Theory of matter in Weyl spacetime

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Weyl's geometry of spacetime is reconsidered based on a novel geometric coupling between the Weyl vector and fermions. Starting from a manifestly Weyl-invariant action for the metric, Weyl field, spinors, and other fields, we introduce and define this coupling: the square root of the scalar Weyl curvature. An important consideration in this development is the reduction of the Weylian to an effective Riemannian structure for spacetime, without trivializing the role of the Weyl field. The aim of this reduction is to make contact with Einstein gravity for the metric sector of the theory. This is achieved by appealing to a dynamical-symmetry-breaking mechanism. The resulting spacetime is Riemannian, the Weyl field survives as a massive vector, and the geometric coupling to fermions decomposes into an admixture of vector and pseudovector couplings. Internal consistency of the field equations further narrows down the class of fermions which can couple to the Weyl vector: these can only be spinors of a fixed chirality. We tentatively identify them with the standard-model neutrinos. Analysis of couplings to other types of fields and particles indicates that the Weyl field is a form of dark matter.

I. INTRODUCTION

The principle of relativity of magnitude, introduced by Weyl in the first and well-known attempt[1,2] to unify Maxwell electrodynamics and Einstein gravity within a common geometric framework, provides a theoretically appealing concept in its own right. Despite the equally well-known objections raised against Weyl's proposal,[3] many later investigations sought to uncover potential physical applications of Weyl's geometry, by probing the nature of this particular generalization of Riemannian spacetime in a number of different ways.[4–9]

Though many of these considerations have raised some interesting issues, the basic question remains: Is spacetime Weylian, and if so, why does it appear to be Riemannian? One immediate response may be to dispense with Weyl's geometry altogether, for there appears to be no *need* for it. Furthermore, electromagnetism has been successfully unified with the weak, not gravitational, interaction. Nevertheless, the first part of this question has been answered, and in the affirmative, in fundamental work that is quite distinct from the work contained in Refs. 4–9. After 1970, a number of axiomatic approaches for deducing spacetime structure were developed and carried out which use basic concepts such as light rays and freely falling test particles. The remarkable fact is these considerations all end by assigning a Weylian, not Riemannian, structure to spacetime.[10] This presents an interesting dilemma, for our physical spacetime appears Riemannian. One may take the point of view that the axiomatic approach is incomplete, or that the conclusion drawn therefrom is valid. We have chosen the latter.

The challenge then is to close this "gap"; that is, describe how our Riemannian world can be deduced starting from the *a priori* Weylian structure that the axiomatic arguments tell us we must start with. In addition, this reduction must be undertaken without trivializing the form of the Weyl vector (a Weyl vector is trivial if it can be written as the gradient of a scalar).

In this paper we present a response to this challenge. Our aim is to close the gap starting from a Weyl-invariant action incorporating matter and other fields. The inclusion of matter is crucial: without it, there is no practical way to probe the spacetime structure, and hence, no possibility to go beyond where the axiomatic arguments leave off. For this reason, we have placed special emphasis on the issue of matter couplings. Our treatment proceeds in two basic steps. We first discuss the vacuum sector and require conventional Einstein gravity to emerge in a natural way. An elegant way to do this is to introduce a suitable gauge-invariant constraint.[5] The associated Lagrange multiplier field, *nonconformally* coupled to the curvature scalar, is allowed to be dynamical as well. Manifest contact to ordinary gravity can be achieved by exploiting the gauge invariance and picking a gauge. This introduces a scale into the problem which gets related to Newton's constant. This gauge-fixing scheme, as outlined here, was introduced by Dirac some time ago.[5] Alternatively, and this is the viewpoint we favor here, one can appeal to a *dynamical*-symmetry-breaking mechanism,[11] which in the end has the same effect of gauge fixing, i.e., explicit contact to Einstein gravity is achieved. The important difference in this latter case is that the gravitational constant is dynamically generated.[12] The possibility for computing a unique, gauge-invariant effective action for metric theories suggests this attractive alternative to the *ad hoc* gauge fixing, and can lead to a deeper understanding of the reduction problem.[13–15]

A most important consequence of this reduction to an effective Riemannian spacetime is the constraint that the Weyl vector satisfies a Lorentz condition. This constraint, which follows from the field equations in the broken phase, plays a valuable role in the second step, which is devoted to an analysis of Weyl couplings to other fields and particles. The novel feature to emerge from this consideration is the existence of a geometric interaction term coupling spinors to the Weyl vector. The Lorentz condition mentioned above allows us to define this interaction term: the square root of the scalar Weyl curvature. We define this square root in the sense of Dirac and Fock and Iwanenko.[16] Imposing Hermiticity and requiring consistency of the coupled field equations pins down both the explicit form of the interaction *as well as* the class of spinors that couple to the Weyl vector. Indeed, the spinors turn out to be chiral, and they couple via an admixture of vector and pseudovector interactions. We identify tentatively the fermions with the left-handed neutrinos of the standard model. We thus have a probe with which to detect the possible Weylian structure of spacetime. The pattern of couplings to other fields can be completed: scalars couple minimally as well as geometrically (the analog of the spinor interaction), while other spin-1 gauge bosons have no coupling at all. However the photon could couple to the Weyl vector provided the latter is strictly massless.

This paper is organized as follows. We begin in Sec. II with the Weyl-invariant action for the vacuum sector (with a constraint). Reduction to a Riemannian spacetime is achieved by dynamical symmetry breaking. As a consequence, the Weyl vector survives as a massive spin-1 field propagating in a Riemannian background. Its mass and Newton's constant are derivable from the parameters of the invariant vacuum action. The detailed treatment of allowed interactions between Weyl vector and other fields is the topic of Sec. III. We begin with spinors, review the absence of minimal coupling, and define our geometric interaction term. Once this is established, we complete the pattern of couplings by inclusion of scalars and other spin-1 gauge bosons. With this analysis complete, we write down the most general effective action involving metric, residual Weyl fields and their respective couplings to other fields and summarize the physical parameters in this theory.

