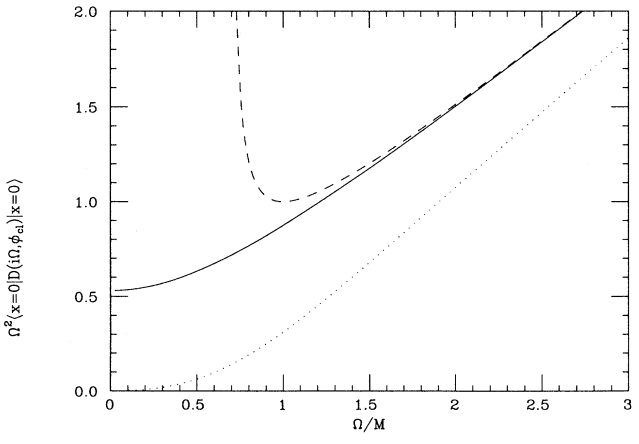

FIG. 2. Values of $\Omega^2\langle x=0|D(i\Omega,\phi_{\mathrm{cl}})|x=0\rangle$ as a function of ω for the kink soliton. The solid line is the exact value, the dashed line is the WKB approximation, and the dotted line is the contribution of the counterterm.

TABLE III. Comparison of the WKB-improved $[E_{\mathrm{GF}}(\Lambda,\phi_{\mathrm{cl}})+E_{\mathrm{GWKB}}(\Lambda,\phi_{\mathrm{cl}})]$ and derivative-expansion-improved $[E_{\mathrm{GF}}(\Lambda,\phi_{\mathrm{cl}})+E_{\mathrm{GWKB}}(\Lambda,\phi_{\mathrm{cl}})+E_{\mathrm{DE}}(\Lambda,\phi_{\mathrm{cl}})]$ calculations with the brute-force calculation $[E_{\mathrm{GF}}(\Lambda,\phi_{\mathrm{cl}})]$ for the one-boson-loop correction to the energy of the kink soliton using the Green's-function technique for various values of Λ.

$\Lambda/\sqrt{2}$	$-E_{\mathrm{GWKB}}-E_{\mathrm{GF}}$	$-E_{\mathrm{DE}}-E_{\mathrm{GWKB}}-E_{\mathrm{GF}}$	$-E_{\mathrm{GF}}$
0.5			0.203
1.0	0.479	0.468	0.326
1.5	0.473	0.471	0.387
2.0	0.472	0.471	0.418
4.0	0.471	0.471	0.456
8.0	0.471	0.471	0.467
16.0	0.471	0.471	0.470
32.0	0.471	0.471	0.471

V. DERIVATIVE EXPANSIONS FOR GREEN'S-FUNCTION CALCULATIONS

The WKB expression derived in the preceding section agrees with the local density approximation when $\Lambda=0$. The local density approximation can be improved by adding in derivative terms generated by a derivative expansion of the effective action. This suggests derivative corrections to our WKB approximation such that our expressions will agree with the full derivative expansion when $\Lambda=0$. Such a procedure is straightforward if, instead of correction the WKB approximation, we apply the standard techniques for deriving a derivative expansion of the effective action to the matrix elements of the Green's functions corresponding to the short-wavelength modes.[3] The derivative expansion for the Green's function D is

$$\int_{-\infty}^{\infty} dx \left\langle x \left| \frac{1}{\nabla^2-\Omega^2-W(x)} \right| x \right\rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x|p\rangle \frac{1}{-p^2-\Omega^2-W\left[i\frac{d}{dp}\right]} \langle p|x\rangle$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{-p^2-\Omega^2-W\left[x+i\frac{d}{dp}\right]} 1$$

$$= \int_{-\infty}^{\infty} dx - \frac{1}{2\sqrt{\Omega^2+W(x)}} + \frac{5}{64} \frac{1}{[\Omega^2+W(x)]^{7/2}} \left[\frac{dW}{dx}\right]^2 + O\left(\left[\left[\frac{d}{dx}\right]^4\right]\right) . \quad (5.1)$$

We insert this result into Eq. (2.14) to approximate the contribution to the boson-loop energy for $\Omega > \Lambda$. Keeping only the first term in this expansion duplicates the results of the preceding section. Including the second term represents the leading derivative correction to these results, which allows us to write the derivative expansion approximation to the boson-loop energy as

$$E_{\rm vac}(\phi_{\rm cl}) = \lim_{\Lambda\to\infty} [E_{\rm GF}(\Lambda,\phi_{\rm cl}) + E_{\rm GWKB}(\Lambda,\phi_{\rm cl}) + E_{\rm DE}(\Lambda,\phi_{\rm cl})] \quad (5.2)$$

where

$$E_{\rm DE}(\Lambda,\phi_{\rm cl}) = \frac{5}{64\pi} \int_{-\infty}^{\infty} dx \int_{\Lambda}^{\infty} d\Omega \frac{\Omega^2}{[\Omega^2+V''(\phi_{\rm cl})]^{7/2}} \left[\frac{dV''(\phi_{\rm cl})}{dx}\right]^2 + O((d/dx)^4) . \quad (5.3)$$

For $\Lambda=0$, $E_{\rm DE}(0,\phi_{\rm cl})$ reproduces the derivative expansion correction to the total energy, Eq. (2.21).

