Stochastic quantization for the axial model

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We use bosonization ideas to solve the axial model in the stochastic quantization framework. We obtain the fermion propagator of the theory decoupling directly the Langevin equation, instead of the Fokker-Planck equation. In the Appendix we calculate explicitly the anomalous divergence of the axial-vector current by using a regularization that does not break the Markovian character of the stochastic process.

I. INTRODUCTION

The stochastic quantization (SQ) method, introduced by the pioneering paper of Parisi and Wu,[1] is a relatively recent alternative approach for studying quantum field theories (QFT's). The most important advantage of this new method lies in the fact that it does not require the introduction of gauge-fixing terms in non-Abelian gauge theories, so that Gribov ambiguities[2] can be avoided.

However, since SQ was invented a lot of work has been done and a great variety of applications have been made. An excellent review is offered by Damgaard and Huffel[3] and the interested reader may find there also a detailed list of references.

Although SQ is a powerful technique when applied perturbatively, our present interest concerns nonperturbative calculations. In this paper we shall solve exactly the Langevin equation for the axial model.[4] Our solution makes use of a variable transformation in the fermionic fields in order to decouple completely its Langevin equation. This idea, used by Roskies and Schaposnik[5] in the solution of the Schwinger model,[6] has been extensively used in the path-integral formalism to compute anomalies and to solve other two-dimensional models either in flat space-time[7] or in curved space-time.[8] It is based essentially in the path-integral calculation of anomalies introduced by Fujikawa.[9] In solving the axial model, we need the explicit expression for the axial anomaly. This calculation is also performed in the SQ method and, though the anomalous theories were widely studied in this framework,[10] we present this calculus using a different stochastic regularization scheme, namely, that proposed by Bern *et al.*[11]

This paper is organized as follows. In Sec. II we sketch the essential steps of the SQ method. In Sec. III we obtain the exact solution for the axial model. We make some final remarks and conclusions in Sec. IV and a detailed exposition of the evaluation of the anomaly is found in the Appendix.

II. STOCHASTIC QUANTIZATION METHOD

The underlying idea of this approach is that the Euclidean vacuum expectation values in the path-integral formalism are closely related to the Boltzmann distribution of a statistical system in equilibrium. Consider, for instance, the following correlation function of a QFT in Euclidean space-time:

$$\langle \Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\rangle = \frac{\int D\phi\, \phi(x_1)\phi(x_2)\cdots\phi(x_n)e^{-S_E/h}}{\int D\phi\, e^{-S_E/h}} . \quad (1)$$

If in the last expression we make the identification $1/\hbar=1/kT$, we can interpret this equation as a statistical expectation value with respect to a system in equilibrium at a temperature T.

The essential idea of the SQ method is to consider $\exp(-S_E)/\int D\phi \exp(-S_E)$ as a stationary distribution of probabilities of a stochastic process. From now on we shall use a system of units in which $\hbar=1=kT$. In order to implement the SQ program, we supplement the field $\phi(x)$ with an additional coordinate, a "fictious" time $\tau[\phi(x)\to\phi(x,\tau)]$, and imagine the system embedded in a $(D+1)$-dimensional space. Then, the system should reach an equilibrium distribution for large fictitious time

τ. In other words, as $\tau\rightarrow\infty$ a thermodynamic equilibrium should be established.

Hence, we need an equation to govern the τ evolution of the system. It must be a relaxation differential equation to guarantee a thermodynamic equilibrium distribution as $\tau\rightarrow\infty$. Of course, there is a large class of equations of this type. The Langevin equation is a possible choice. For instance, for a massive scalar field let S_E be the appropriate generalization of the usual Euclidean action $S_E=\int d\tau\int d^Dx\,\mathcal{L}(\phi(x\tau),\partial_\mu\phi(x,\tau))$. Then, the Langevin equation reads

$$\frac{\partial\phi(x,\tau)}{\partial\tau}=-\frac{\delta S_E}{\delta\phi(x,\tau)}+\eta(x,\tau)\ , \tag{2}$$

where the random noise field $\eta(x,\tau)$ satisfies the stochastic expectation values

$$\begin{aligned}&\langle\eta(x,\tau)\rangle_\eta=0\ ,\\&\langle\eta(x_1,\tau_1)\eta(x_2,\tau_2)\rangle_\eta=2\delta^D(x_1-x_2)\delta(\tau_1-\tau_2)\ ,\\&\qquad\vdots\\&\langle\eta(x_1,\tau_1)\cdots\eta(x_m,\tau_m)\rangle_\eta=0\quad\text{for } m \text{ odd}\ ,\\&\langle\eta(x_1,\tau_1)\cdots\eta(x_m,\tau_m)\rangle_\eta\\&\qquad=\sum_{\substack{\text{all possible}\\\text{permutations}}}\langle\eta(x_i,\tau_i)\eta(x_j,\tau_j)\rangle_\eta\\&\qquad\qquad m\text{ even}\ .\end{aligned} \tag{3}$$