Conclusions, a discussion of our results, and speculative remarks are presented in Sec. IV. A resume of Weyl geometry and technical details concerning spinors in Weylian spacetime are collected in two Appendixes.

II. VACUUM SECTOR–REDUCTION FROM WEYLIAN TO RIEMANNIAN SPACETIME

Gauge invariance severely limits the form of any action built up from the Weyl curvature tensor and its various contractions. Indeed, let

$$A = \int d^4x \sqrt{-g}\, \mathcal{L} \tag{1}$$

be any gauge- (and general coordinate) invariant action containing the metric and the Weyl vector (the following remarks also hold for invariant actions depending on other fields). Requiring that the action has weight zero (see Appendix A for technical details and definitions), i.e.,

$$\omega(\sqrt{-g}\mathcal{L}) = 0, \tag{2}$$

implies the Lagrangian density must satisfy the condition

$$\omega(\mathcal{L}) = -2, \tag{3}$$

which follows immediately from the fact $\omega(\sqrt{-g}) = d/2$ in d dimensions and the property $\omega(ST) = \omega(S) + \omega(T)$. From the physical standpoint the vacuum action must contain kinetic terms for both the Weyl field and the metric. By inspection of (A9) the former is supplied by $\mathcal{L}_W \sim W_{\mu\nu}W^{\mu\nu}$, which has the proper weight, while an obvious candidate for the latter is the Weyl curvature scalar, $\overline{R}$ defined in (A10). However, $\omega(\overline{R}) = -1$, and so violates condition (3). The next best choice is to take $\overline{R}^2$, but with the consequence that the Riemannian curvature scalar appears quadratically in the action, and contact with Einstein gravity is not obvious. Weyl himself recognized the possibility of fixing a gauge wherein $\overline{R}$ =const,[2] and thus restricts the allowed field configurations for $g_{\mu\nu}$ and W_μ. This has the desired effect of linearizing the quadratic curvature term in A. The above constant is identified with Newton's constant, as it must be for phenomenological consistency. A substantial improvement has been offered by Dirac, by introducing instead an invariant constraint.[5] This minimal action contains

$$\mathcal{L} \sim aW_{\mu\nu}W^{\mu\nu} + \xi\Phi^2(x)\overline{R}, \tag{4}$$

where Φ^2 is a Lagrange multipier field, ξ is a dimensionless coupling and a is a constant. The scalar field has weight $\omega(\Phi) = -\frac{1}{2}$, so (4) satisfies (3). Since Φ is a local field associated with additional properties of the vacuum, we may suppose it propagates. Therefore, let us postulate the following action describing the geometric properties of a Weyl-invariant vacuum (i.e., no other matter fields are present):

$$\mathcal{L}^{\text{vac}}_{g,W,\Phi} = aW_{\mu\nu}W^{\mu\nu} + \xi\Phi^2\overline{R} + (D_\mu\Phi)(D^\mu\Phi) + V(\Phi), \tag{5}$$

where $D_\mu\Phi = (\partial_\mu\Phi - \frac{1}{2}W_\mu\Phi)$ is the Weyl-covariant derivative [see (A4) for the general properties of D] and $V(\Phi) = \lambda\Phi^4$ is the only possible gauge-invariant potential. Now the scalar curvature comes in linearly in *all* gauges, which represents a substantial improvement over Weyl's approach.

Let us mention briefly what class of allowed terms we have not included in the classical action above. Because of (3), only a finite number of quadratic curvature terms are allowed, and we summarize them in

$$\mathcal{L}_{\text{quadratic}} = b\overline{R}^2 + c\overline{R}_{\mu\nu}\overline{R}^{\mu\nu} + \overline{R}_{\mu\nu\alpha\beta}(d_1\overline{R}^{\mu\nu\alpha\beta} + d_2\overline{R}^{\alpha\beta\mu\nu} + d_3\overline{R}^{\mu\alpha\nu\beta}). \tag{6}$$

We have already ruled out the first term. The remaining forms are analogous to quadratic curvature terms for generalized gravity: They all introduce higher-derivative terms, and are hence unfavorable for constructing a minimal classical action.[17]

One may exploit the gauge invariance of (5) in order to make explicit contact with classical Einstein gravity. This is already spelled out in detail in Dirac's paper,[5] where Φ is gauge fixed to be constant. From this point of view, our spacetime turns out as an effective Riemannian spacetime that can be understood as representing a preferred gauge slice of the underlying Weyl geometry. As soon as this gauge is taken, the Weyl vector appears as an additional massive field propagating in spacetime.

However, while the gauge fixing leads to a completely consistent classical theory describing a Riemannian spacetime containing a massive vector field, and so closes the gap in the sense outlined earlier, the choice of gauge is *a posteriori*: The constant to which Φ is fixed must be related to Newton's constant. Is it possible to understand the preference of this gauge slice over any other? In fact, with the scalar field Φ present in the vacuum action, it is tempting to imagine that it might have a much deeper role in the context of closing the gap. A Higgs-type mechanism is out of the question, since the potential $V(\Phi)$ only has a minimum at $\Phi = 0$. However this does not preclude a *dynamical mechanism* for symmetry breaking; the success of this approach hinges only on the computability of an effective potential.[11] Reparametrization-invariant effective potentials have in fact been computed at the one-loop approximation for Brans-Dicke-type theories and show that Einstein gravity can be dynamically induced.[15] We therefore conjecture that an analogous calculation based upon the vacuum sector (5) will demonstrate that a Riemannian spacetime can be induced from a Weylian geometry. Taking this as a working hypothesis, the conclusions drawn concerning the structure of the vacuum sector based on gauge-fixing arguments remain unchanged. From (5) it is straightforward to derive the Euler-Lagrange equations for the metric, scalar, and Weyl vector. In the broken phase they reduce to following set of Einstein-Proca equations (omitting possible quantum fluctuations of Φ):