The above expression generates the exact boson-loop energy as $\Lambda\to\infty$. If the derivative expansion is convergent, then it will also generate the exact result for $\Lambda=0$ as long as a sufficient number of derivative terms are included. If the derivative expansion is not convergent, then we need to increase Λ to a nonzero value in order to evaluate the boson-loop energy. The criterion for convergence of the derivative expansion for the energy is roughly

$$\frac{1}{[\Lambda^2+V''(\phi_{\rm cl})]^{3/2}} \frac{dV''(\phi_{\rm cl}(x))}{dx} \ll 1 , \quad (5.4)$$

which becomes increasingly satisfied as Λ increases. Once the expansion converges, then a value of Λ corresponding to this convergence should yield the correct full boson-loop energy when inserted into Eq. (5.2). The utility of this method then depends on how the point at which the derivative expansion converges compares to the point at which using the pure WKB expression gives the desired accuracy.

We illustrate the derivative-expansion-improved calculation with several examples. First we calculate the energy of the kink soliton using the derivative-improved formula, Eq. (5.2). The results are shown in the third column of Table III. Observe that a fixed Λ, the WKB result lies below the actual answer and the derivative-improved results lies above the actual answer as is typical of the behavior of the derivative expansions.[5,8] The derivative expansion is apparently convergent for $\Lambda > \sqrt{2}$ allowing the derivative expansion expression to give a more accurate result than the bare WKB calculation. Hence, in this case the derivative-improved formula converges more rapidly than the WKB formula.

This need not always be the case, however. Consider the background fields of Eq. (3.19) in which the radius and strength of the field are fixed and the thickness is varied. This allows us to control the slope of the surface, and hence the convergence of the derivative expansion. The convergence of the full derivative expansion, Eq. (2.21), has been previously studied for these background fields by Li, Perry, and Wilets.[8] Results of calculations using our high-energy derivative expansion for these fields are shown in Table IV.

For the largest thickness tabulated ($T=1.0$), the derivative expansion converges quickly at small Λ and therefore the derivative-improved series converges faster than the WKB series. In fact, if we add in the next-order

TABLE IV. Comparison of WKB-improved $[E_{\mathrm{GWKB}}(\Lambda,\phi_{\mathrm{cl}})+E_{\mathrm{GF}}(\Lambda,\phi_{\mathrm{cl}})]$ and derivative-expansion-improved $[E_{\mathrm{DE}}(\Lambda,\phi_{\mathrm{cl}})+E_{\mathrm{GF}}(\Lambda,\phi_{\mathrm{cl}})]$ calculations as a function of Λ for the background field of Eq. (3.19) with $\phi_b=0.2$ and $R=1$.

T	Λ	$-E_{\mathrm{GWKB}}-E_{\mathrm{GF}}$	$-E_{\mathrm{DE}}-E_{\mathrm{GWKB}}-E_{\mathrm{GF}}$
1.0	0.0	0.0148	0.0138
	1.0	0.0143	0.0140
	2.0	0.0141	0.0140
	3.0	0.0140	0.0140
0.5	0.0	0.0242	0.0191
	1.0	0.0222	0.0209
	2.0	0.0215	0.0213
	3.0	0.0213	0.0213
0.1	0.0	0.0290	−0.0857
	1.0	0.0257	−0.0042
	2.0	0.0242	0.0198
	3.0	0.0239	0.0228
	4.0	0.0238	0.0234
	5.0	0.0237	0.0236
	6.0	0.0237	0.0237

derivative correction[8] then we get the correction answer to the precision shown in the table even with $\Lambda=0$. As the thickness is decreased, the convergence of the derivative expansion becomes worse at small Λ, and for $T=0.5$, the full derivative expansion is barely divergent. The first-order derivative corrections bring slightly better agreement with the actual answer, but the higher-order corrections make the agreement worse. Hence, we need to use a finite Λ to get a convergent result. Eventually the derivative expansion begins to converge and the derivative-improved result converges to the actual answer faster than the WKB result. For $T=0.1$ the derivative expansion is badly divergent at $\Lambda=0$ and, at the level of precision displayed in the table, the WKB-improved calculation converges to the correct value before the derivative expansion has begun to converge. In this case, the WKB-improved calculation converges to the actual answer faster than does the derivative-improved calculation.

In summary, this method involves computing, for each Λ, a derivative expansion for the contribution of energies higher than Λ to the boson-loop energy integral. This expansion becomes increasingly convergent as Λ is increased and, once it has converged to the desired level of precision, it can be combined with a brute-force calculation of the contribution of the energies less than Λ to give a final value at that same level of precision. Since the derivative expansion is generally an asymptotic series, in some cases one will get more accurate results by only keeping lower-order terms and obtain faster convergence to a given level of precision by using this lower-order series.

This method is very straightforward to generalize to fermion loops and to higher dimensions. The same method used above for the $(1+1)$-dimensional boson loop to derive the derivative expansion for energies above Λ works for these cases as well.

VI. SUMMARY

We have discussed various approximate methods for treating the perturbations of the short-wavelength vacuum modes by a background field. These methods allow us to significantly accelerate the calculation of one-loop corrections to the energy. Several approaches were implemented successfully.

The first method improved the convergence of mode-sum calculations by using the WKB approximation to calculate the phase shift of the short-wavelength vacuum modes. The other methods improved the convergence of the Green's-function method by using either the WKB approximation for the high-energy wave functions or a derivative expansion for the high-energy Green's functions. The latter method is particularly attractive since it reduces to the conventional derivative expansion when applied at all energies. When this expansion diverges, then our model requires the use of exact solutions up to an energy at which the expansion begins to converge, above which we can use our derivative expansion to calculate with high accuracy the remaining contribution to the vacuum energy.

All of these methods involve analytically determined local expressions that are added to the results of a brute-force calculation to rapidly improve the convergence. Thus, it is easy to modify any existing brute-force code to incorporate these improvements.

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