The idea is that quantum effects are incorporated into the system through the noise term (the "fluctuation" term). The term $-\delta S_E/\delta\phi$ is responsible for the convergence towards equilibrium as $\tau\rightarrow\infty$ (the "dissipation" term).

A general stochastic expectation value for a functional of $\phi^\eta(x,\tau)$ is defined as

$$\begin{aligned}\langle F(\phi^\eta(x,\tau))\rangle_\eta&=\frac{\int D\eta\,F(\phi^\eta(x,\tau))\exp\left[-\frac{1}{4}\int d^Dx\int d\tau\,\eta^2(\tau)\right]}{\int D\eta\exp\left[-\frac{1}{4}\int d^Dx\int d\tau\,\eta^2(\tau)\right]}\\&=\int D\phi\,P(\phi,\tau)F(\phi^\eta)\ ,\end{aligned} \tag{4}$$

where in the last equation we wrote $\phi^\eta(x,\tau)$ in order to emphasize that the field $\phi(x,\tau)$ has gained a functional dependence in $\eta(x,\tau)$ through the Langevin equation. (We are dealing with Markovian processes, which are the most important for the study of physical systems. This definition ensures the Markovian character of the process, see Ref. 3 for more details.) In the last step of Eq. (4) we expressed the stochastic expectation value as a functional integral over the field ϕ itself, instead of the noise field η, through the introduction of the associated distribution of probability $P(\phi,\tau)$. This is an alternative procedure to study stochastic processes and it can be shown that $P(\phi,\tau)$ satisfies the so-called Fokker-Planck equation. It is worth mentioning that the equivalence between the SQ method and other quantization approaches, say, the path-integral formalism, can be better established in the Fokker-Planck formulation. One can show that the approach to the equilibrium configuration is reflected by

$$\lim_{\tau\rightarrow\infty}P(\phi,\tau)\equiv P^{\text{eq}}(\phi)=\frac{e^{-S_E}}{\int D\phi\,e^{-S_E}}\ . \tag{5}$$

Of course, the uniqueness of the solution $P^{\text{eq}}(\phi)$ must be assured to avoid ambiguities in the quantization process. The fundamental result relating Euclidean quantum expectation values to stochastic expectation values is then given by

$$\langle F(\Phi(x))\rangle=\lim_{\tau\rightarrow\infty}\langle F(\phi^\eta(x,\tau))\rangle_\eta\ . \tag{6}$$

It is worth emphasizing that in the left-hand side of (6) we have vacuum expectation values of time-ordered quantum operators in the Euclidean space-time, while in the right-hand side we have the equilibrium value of the stochastic expectation value of $F(\phi^\eta)$. However, as will become clear in our exposition, we shall not work with the Fokker-Planck equation, and the interested reader is referred to Ref. 3 for a detailed explanation concerning this alternative formulation.

The naive generalization of the SQ method introduced above for fermions may present some problems. This can even be seen in the free fermion theory, where the particular case of a massless fermion leads to a Langevin equation with no drift term. In other words, a thermodynamic equilibrium is never achieved in this case and, as a consequence, quantization is never implemented.

One way to circumvent this problem is to generalize the Langevin equation for ψ (and a similar procedure is valid also for $\bar{\psi}$) by introducing a kernel $K(x,y)$:[12]

$$\frac{\partial\psi(x,\tau)}{\partial\tau}=-\int d^Dy\,K(x,y)\frac{\delta S_E}{\delta\bar{\psi}(y,\tau)}+\theta(x,\tau) \tag{7}$$

and a similar equation for $\bar{\psi}(x,\tau)$. Although it is not necessary, this kind of generalization is also allowed in the bosonic case.