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}(R + \Lambda) = -8\pi G_N T_{\mu\nu}(g, W), \tag{7}$$

$$\nabla_\mu W^{\mu\nu} = m_W^2 W^\nu \tag{8}$$

Only the expressions for the phenomenological constants like Newton's constant G_N, and the Proca mass m_W, will now be expressed in terms of the scalar field vacuum expectation value $\langle\Phi\rangle = v$ and the other fundamental, dimensionless parameters of the action (5):

$$G_N = -\frac{1}{16\pi\xi v^2}, \quad \Lambda = \frac{V_{\text{eff}}(v)}{2\xi v^2}, \quad m_W^2 = \frac{v^2}{2a}\left(\frac{1}{4} + \frac{3\xi}{2}\right). \tag{9}$$

We note that $G_N > 0$ and m_W is real, provided $-\frac{1}{6} \leq \xi < 0$. The dynamically induced cosmological constant Λ is proportional to the effective scalar potential evaluated at its minimum, $V_{\text{eff}}(v)$. The calculation of (1-loop) effective potentials[15] indicates we can expect $V_{\text{eff}}(v) < 0$, thus $\Lambda > 0$. The vector field W_μ survives this reduction to Riemannian spacetime. The residuum of what initially began as an integral component of the connection (A6) is manifested as a massive spin-1 field.

The corresponding Proca equation for W_μ leads to an important covariant constraint that will be of later use. Taking the Riemann covariant derivative of (8) implies that

$$\nabla_\mu W^\mu = 0, \tag{10}$$

which represents the general coordinate invariant version of the Lorentz condition for the vector field W_μ. The validity of this condition will be crucial in our discussion of spinor couplings in (effective) Riemannian spacetime, which will be taken up in the next section.

III. COUPLING BETWEEN WEYL'S GAUGE FIELD AND OTHER FIELDS

A. Geometric spinor interaction

The spinor calculus in a Weyl geometry allows one to extend the action of Weyl transformations to fermions.[6,18] The assignment of weights and the construction of a spinor covariant derivative proceeds as in the case of vectors and tensors, but with one important difference: Applying the condition $\omega(\sqrt{-g}\mathcal{L}) = 0$ to the spinor sector of a Weyl invariant action requires the coupling between the Weyl vector and spinors in the covariant spinor derivative to identically vanish (see the Appendix B for the elaboration of this point). As the net result of this, the Weyl covariant spinor derivative reduces to the ordinary Riemann covariant part only, which for the case of Dirac spinors is given by

$$D_\mu = (\partial_\mu + \Gamma_\mu), \tag{11}$$

where

$$\Gamma_\mu = \tfrac{1}{2}\,\sigma^{ab}\,e_a^\lambda \nabla_\mu e_{\lambda b} \tag{12}$$

is the standard spin connection for fermions in a Riemannian background.[19,20]

Although spinors do not couple minimally to the Weyl vector, this does not preclude the existence of other types of coupling terms. In fact there exists a class of potentially important couplings of a manifestly geometric character that should not be overlooked. By geometric, we simply mean that the fermion couples to W_μ via some general operator-valued function of the curvature tensor $\overline{R}^{\lambda}_{\mu\nu\kappa}$. We have already encountered one example of a geometric coupling in the case of a scalar field, in the discussion of the vacuum sector.

To find the explicit form of a local geometric coupling to spinors, we consider fermion bilinears involving a function $\mathcal{F}$ of the Weyl curvature:

$$\mathcal{L}^{\rm int}_{W,\psi} = \lambda_W \overline{\psi} \mathcal{F}(\overline{R}^{\lambda}_{\mu\nu\kappa})\psi. \qquad (13)$$

It is clear that whatever $\mathcal{F}$ is, it must be a general coordinate scalar as well as a second-rank tensor in spinor space. We will not consider the trivial case where $\mathcal{F}$ is proportional to the identity, for such an interaction would *not* be local. We shall return to this point below. Then, the simplest candidates for $\mathcal{F}$ are built from a single factor of the curvature together with the only remaining coordinate tensors at our disposal: the metric, the Kronecker and the totally antisymmetric Levi-Civita symbols. The only coordinate scalars of this form are

$$\mathcal{F}(\overline{R}^{\lambda}_{\mu\nu\kappa}) = \begin{cases} (\overline{R})^p, \\ (\epsilon^{\lambda\mu\nu\kappa}\overline{R}_{\lambda\mu\nu\kappa})^r, \end{cases} \qquad (14)$$

where p and r are uniquely determined by (3). However, the second form vanishes identically, by virtue of (A12). This leaves the first form. Gauge invariance now requires $p = \frac{1}{2}$, which follows directly from the weight $[\omega(\psi) = \omega(\overline{\psi}) = -\frac{3}{4}]$ for Dirac spinors in four dimensions (see Appendix B).

It is pleasing that gauge invariance singles out this particular value for p. We interpret the square root $(\overline{R})^{1/2}$ in the sense of Dirac or by following the linearized geometry concept of Fock and Iwanenko,[16] which has also recently been exploited for linearizing wave equations.[21–23] Thus, linearizing $\overline{R}$ as an operator should lead to the nontrivial spinor structure we require in order to complete the specification of $\mathcal{F}(\overline{R}^{\lambda}_{\mu\nu\kappa})$. The task is to find a consistent factorization of $\overline{R}$ as an operator acting on spinors, i.e., we seek an operator $\mathcal{R}$ such that

$$(\mathcal{R} \otimes \mathcal{R}) = \overline{R}\mathbf{1}. \qquad (15)$$

Let us note that the factorization of the curvature scalar (A10) at the gauge-invariant level is a difficult exercise. We do however know that it is only of relevance to define the square root at the level of an effective Riemannian spacetime, in order to reveal the interactions between spinors and W_μ in physical spacetime. This also allows us to exploit the Lorentz condition (10), which simplifies finding the factorization of (15). Consequently, we focus on linearizing

$$\overline{R}_{|\nabla\cdot W=0} = (\tfrac{3}{2}W_\mu W^\mu + R), \qquad (16)$$

over the space of spinor operators.