In order to generate an appropriate Fokker-Planck equation, the basic stochastic correlation functions for the noise fields present in (7) are now defined by

$$\langle\theta_\alpha(x,\tau)\bar{\theta}_\beta(x,',\tau')\rangle_\theta=2[K(x,x')]_{\alpha\beta}\delta(\tau-\tau')\ . \tag{8}$$

One should stress the fact that for a fermionic theory, $\delta S_E/\delta\psi$ may assume negative values. This means that we must choose $K(x,y)$ such that the negative values of $\delta S_E/\delta\psi$ correspond to the negative values of $K(x,y)$. An obvious choice is the Dirac operator which generates a positive defined operator in the drift term of Langevin equation (7).

As we shall see in the next section, we can choose conveniently more sophisticated kernels in order to introduce into the theory arbitrary parameters. These parameters in some sense reflect an arbitrariness in the regularization procedure.

Another relevant benefit of the SQ method is that it is possible to implement various regularization schemes that arise naturally from the properties of the stochastic process. The oldest one, proposed by Sakita[14] and Breit *et al*;[15] is based on the spreading of the second correlation function of the noise. With this, the stochastic process becomes non-Markovian, then being too difficult to study the convergence to the thermodynamic equilibrium. Furthermore, this scheme is not appropriate when gauge symmetry is relevant.[16] There is an alternative method, proposed by Bern *et al.*,[11] which essentially regularizes the quantum fluctuation at the Langevin equation level. This is performed by a regularizing operator R_Λ acting on the white-noise source, preserving the Markovian character of the stochastic process. We will apply this method in our calculus of the axial anomaly, presented in the Appendix.

III. THE AXIAL MODEL

This model describes a massive pseudoscalar field Φ interacting with a massless fermionic field Ψ through a derivative coupling. Its Euclidean action is given by

$$S(\Psi,\bar\Psi,\Phi)=\int d^2x[-\tfrac{1}{2}\Phi(\Box-m^2)\Phi+i\bar\Psi\gamma_\mu\partial_\mu\Psi -g\bar\Psi\gamma_5\gamma_\mu(\partial_\mu\Phi)\Psi] . \quad (9)$$

Before we proceed, it is convenient to introduce our conventions, as well as some definitions and relations. In a two-dimensional Euclidean space-time γ matrices satisfy

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu} . \quad (10)$$

A possible representation is

$$\gamma_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \gamma_2=i\gamma_0=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}, \quad (11a)$$

$$\gamma_5=\gamma_0\gamma_1=i\gamma_1\gamma_2=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} . \quad (11b)$$

From the above equations the important relations (valid only in a space-time of two dimensions) follow immediately:

$$\gamma_\mu\gamma_5=i\epsilon_{\mu\nu}\gamma_\nu , \quad (12a)$$

$$j^5_\mu\equiv\bar\Psi\gamma_\mu\gamma_5\Psi=i\epsilon_{\mu\nu}\bar\Psi\gamma_\nu\Psi\equiv i\epsilon_{\mu\nu}j_\nu . \quad (12b)$$

Using these results we may cast the action into the form

$$S(\Psi,\bar\Psi,\Phi)=-\int d^2x(\mathcal{L}_\Phi-\bar\Psi \not{D}\Psi) , \quad (13)$$

where

$$\mathcal{L}_\Phi=\tfrac{1}{2}\Phi(\Box-m^2)\Phi \quad (14)$$

and we defined the "Dirac" operator D as

$$\not{D}\equiv\gamma_\mu[\partial_\mu-g\epsilon_{\mu\nu}(\partial_\nu\Phi)] . \quad (15)$$

Choosing different "fictitious" times for the scalar and fermion fields, say τ and σ, respectively, we get the Langevin equations

$$\begin{aligned}\frac{\partial\psi_\alpha(x,\sigma)}{\partial\sigma}&=-(\not{D}_a\not{D}\psi)_\alpha+\theta_\alpha(x,\sigma) ,\\ \frac{\partial\bar\psi_\beta(x,\sigma)}{\partial\sigma}&=-((\not{D}\not{D}_a)^T\bar\psi)_\beta+\bar\theta(x,\sigma) ,\\ \frac{\partial\phi(x,\tau)}{\partial\tau}&=(\Box-m^2)\phi+g\epsilon_{\mu\nu}\partial_\nu(\bar\psi(x,\sigma)\gamma_\mu\psi(x,\sigma))\\ &\quad+\eta(x,\tau) .\end{aligned} \quad (16)$$

where we introduced the operator $\not{D}_a\equiv\gamma_\mu[\partial_\mu-ag\epsilon_{\mu\nu}(\partial_\nu\Phi)]$ to permit the appearance of an arbitrary parameter in the solution (see the Appendix for the details).

The basic stochastic expectation values for the scalar and fermionic noise fields are given, respectively, by

$$\langle\eta(x,\tau)\eta(x',\tau')\rangle\eta=2\delta(x-x')\delta(\tau-\tau') , \quad (17a)$$

$$\langle\theta_\alpha(x,\sigma)\bar\theta_\beta(x',\sigma')\rangle_{\theta\bar\theta}=2\not{D}_{\alpha\beta}\delta(x-x')\delta(\sigma-\sigma') , \quad (17b)$$

$$\langle\theta(x,\sigma)\eta(x',\tau)\rangle_{\theta\eta}=0=\langle\bar\theta(x,\sigma)\eta(x',\tau)\rangle_{\theta\eta} . \quad (17c)$$

It is very important to emphasize at this step that the fermionic term present in the Langevin equation for $\phi(x,\tau)$ [the last of Eq. (16)] depends on the fictitious time σ, and not τ.

Substituting, then, (12b) into the third equation of (16), calculating its stochastic expectation value in the noise fields θ and $\bar\theta$, and taking the limit $\sigma\to\infty$, we get

$$\frac{\partial\phi}{\partial\tau}=(\Box-m^2)\phi-ig\langle\partial_\mu(\bar\Psi\gamma_\mu\gamma_5\Psi)\rangle_\Psi+\eta(x,\tau) , \quad (18)$$

where we have used Eq. (6) to write the stochastic expectation value $\langle\partial_\mu(\bar\psi\gamma_\mu\gamma_5\psi)\rangle_\theta$ in the limit $\sigma\to\infty$ as the quantum correlation function $\langle\partial_\mu(\bar\psi\gamma_\mu\gamma_5\psi)\rangle_\psi$ in Euclidean space-time. The anomalous divergence of the axial-vector current can be calculated with the stochastic regularization prescription (see the Appendix for a careful calculation). In the limit of large σ the result is given by

$$\langle\partial_\mu[\bar\Psi(x)\gamma_\mu\gamma_5\Psi(x)]\rangle_\Psi=\lim_{\sigma\to\infty}\langle\partial_\mu[\bar\psi(x,\sigma)\gamma_\mu\gamma_5\psi(x,\sigma)]\rangle_\theta =-(1+a)\frac{g^2}{\pi}\Box\phi . \quad (19)$$

Substituting (19) into (18) we get

$$\frac{\partial\phi}{\partial\tau}=(Z_\phi^{-1}\Box-m^2)\phi+\eta(x,\tau)\ , \tag{20}$$

where we defined the constant $Z_\phi=1/(1-g^2/\pi)$. The Green's function associated with (20) is simply given by

$$G_\phi(x,t)=H(t)\int\frac{d^2k}{(2\pi)^2}e^{ikx}e^{-(Z_\phi^{-1}k^2+m^2)t}\ , \tag{21}$$

where $H(t)$ is the step function. We are now able to compute the propagator of the scalar field ϕ. Using the fundamental relation (8) as well as the previous Green's function we obtain

$$\begin{aligned}\Delta(x,x')&\equiv\langle\Phi(x)\Phi(x')\rangle\\&=\lim_{t=t'\to\infty}\langle\phi(x,t)\phi(x',t')\rangle_\eta\\&=\lim_{t=t'\to\infty}\int d\tau\,d\tau' d^2y\,d^2y' G_\phi(x-y;t-\tau)G_\phi(x'-y';t'-\tau')\langle\eta(y,\tau)\eta(y',\tau')\rangle_\eta\\&=Z_\phi\int\frac{d^2k}{(2\pi)^2}\frac{e^{ik(x-x')}}{k^2+Z_\phi m^2}\ .\end{aligned} \tag{22}$$