Concentrating now on the interaction Lagrangian of the form

$$\mathcal{L}^{\rm int}_{\psi,W} = \lambda_W \overline{\psi}\mathcal{R}\psi, \qquad (17)$$

the most general ansatz for $\mathcal{R}$ is given by

$$\mathcal{R} = \eta(\ \lambda_V\gamma^\mu A_\mu + \lambda_A\gamma_5\gamma^\mu B_\mu + \lambda_S D\mathbf{1} + \lambda_P\gamma_5 C + \lambda_T\sigma^{\mu\nu}B_{\mu\nu}), \qquad (18)$$

where the λ's are (in general complex) coupling constants and η denotes an overall phase. This operator expansion makes use of the completeness of the Dirac algebra. The Dirac matrices are understood to be the curved-space analog of those forming the usual Dirac algebra (e.g., $\gamma^\mu(x) = \gamma^a e^\mu_a(x)$, e^μ_a is a vierbein, $\gamma_5(x) = [\det(e^\mu_a)]^{-1}\gamma_5^{\rm flat}$) together with coefficients that are tensor functions of the metric and Weyl field [e.g., $A_\mu = A_\mu(g, W)$, etc.]. Expanding the product $\mathcal{R}\otimes\mathcal{R}$ and equating it with (16) leads to the following set of constraints for the coefficient functions:

$$\overline{R}_{|\nabla\cdot W=0} = \eta^2(\ \lambda_V^2 A_\mu A^\mu - e^{-2}\lambda_A^2 B_\mu B^\mu + \lambda_S^2 D^2 + e^{-2}\lambda_P^2 C^2 + 2\lambda_T^2 B_{\mu\nu}B^{\mu\nu}), \qquad (19)$$

$$(\lambda_S\lambda_P CD - ie\lambda_T^2 B_{\mu\nu}\tilde{B}^{\mu\nu}) = 0, \qquad (20)$$

$$(\lambda_V\lambda_S DA_\mu - e^{-1}\lambda_P\lambda_T\epsilon_\mu{}^{\alpha\beta\nu}B_\alpha B_{\beta\nu}) = 0, \qquad (21)$$

$$(\lambda_S\lambda_A DB_\mu - e\lambda_V\lambda_T\epsilon_\mu{}^{\alpha\beta\nu}A_\alpha B_{\beta\nu}) = 0, \qquad (22)$$

$$(2\lambda_S\lambda_T DB_{\mu\nu} - ie^{-1}\lambda_P\lambda_T\epsilon^{\alpha\beta}{}_{\mu\nu}CB_{\alpha\beta} - e^{-1}\lambda_V\lambda_A\epsilon^{\alpha\beta}{}_{\mu\nu}B_\alpha A_\beta) = 0, \qquad (23)$$

where e denotes the vierbein determinant. What can we say regarding the coefficient functions appearing in the above equations? First of all, $\overline{R}$ is a function of W_μ and $g_{\mu\nu}$, so the coefficients must be defined in terms of these same quantities. Second, inspection of (16) implies $\lambda_T = \lambda_s = 0$. The only rank-two antisymmetric tensor built from the geometric fields is $W_{\mu\nu}$ (or a multiple thereof, where the proportionality factor can be a scalar function) so that $B_{\mu\nu} \sim W_{\mu\nu}$, but its square does not appear in $\overline{R}$, so $\lambda_T = 0$. Furthermore, maintaining a strictly local effective coupling requires $\lambda_S = 0$, for otherwise we should take $D \sim \sqrt{\frac{3}{2}W_\mu W^\mu + R}$, and expanding this (naive) square root generates an infinite series in these fields. Under these conditions, the set of equations (19) to (23) reduces to

$$\tfrac{3}{2}W_\mu W^\mu + R = \eta^2(\ \lambda_V^2 A_\mu A^\mu - e^{-2}\lambda_A^2 B_\mu B^\mu + e^{-2}\lambda_P^2 C^2) \qquad (24)$$

and

$$\epsilon^{\mu\nu\alpha\beta}A_\mu B_\nu = 0. \qquad (25)$$

The most general solution of the constraint (25) is obtained taking $A_\mu \sim B_\mu$. The only vector field in the problem is W_μ, so without loss of generality we now set

$$B_\mu = eA_\mu \equiv eW_\mu. \qquad (26)$$

Substituting this into (24) leads to the relations

$$\eta^2(\lambda_V^2 - \lambda_A^2) = \tfrac{3}{2} \qquad (27)$$

and

$$\eta^2\left(\lambda_P^2 C^2\right) = e^2 R. \qquad (28)$$

Additional relations between all these coupling parameters are obtained by the requirement of Hermiticity of the interaction Lagrangian:

$$\lambda_W^* \beta \mathcal{R}^\dagger \beta = \lambda_W \mathcal{R}, \qquad (29)$$

where β, following the convenient representation of the Dirac algebra in curved space according to Bargmann[24] and Schmutzer,[20] is a Hermitian matrix used to define the adjoint spinor, i.e., $\overline{\psi} = \psi^\dagger \beta$. Hermiticity of the geometric coupling implies the conditions

$$\begin{aligned} (\lambda_W \eta)^* \lambda_V^* &= (\lambda_W \eta)\lambda_V, \\ (\lambda_W \eta)^* \lambda_A^* &= (\lambda_W \eta)\lambda_A, \\ (\lambda_W \eta)^* \lambda_P^* &= -(\lambda_W \eta)\lambda_P. \end{aligned} \qquad (30)$$

We are now in the position to specify the interaction Lagrangian (17) completely. It might seem initially as if there would exist a variety of possible combinations of the parameters consistent with the Hermiticity, (30), and the matching conditions, (27) and (28), respectively. However, it turns out that for the case of a positive curvature scalar ($R > 0$) only one independent combination exists: λ_V, λ_A real, and λ_P pure imaginary, together with the phase $\eta = e^{i\pi/2}$. (We should note that a mathematical solution for these parameters can also be found if $R < 0$.) Therefore, we can choose λ_W such that the product $(\lambda_W \eta) \equiv \lambda$ is real. The explicit, factorized and Hermitian geometric interaction is then given by

$$\mathcal{L}^{\rm int}_{\psi,W} = \lambda \overline{\psi}[(\lambda_V + e\lambda_A\gamma_5)\gamma^\mu W_\mu + ie\gamma_5 R^{1/2}]\psi, \qquad (31)$$

where $R^{1/2}$ is the ordinary square root of the Riemann scalar. The remaining freedom $(\lambda_A^2 - \lambda_V^2) = \frac{3}{2}$ for the relative coupling strengths reveals the predominance of the pseudovector over the vector coupling.