It is clear that the field ϕ has suffered a finite-mass renormalization, the renormalized mass being $m_R^2=Z_\phi m^2$. The constant Z_ϕ is identified as the wave-function renormalization constant, since

$$(\Box-m_R^2)\Delta(x,x')=-Z_\phi\delta(x-x')\ . \tag{23}$$

In order to solve the Langevin equation for the fermion field we make the transformations

$$\begin{aligned}\psi(x,\sigma)&=e^{ig\gamma_5\phi(x,\tau)}\chi(x,\sigma)\ ,\\ \bar\psi(x,\sigma)&=\bar\chi(x,\sigma)e^{ig\gamma_5\phi(x,\tau)}\ .\end{aligned} \tag{24}$$

Observing that $\phi(x,\tau)$ depends on τ, while $\psi(x,\sigma)$ and $\bar\psi(x,\sigma)$ depend on σ, the Langevin equation becomes

$$e^{ig\gamma_5\phi}\frac{\partial\chi}{\partial\tau}=-\not{D}_a\not{D}e^{ig\gamma_5\phi}\chi+\theta\ . \tag{25}$$

Multiplying the last equation on the left-hand side by $e^{-ig\gamma_5\phi}$, we get

$$\frac{\partial\chi}{\partial\tau}=-2\not{D}_{a+1}(\gamma_\mu\partial_\mu)\chi+\tilde\theta\ , \tag{26}$$

where we used the identities

$$\begin{aligned}\not{D}e^{ig\gamma_5\phi}&=e^{-ig\gamma_5\phi}\gamma_\mu\partial_\mu\ ,\\ e^{-ig\gamma_5\phi}\not{D}_a e^{-ig\gamma_5\phi}&=\not{D}_{a+1}\end{aligned} \tag{27}$$

and defined the new noise fields $\tilde\theta$ and $\bar{\tilde\theta}$, respectively, by

$$\tilde\theta=e^{-ig\gamma_5\phi}\theta,\quad \bar{\tilde\theta}=\bar\theta e^{-ig\gamma_5\phi}\ . \tag{28}$$

The stochastic expectation values for these new noise fields follow as a direct consequence from the original definition for the basic correlation between θ and $\bar\theta$, given by Eq. (17), and relations (28). They are simply given by

$$\langle\tilde\theta_\alpha(x,\sigma)\bar{\tilde\theta}_\beta(x',\sigma')\rangle_{\tilde\theta}=-2(\not{D}_{a+1})_{\alpha\beta}\delta(x-x')\delta(\sigma-\sigma')\ . \tag{29}$$

Equation (26) shows that χ describes a free fermion field. However, our interest lies on the original fermionic field ψ, and we need to calculate the propagator

$$\begin{aligned}\langle\Psi(x)\bar\Psi(0)\rangle&=\lim_{\substack{\eta\to\infty\\ \sigma\to\infty}}\langle e^{ig\gamma_5\phi(x,\tau)}\chi(x,\sigma)\bar\chi(x,\sigma)e^{ig\gamma_5\phi(0,\tau)}\rangle_{\eta\theta}\\&=\lim_{\substack{\eta\to\infty\\ \sigma\to\infty}}\langle e^{ig\gamma_5[\phi(x,\tau)-\phi(0,\tau)]}\rangle_\eta\lim_{\sigma\to\infty}\langle\chi(x,\sigma)\bar\chi(0,\sigma)\rangle_\theta\ ,\end{aligned} \tag{30}$$

where we used the field transformations (24), the anticommutation rule between γ_5 and γ_0, and the fact that the noise fields η and θ are not correlated with each other [see Eq. (17c)].

Because of the fundamental relation (6) the second term on the right-hand side of (30) is simply the well-known propagator for a free fermionic field, given by

$$\begin{aligned}\lim_{\sigma\to\infty}\langle\chi(x,\sigma)\bar\chi(0,\sigma)\rangle_{\tilde\theta}&=\langle\chi(x)\bar\chi(0)\rangle_\chi\\&=\frac{1}{2\pi}\frac{x_\mu\gamma_\mu}{x^2+i\epsilon}.\end{aligned} \tag{31}$$

In order to evaluate the first term on the right-hand side of (30), we must make a power expansion in the coupling constant g, use that $\gamma_5^2=1$ and the stochastic expectation values for a product of various noise fields [see Eq. (4)]. This will reproduce Wicks theorem[13] in the SQ scheme. After a straightforward calculation we obtain

$$\lim_{\eta\to\infty} \langle e^{ig\gamma_5[\phi(x,\tau)-\phi(0,\tau)]}\rangle_\eta = e^{g\Delta(x,0)} , \qquad (32)$$

where $\Delta(x,0)$ is given by (22).

Inserting (32) and (31) into (30) we finally obtain

$$S_F(x,0)\equiv\langle\Psi(x)\bar{\Psi}(0)\rangle = \exp[g\Delta(x,0)]\frac{1}{2\pi}\frac{x_\mu\gamma_\mu}{x^2+i\epsilon} \qquad (33)$$

which coincides with previous results found in literature obtained by other quantization methods (see Refs. 4 and 7).

IV. CONCLUSIONS AND FINAL REMARKS

In this paper we solved exactly the axial model using the stochastic quantization method. Our solution was based essentially in a field transformation in order to turn the theory into a free one. This procedure is very commonly used in the path-integral solution of two-dimensional models.

In contrast with previous work on the solutions for both the Schwinger and Thirring models, see Webb in Ref. 10, we applied the field transformation directly in the Langevin equations, instead of the Fokker-Planck equation.

Regarding the computation of the anomalous divergence of the axial-vector current, we have used the prescription proposed by Bern *et al.*,[11] regularizing the fermionic noise with the exponential of the operator that appears in the drift terms of each fermionic Langevin equation. We present a detailed discussion in the Appendix in order to clarify some mathematical steps in the application of the stochastic regularization prescription for the calculation of anomalies. It is worth saying that with the choice we made for the regularizing operator, in the Langevin equation for the fermion field, we obtain essentially Fujikawa's regularized expression for the anomaly, compare Eq. (A16) with Fujikawa's formula given in Ref. 9. By choosing other functions of the operator that appear in the drift term, we will arrive at other regularization schemes.[17] As a final comment, we would like to point to the fact that different regularization prescriptions (through different choices for the regulator operators in Fujikawa's method, for instance) can also be realized, in the SQ method, through different choices of the drift term in the Langevin equation. Of course these different Langevin equations must correspond to Fokker-Planck equations with the same equilibrium distribution.

The physical interpretation of this fact is that different regularization schemes correspond, then, to different "trajectories" (or distinct evolutions) to the equilibrium. But once you get the same equilibrium distribution and observe that a Markovian process does not keep any memory of the past at all, the physical results will be the same. Curiously, this subject has not attracted much attention in literature, but we think it deserves a deeper understanding. One of us has in fact initiated this kind of study and has calculated both the covariant and the consistent anomaly in four dimensions using different regularization prescriptions in the stochastic quantization scheme.

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APPENDIX

We shall evaluate here the vacuum expectation value of $\partial_\mu j_5^\mu(x)$ as the large-σ limit of the stochastic expectation value $\langle\partial_\mu j_5^\mu(x)\rangle_\theta$ with the use of the stochastic regularization prescription.[11] Starting with the fermionic action

$$S_F=\int d^2x\, \bar{\psi}\{\gamma_\mu[i\partial_\mu - g\gamma_5(\partial_\mu\phi)]\}\psi \qquad (A1)$$

it is easy to see that the infinitesimal chiral transformations

$$\delta\psi=i\epsilon(x)\gamma_5\psi,\quad \delta\bar{\psi}=i\epsilon(x)\bar{\psi}\gamma_5 \qquad (A2)$$

lead to the variation

$$\delta S_F=\int d^2x\, \epsilon(x)\partial_\mu(\bar{\psi}\gamma_5\gamma_\mu\psi) = \int d^2x\, \bar{\psi}\gamma_5[\not{D},\epsilon(x)]\psi , \qquad (A3)$$

where in the last mathematical step we wrote δS_F in a convenient form.

The divergence of the axial-vector current follows immediately by a functional differentiation, that is,

$$\frac{\delta S}{\delta\epsilon(x)}=\partial_\mu j_5^\mu(x) . \qquad (A4)$$

In order to introduce an arbitrary parameter into the regularization procedure, we shall rewrite the Langevin equations for the fermion fields as

$$\begin{aligned}\frac{\partial\psi(x,\sigma)}{\partial\sigma} &= -\not{D}_a\not{D}\psi(x,\sigma)+e^{-\not{D}_a\not{D}/\Lambda^2}\theta(x,\sigma) ,\\ \frac{\partial\bar{\psi}(x,\sigma)}{\partial\sigma} &= -\bar{\psi}(x,\sigma)\not{D}\not{D}_a+\bar{\theta}(x,\sigma)e^{-\not{D}\not{D}_a/\Lambda^2} ,\end{aligned} \qquad (A5)$$

where we defined $i\not{D}_a=\gamma_\mu[i\partial_\mu - ag\gamma_5(\partial_\mu\phi)]$.