In order to complete the effective spinor Lagrangian also requires some comments on possible mass terms. Let us consider the most general gauge invariant spinor sector to be appended to the vacuum Lagrangian (5), given by

$$\mathcal{L}_{\psi,W} = \overline{\psi}(i\gamma^\mu D_\mu - \mu)\psi + \mathcal{L}^{\rm int}_{\psi,W} \qquad (32)$$

where μ stands for a masslike parameter, which must have weight $\omega(\mu) = -\frac{1}{2}$. We stress that although μ may vary as a function of spacetime, the ratio of any two such "mass" fields is always meaningful in the sense that μ_1/μ_2 is *constant*.[25] The measurement of mass always requires a reference mass for comparison; hence, ratios of this sort are what we have in mind whenever we discuss values for dimensionful physical quantities. However, only the form of the effective theory in Riemannian spacetime is relevant. Consequently, we need not conjure up a new field $\mu(x)$, since the gauge invariant Yukawa term

$$\mathcal{L}_{\rm Yuk} = \Gamma_\psi\, \Phi^2\, \overline{\psi}\psi \qquad (33)$$

implies a constant fermion mass

$$m_0 = \Gamma_\psi\, v^2, \qquad (34)$$

after dynamical symmetry breaking, where Γ_ψ is the Yukawa coupling. According to this argument different, constant fermion masses on the effective level arise due to different Yukawa couplings between spinors and the scalar field Φ. However, it would be more satisfying if one could determine the masses, and thus the nature of the spinors involved in this effective theory:

$$[i\gamma^\mu D_\mu - m_0 + \lambda(\lambda_A e\gamma_5 + \lambda_V)\gamma^\mu W_\mu + ie\lambda\gamma_5 R^{1/2}]\psi = 0, \qquad (35)$$

defined by the above equation of motion, which follows from (32) after symmetry breaking. This can in fact be done.

Because of the geometric coupling terms, the Proca equation contains fermionic source currents

$$j_V^\mu = (\lambda\lambda_V)\overline{\psi}\gamma^\mu\psi \qquad (36)$$

and

$$j_A^\mu = (\lambda\lambda_A)e\overline{\psi}\gamma_5\gamma^\mu\psi, \qquad (37)$$

yielding the effective field equation

$$\nabla_\mu W^{\mu\nu} = m_W^2 W^\nu + j_V^\nu + j_A^\nu. \qquad (38)$$

Taking the (Christoffel) covariant divergence of (38) implies

$$\nabla_\mu j_A^\mu = 0, \qquad (39)$$

since we already know that the W_μ field satisfies the Lorentz condition $\nabla_\mu W^\mu = 0$, and the vector current is divergenceless, as one can show directly from the spinor equation of motion (35). On the other hand, calculating the divergence of the pseudovector current using the spinor field equation, one obtains

$$\nabla_\mu j_A^\mu = 2m_0\, e\overline{\psi}\gamma_5\psi - 2iR^{1/2}\,\overline{\psi}\psi. \qquad (40)$$

What does this quasi-inconsistency imply? It reveals the nature of the spinors involved in the theory. The inhomogeneity vanishes identically provided the spinors are of the form $\psi \sim (\mathbf{1} \pm e\gamma_5)\psi'$. In other words the Weyl vector couples only to strictly chiral fermions. The terms on the right-hand side of (40) vanish if ψ is replaced either by ψ_L or by ψ_R, but not both. This then rules out couplings to fermions that acquire masses by other mechanisms, such as electrons (and quarks), which exist in nature with both chiralities. Since the standard model contains left-handed neutrinos among its physical spectrum, we therefore postulate a new neutrino-neutrino interaction mediated by a (massive) Weyl vector. We note that the curvature term $\sim \gamma_5 R^{1/2}$ then drops out. We are thus left with

$$\mathcal{L}^{\rm int}_{\psi_L,W} = \sum_{i,j} \lambda_{ij} \overline{\psi}^i_L (\lambda_V + \lambda_A e\gamma_5)\, \not{W} \psi^j_L , \qquad (41)$$

where i,j run over the three flavors (e,μ,τ) and $\psi^i_L = \frac{1}{2}(1-e\gamma_5)\psi^i$ is the ith neutrino species. The coupling matrix $(\lambda_{ij}) \equiv \hat{\lambda}$ allows for flavor mixing among the various species, since the divergencelessness of j^μ_A is still maintained for multiple flavors. The abstract interaction introduced in (13) is thus represented in concrete terms by the explicit coupling operator in (41).

B. Couplings to other fields

Scalar fields, as we have already seen, couple to Weyl's field through the covariant derivative. All scalars ϕ (in four dimensions) have weight $\omega(\phi) = -\frac{1}{2}$, so the minimal coupling is universal. They may in addition also couple to W_μ in the nonminimal, i.e., geometric fashion: $\mathcal{L}^{\rm int}_\phi = \xi_\phi\, \phi^2 \overline{R}$, where ξ_ϕ can in general depend on the "flavor" of the scalar field involved. If any $\xi_\phi = -\frac{1}{6}$, the minimal and geometric interactions between W_μ and ϕ mutually cancel, but the scalar sector contained in (5) still remains gauge invariant.

Additional interactions with other gauge fields can be addressed immediately. In the case of Yang-Mills (YM) fields, the only possible coupling to W_μ is geometric, and the only locally scale and YM gauge-invariant term of this form is

$$\mathcal{L}^{\rm int} = \zeta {\rm Tr}(W_{\mu\nu}\mathcal{F}^{\mu\nu}), \qquad (42)$$

where $\mathcal{F}_{\mu\nu} = g^a F^a_{\mu\nu}, a = 1,\ldots,\dim(\mathcal{G})$ is the field-strength tensor. However, this interaction Lagrangian is *not* invariant under the action of the symmetry group $\mathcal{G}$. Furthermore the trace over the group generators g^a identically vanishes. Hence there are no couplings between W_μ and non-Abelian gauge fields.