This scheme does not modify the correlation function for the noise, preserving its Markovian character:

$$\langle\theta(x,\sigma)\rangle_\theta=\langle\bar{\theta}(x,\sigma)\rangle_\theta=0 , \qquad (A6)$$

$$\langle\theta_\alpha(x,\sigma)\bar{\theta}_\beta(x',\sigma)\rangle_\theta=2(\not{D}_a)_{\alpha\beta}\delta(x-x')\delta(\sigma-\sigma') . \qquad (A7)$$

Let us, then, start by calculating $\langle \delta S_F(\psi^\theta,\bar\psi^\theta,\sigma)\rangle_\theta$. After this we must take the limit $\sigma\to\infty$, as well as $\Lambda\to\infty$ in order to obtain the correspondent quantum expectation value.

The solutions of the Langevin equations (A5) can be written in the form

$$\psi^\theta(x,\sigma)=\int_0^\sigma dt_1 e^{-\not{D}_a\not{D}(\sigma-t_1+\Lambda^{-2})}\theta(x,t_1)\ ,$$
$$\bar\psi^\theta(x,\sigma)=\int_0^\sigma dt_2\bar\theta(x,t_2)e^{-\not{D}\not{D}_a(\sigma-t_2+\Lambda^{-2})}\ . \tag{A8}$$

Inserting (A8) into (A3) we get

$$\begin{aligned}\langle \delta S_F(\psi^\theta,\bar\psi^\theta)\rangle_\theta&=\left\langle i\int d^2x\int_0^\sigma dt_1dt_2\bar\theta(x,t_1)e^{-\not{D}\not{D}_a(\sigma-t_1+\Lambda^{-2})}\gamma_5[\not{D},\epsilon(x)]e^{-\not{D}_a\not{D}(\sigma-t_2+\Lambda^{-2})}\theta(x,t_2)\right\rangle_\theta\\&=i\int d^2x\, d^2x'\delta^{(2)}(x-x')\int_0^\sigma dt_1dt_2 e^{-\not{D}'\not{D}_a'(\sigma-t_1+\Lambda^{-2})}\gamma_5[\not{D},\epsilon(x)]e^{-\not{D}_a\not{D}(\sigma-t_2+\Lambda^{-2})}\langle\theta(x,t_2)\bar\theta(x',t_1)\rangle_\theta\ ,\end{aligned} \tag{A9}$$

where we made the substitution $\bar\theta(x,t_1)=\int d^2x'\delta^{(2)}(x-x')\bar\theta(x',t_1)$ in order to put $\bar\theta$ and θ together, and we are using the notation that the operators $\not{D}'$ and $\not{D}_a'$ act on $\bar\theta(x',t_1)$.

Substituting (A7) into (A9), we obtain, after some mathematical manipulations,

$$\langle \delta S_F(\psi^\theta,\bar\psi^\theta,\sigma)\rangle_\theta=2i\int_0^\sigma dt_1dt_2\mathrm{Tr}\{e^{-\not{D}\not{D}_a(\sigma-t_1+\Lambda^{-2})}\gamma_5[\not{D},\epsilon(x)]e^{-\not{D}_a\not{D}(\sigma-t_2+\Lambda^{-2})}\not{D}_a\}\ , \tag{A10}$$

where Tr means a functional trace. Using its cyclic properties we get, after some calculations,

$$\begin{aligned}\langle \delta S_F(\psi^\theta,\bar\psi^\theta,\sigma)\rangle_\theta=2i\int_0^\sigma dt_1dt_2\delta(t_1-t_2)\int d^2x\, d^2x'\delta^{(2)}(x-x')\\\times\epsilon(x)\mathrm{tr}[(e^{-\not{D}_a\not{D}(2\sigma-t_1-t_2+2\Lambda^{-2})}\not{D}_a\not{D}+\not{D}\not{D}_a e^{-\not{D}\not{D}_a(2\sigma-t_1-t_2+2\Lambda^{-2})})\gamma_5]\delta^{(2)}(x-x')\ ,\end{aligned} \tag{A11}$$