The situation for *Abelian* vector bosons is somewhat more subtle. If $F_{\mu\nu}$ denotes the field strength of a U(1) gauge field, then

$$\mathcal{L}^{\rm int}_{\rm Abelian} = \zeta W_{\mu\nu} F^{\mu\nu} \qquad (43)$$

is simultaneously locally scale and U(1) invariant. To be specific, take $F_{\mu\nu}$ to be the electromagnetic field-strength tensor. Then (43) implies a coupling between photons and W_μ, where ζ denotes the corresponding strength. Nevertheless, after dynamical symmetry breaking, the Weyl vector becomes massive, while the photon remains massless. Such a coupling would allow real photons to transform into a real W_μ, thus violating energy-momentum conservation. We therefore exclude it as a potential interaction.

We are now in the position to write down the complete (classical) effective theory representing matter interacting with the Weyl vector. The effective vacuum sector describes Einstein gravity with a dynamically induced cosmological term:

$$A^{\rm vac}_{\rm eff} = -\frac{1}{16\pi G_N} \int d^4x\sqrt{-g}\,(R+\Lambda), \qquad (44)$$

where the induced Newton's constant G_N and cosmological term Λ are computed in terms of the scalar field vacuum expectation value and the dimensionless parameters of the invariant vacuum according to Eq. (9). The complete interacting matter sector is given by

$$A^{\rm matt}_{\rm eff} = \int d^4x\sqrt{-g}\,\left(\mathcal{L}_W + \mathcal{L}_{\psi_L,W}\right), \qquad (45)$$

where

$$\mathcal{L}_W = a(W_{\mu\nu}W^{\mu\nu} - 2m_W^2 W_\mu W^\mu)$$

is the Proca sector together with the mass parameter given in (9), and

$$\mathcal{L}_{\psi_L,W} = \overline{\Psi}_L(\hat{1} i \not{D} - i\hat{\lambda}\mathcal{R})\Psi_L$$

gives the neutrino sector including the coupling to Weyl's field. We have introduced the compact notation $\Psi_L = (\psi_{\nu_e}, \psi_{\nu_\mu}, \psi_{\nu_\tau})$ to denote the three flavors of left-handed neutrinos. The spinor coupling is

$$\mathcal{R} = i[(\lambda_V + \lambda_A e\gamma_5)\, \not{W} + ie\gamma_5\sqrt{R}], \qquad (46)$$

where λ_V, λ_A are real couplings subject to the constraint

TABLE I. Coupling pattern between Weyl vector and matter fields.

Field	Minimal coupling	Geometric coupling
Spin-0		
(scalars): ϕ	$\omega(\phi)W_\mu\phi$	$\xi_\phi\phi^2\overline{R}$
Spin-$\frac{1}{2}$		
(neutrinos): Ψ_L	No	$-i\hat{\lambda}\overline{\Psi}_L\mathcal{R}\Psi_L$
(quarks, massive leptons): Ψ	No	No
Spin-1		
(Yang-Mills): B^a_μ	No	No
(photon): A_μ	No	$\zeta F_{\mu\nu}W^{\mu\nu}$ (only if $m_W = 0$)

$$(\lambda_A^2 - \lambda_V^2) = \tfrac{3}{2}. \qquad (47)$$

One is free to include other gauge and matter fields in $A_{\rm eff}^{\rm matt}$, e.g., coming from the QCD Lagrangian, but these extra fields (the quarks and gluons in this case) do not couple to W_μ. The status of couplings is summarized in Table I, which exhibits the differences between minimal and geometric couplings for fundamental fields. Because of this pattern, the residuum of the Weyl vector is a form of dark matter, and so contributes to the closure density of the Universe.

IV. SUMMARY AND DISCUSSION

Motivated in part by axiomatic attempts to derive the properties of Riemannian geometry from locally observable properties of the world, we have reconsidered Weyl's geometry from the viewpoint of interacting field theory. Given that these axiomatic approaches all end up by assigning a Weylian structure to spacetime, it is a challenge to find a way to restrict this geometry so that we can "explain" or derive the apparent Riemannian structure of general relativity. In responding to this, we have taken the point of view that this desired reduction must proceed dynamically, and since this presupposes the existence of interactions, we have been led therefore to phrase our approach to the problem in terms of matter fields in a Weylian spacetime.

That interactions must play a decisive role in this reduction is already evident from the analysis of the pure vacuum sector. Although one is free to exploit the gauge invariance to choose a gauge wherein the metric obeys Einstein's equations, this contact to ordinary gravitation becomes explicit only for a particular gauge choice, and is thus conceptually unsatisfactory. However, introduction of a gauge-invariant vacuum constraint has the virtue of leading to metric field equations proportional to the Einstein tensor in *all* gauges. Though here one is still at liberty to fix a gauge, the constraint represents an explicit interaction term between the geometric fields and the scalar (Lagrange multiplier) field. Then, a unique gauge invariant kinetic and potential term can be included to form a complete action. This then raises the possibility that the Riemannian structure we perceive today is induced via a dynamical mechanism which breaks the Weyl invariance of the vacuum. If this turns out to be the case, this would imply that the Riemannian structure observable in our world may be understood as a "low-energy" consequence, or phase, of a more symmetric (i.e., Weylian) spacetime geometry, which is manifested only at high energies or temperatures.

The issue of dynamically closing the gap can be addressed quantitatively. The proper approach is to compute the Vilkovisky-DeWitt effective potential for the scalar field Φ.[13,14] We expect this potential will exhibit a nontrivial minimum vacuum configuration for Φ, the value of which is given in terms of the dimensionless vacuum parameters (a, ξ, λ_Φ). In fact, the effective potential for a scalar, nonminimally coupled to the Riemannian curvature scalar (which is identical to our constrained vacuum action if $W_\mu = 0$) has been calculated recently.[15] Accordingly, the 1-loop effective potential indeed possesses a nontrivial minimum:

$$V_{\rm VD}\,|_{\langle\Phi\rangle} = -\frac{1}{2}\left(\frac{3}{16\pi^2}\right)\frac{(\lambda\langle\Phi\rangle^2)^2}{144\xi^2} < 0.$$

Furthermore, these calculations do not require the theory to be renormalizable. Thus, lack of renormalizability does not prevent us from defining the effective Riemannian theory (44)–(46). A detailed calculation of $V_{\rm VD}$ for the full gauge invariant vacuum (5) is outside the scope of the present paper, and will be reported elsewhere.[26]

Since the constraint discussed above gives rise to an interaction, it is natural to explore the pattern of gauge-invariant interactions involving other types of fields, which we have done for spinors and vectors. The lack of minimal coupling of spinors to W_μ at the gauge-invariant level is intriguing, and provided the motivation to look for so-called geometric or nonminimal couplings. Our explicit geometric spinor coupling is completely analogous to the geometric scalar coupling that appears in the constraint. To represent this coupling in terms of explicit spinor operators, we have made use of Dirac's linearization scheme for defining the square root. Requiring a Hermitian interaction and demanding internal consistency of the effective field equations has led to a unique coupling, as well as to the class of fermions permitted to couple in this way. The latter turn out to be strictly massless, and of a given chirality, while the former geometric interaction can be written as an (A-V)-type coupling. Consistency is maintained for multiple flavors, and we identify these fermions with the three species of standard model neutrinos. Thus, we predict a new neutrino-neutrino force mediated by a vector particle which is the low-energy remnant of Weyl's vector. The couplings appear as phenomenological parameters. Speculations about the existence of new galactic-range forces coupling neutrinos have already been raised in connection with neutrinos detected from supernova 1987A.[27] Thus, this presumably feeble ν-ν interaction may be the only signature of an underlying Weylian structure of spacetime. It is rather striking to realize that Weyl geometry, originally invented to unify two known interactions, namely gravity and electromagnetism, may instead give rise to an additional force interfering with the weak interaction.

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APPENDIX A: RESUME OF WEYL GEOMETRY

We recall the basic definitions and properties of Weyl geometry and establish the notation employed in this paper.

A Weyl space is a conformal manifold consisting of a real four-dimensional manifold and a conformal equivalence class of local Lorentz metrics. In addition to the metric tensor $g_{\mu\nu}$, a Weyl space contains an equally fundamental vector field W_μ, which is the gauge field of local scale transformations. Thus, in addition to the usual general coordinate transformations, one has Weyl transformations, under which the metric and Weyl vector transform as

$$g'_{\mu\nu} = e^{\omega(g)\Lambda(x)} g_{\mu\nu}, \tag{A1}$$

$$W'_\mu = W_\mu - \partial_\mu \Lambda(x), \tag{A2}$$

respectively; $\Lambda(x)$ is a real-valued function and $\omega(g)$, the Weyl weight of the metric, is a real constant. It will prove convenient to set $\omega(g) = 1$, though any other choice is equally good. Once $\omega(g)$ is fixed, the Weyl weights of other fields may be determined unambiguously. The inverse metric has weight $\omega(g^{-1}) = -\omega(g)$. The Weyl-gauge transformation extends to any tensor or spinor field T in Weyl space:

$$T'(x) = e^{\omega(T)\Lambda(x)} T(x), \tag{A3}$$

where the weight $\omega(T)$ is real.

Weyl geometry contains a unique, gauge-invariant affine connection $\overline{\Gamma}$ and a doubly covariant derivative D that is linear, Leibniz, general coordinate and Weyl covariant. Thus for a general tensor field T,

$$D_\mu T^{\alpha\cdots}_{\beta\cdots} = \partial_\mu T^{\alpha\cdots}_{\beta\cdots} - \overline{\Gamma}^\lambda_{\beta\mu} T^{\alpha\cdots}_{\lambda\cdots} + \overline{\Gamma}^\alpha_{\lambda\mu} T^{\lambda\cdots}_{\beta\cdots} + \cdots + \omega(T) W_\mu T^{\alpha\cdots}_{\beta\cdots}, \tag{A4}$$

so that $\omega(DT) = \omega(T)$. The metric, Weyl vector and connection are not independent, but are correlated by the (doubly) covariant constancy of the metric:

$$D_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \overline{\Gamma}^\lambda_{\alpha\mu} g_{\lambda\beta} - \overline{\Gamma}^\lambda_{\beta\mu} g_{\alpha\lambda} + W_\mu g_{\alpha\beta} = 0. \tag{A5}$$

Solving for $\overline{\Gamma}$ (assuming vanishing torsion) yields

$$\overline{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \tfrac{1}{2}\left(\delta^\alpha_\mu W_\nu + \delta^\alpha_\nu W_\mu - g_{\mu\nu} W^\alpha\right), \tag{A6}$$

where $\Gamma^\alpha_{\mu\nu}$ is the ordinary Christoffel connection. From (A1), (A2), and (A6) it is easy to check that $\omega(\overline{\Gamma}) = 0$. The explicit dependence of the Weyl connection on W_μ shows convincingly that this vector is an integral part of the geometry of Weyl space.

The curvature tensor in Weyl space is given by

$$\overline{R}^\lambda_{\mu\nu\kappa} = \partial_\kappa \overline{\Gamma}^\lambda_{\mu\nu} - \partial_\nu \overline{\Gamma}^\lambda_{\mu\kappa} + \overline{\Gamma}^\eta_{\mu\nu}\overline{\Gamma}^\lambda_{\kappa\eta} - \overline{\Gamma}^\eta_{\mu\kappa}\overline{\Gamma}^\lambda_{\nu\eta}, \tag{A7}$$

and is gauge invariant: $\omega(\overline{R}^\lambda_{\mu\nu\kappa}) = 0$. This tensor possesses fewer algebraic symmetries than its Riemannian counterpart, and this fact has important consequences when considering invariant actions built up from it. There are *two* distinct rank-two contractions

$$\overline{R}_{\mu\kappa} = \overline{R}^\lambda_{\mu\lambda\kappa} \tag{A8}$$

and

$$W_{\mu\nu} = \tfrac{1}{2}\overline{R}^\lambda_{\lambda\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu, \tag{A9}$$

while the analog of the scalar curvature is

$$\overline{R} = g^{\mu\nu}\overline{R}_{\mu\nu} = R + 3\nabla_\mu W^\mu + \tfrac{3}{2} W_\mu W^\mu, \tag{A10}$$

where R is the scalar Riemann curvature, and ∇_μ refers to the Christoffel covariant derivative. The associated Weyl weights are $\omega(\overline{R}_{\mu\nu}) = \omega(W_{\mu\nu}) = 0$ and $\omega(\overline{R}) = -1$, respectively.