where tr means simply a trace in the spinor indices. Making use of the Fourier representation of the Dirac delta function $\delta^{(2)}(x-x')$, and also having in mind that

$$\not{D}e^{ik(x-x')}=e^{ik(x-x')}(\not{D}-k_\mu\gamma_\mu) \tag{A12}$$

as well as a similar relation for $\not{D}_a e^{ik(x-x')}$, we get

$$\begin{aligned}\langle \delta S_F(\psi^\theta,\bar\psi^\theta,\sigma)\rangle_\theta=\frac{-2i}{(2\pi)^2}\int_0^\sigma dt_1dt_2\int d^2k\, d^2x\ \epsilon(x)\delta(t_1-t_2)\\\times\mathrm{tr}[(e^{-d_ad(2\sigma-t_1-t_2+2\Lambda^{-2})}d_ad+dd_ae^{-dd_a(2\sigma-t_1-t_2+2\Lambda^{-2})})\gamma_5]\ .\end{aligned} \tag{A13}$$

In obtaining the last equation, we made an integration in the variable x' and defined

$$d\equiv\not{D}-k_\mu\gamma_\mu,\quad d_a\equiv\not{D}_a-k_\mu\gamma_\mu\ . \tag{A14}$$

Instead of integrating over the variables t_1 and t_2, it is convenient to make the change of variables

$$t=\frac{t_1-t_2}{\sqrt2},\quad T=\frac{t_1+t_2}{\sqrt2}\ . \tag{A15}$$

Now, the t and T integrals are immediate, getting

$$\begin{aligned}\langle \delta S_F(\Psi,\bar\Psi)\rangle_\Psi&=\lim_{\sigma\to\infty}\lim_{\Lambda\to\infty}\langle \delta S_F(\psi^\theta,\bar\psi^\theta,\sigma)\rangle_\theta\\&=\lim_{\substack{\sigma\to\infty\\\Lambda\to\infty}}\frac{2i}{(2\pi)^2}\int d^2x\, d^2k\ \epsilon(x)\mathrm{tr}(e^{-d_ad(2\sigma-\sqrt2T+2\Lambda^{-2})}+e^{-dd_a(2\sigma-\sqrt2T+2\Lambda^{-2})})_{T=0}^{T=\sqrt2\sigma}\\&=\lim_{\Lambda\to\infty}\frac{i}{2\pi^2}\int d^2x\, d^2k\ \epsilon(x)\mathrm{tr}(\gamma_5e^{-d_ad/2\Lambda^{-2}}+\gamma_5e^{-dd_a/2\Lambda^{-2}})\\&=\lim_{\Lambda\to\infty}\frac{2i}{(2\pi)^2}\int d^2x\, d^2k\ \epsilon(x)e^{-k^2/2\Lambda^{-2}}\mathrm{tr}(\gamma_5e^{-\Omega/2\Lambda^{-2}}+\gamma_5e^{-\Omega/2\Lambda^{-2}})\ ,\end{aligned} \tag{A16}$$

where we defined

$$d_ad\equiv\Omega+k^2,\quad dd_a\equiv\bar\Omega+k^2\ . \tag{A17}$$

From now on the calculations are more or less standard, see, for instance, Ref. 9 for more details. Using the trace properties of the γ matrices in two dimensions and taking the limit $\Lambda\to\infty$, it can be shown that the only contribution is given by

$$\langle \delta S_F(\Psi,\overline{\Psi})\rangle_\Psi=-\frac{2i}{(2\pi)^2}\int d^2x\,\epsilon(x)\int d^2k\,e^{-k^2}\mathrm{tr}\gamma_5\{\not{D},\not{D}_a\}$$
$$=-(1+a)\frac{g}{2\pi^2}\int d^2x\,\epsilon(x)\mathrm{tr}\gamma_5\{\gamma_\mu\partial_\mu,\gamma_5\partial_\mu\phi\}=-2(1+a)\frac{g}{\pi^2}\int d^2x\,\epsilon(x)\Box\phi\ . \tag{A18}$$

Using (A4) we finally obtain the anomalous divergence for the axial-vector current:

$$\langle \partial_\mu j_\mu^5\rangle_\Psi=(1+a)\frac{g}{\pi^2}\Box\phi\ . \tag{A19}$$

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