Finally we note that from the cyclic identity

$$\overline{R}_{\lambda\mu\nu\kappa} + \overline{R}_{\lambda\kappa\mu\nu} + \overline{R}_{\lambda\nu\kappa\mu} = 0 \tag{A11}$$

we have

$$\epsilon^{\lambda\mu\nu\kappa}\overline{R}_{\lambda\mu\nu\kappa} = 0, \tag{A12}$$

which we will have occasion to use in the paper.

APPENDIX B: ABSENCE OF MINIMAL COUPLING

We first consider fundamental two-spinors ξ^A and $\phi_{\dot{B}}$. Under a Weyl transformation these and the vierbeins e^μ_a transform as

$$\xi'^A = e^{\omega(\xi)\Lambda(x)} \xi^A, \tag{B1}$$

$$\phi'_{\dot{B}} = e^{\omega(\phi)\Lambda(x)} \phi_{\dot{B}}, \tag{B2}$$

and

$$e'^\mu_a = e^{\omega(e)\Lambda(x)} e^\mu_a, \tag{B3}$$

respectively. Up to this point the separate weights $\omega(\xi)$, $\omega(\phi)$, and $\omega(e)$ are completely arbitrary and remain so within the strict framework of Weyl transformations. However, nontrivial relations connecting these weights arise when treating the dynamics of the fields themselves. To this end, consider the spinor kinetic Lagrangian

$$\mathcal{L}_{\rm kin} = \frac{i}{\sqrt{2}}(\xi^{\dot{B}}\sigma^\mu_{A\dot{B}} D_\mu \xi^A + \phi_A \sigma^{\mu A\dot{B}} D_\mu \phi_{\dot{B}}) + \text{H.c.}, \tag{B4}$$

where D_μ is the spinor covariant derivative in Weyl space:[18]

$$D_\lambda \xi^A = \partial_\lambda \xi^A + \overline{\Gamma}^A_{B\lambda}\xi^B + \omega(\xi^A) W_\lambda \xi^A, \tag{B5}$$

where

$$\overline{\Gamma}^A_{B\lambda} = \tfrac{1}{2}\sigma^{A\dot{X}}_a \sigma^b_{B\dot{X}} \overline{\Gamma}^a_{b\lambda} - \tfrac{1}{8}\delta^A_B \overline{\Gamma}^a_{a\lambda}, \tag{B6}$$

are the spin coefficients and

$$\overline{\Gamma}^a_{b\lambda} = e^a_\mu[\partial_\mu e^\mu_b + \overline{\Gamma}^\mu_{\nu\lambda} e^\nu_b + \omega(e) W_\lambda e^\mu_b], \quad \text{(B7)}$$

where $\overline{\Gamma}^\mu_{\nu\lambda}$ is the Weyl connection in (A6). Now, from $\overline{\Gamma}^\mu_{\mu\lambda} = \partial_\lambda \ln(\sqrt{-g}) + 2W_\lambda$, it follows that

$$\overline{\Gamma}^a_{a\lambda} \,|_W = 4[\tfrac{1}{2} + \omega(e)]W_\lambda. \quad \text{(B8)}$$

Next, the remaining W-dependent pieces in the connection are

$$\overline{\Gamma}^a_{b\lambda} \,|_W = [\delta^a_b(\tfrac{1}{2} + \omega(e))W_\lambda + \tfrac{1}{2}(\delta^\mu_\lambda e^a_\mu e^\nu_b W_\nu - e^a_\mu e^\nu_b g_{\lambda\nu} W^\mu)]. \quad \text{(B9)}$$

Then,

$$\tfrac{1}{2}\sigma^\lambda_{A\dot{Z}}(\sigma^{A\dot{X}}_a \sigma^b_{B\dot{X}})\overline{\Gamma}^a_{b\lambda} \,|_W = [\tfrac{5}{4} + \omega(e)]\sigma^\lambda_{B\dot{Z}} W_\lambda, \quad \text{(B10)}$$

where we have used the identities

$$\sigma^{A\dot{X}}_a \sigma^a_{B\dot{X}} = \delta^A_B \delta^{\dot{X}}_{\dot{X}} = 2\delta^A_B \quad \text{(B11)}$$

and

$$\sigma_{bA\dot{Z}} \sigma^b_{B\dot{X}} = \gamma_{BA}\, \gamma_{\dot{X}\dot{Z}}\,. \quad \text{(B12)}$$

Here, $\gamma_{AB} = -\gamma_{BA}$ is the spinor metric used to raise and lower spinor indices and satisfies $\gamma^{AC}\gamma_{BC} = \delta^A_B$. Adding up all the W-dependent contributions in $\sigma^\lambda_{A\dot{Z}} D_\lambda \xi^A$ yields

$$[1 + \tfrac{1}{2}\omega(e) + \omega(\xi^A)]\, \sigma^\lambda_{A\dot{Z}} W_\lambda \xi^A. \quad \text{(B13)}$$

In a similar fashion, we find that all W-dependent pieces in $\sigma^{\mu A\dot{B}} D_\mu \phi_{\dot{B}}$ assemble into

$$[\tfrac{1}{2} - \tfrac{1}{2}\omega(e) + \omega(\phi_{\dot{B}})]\, \sigma^{\lambda A\dot{B}} W_\lambda \phi_{\dot{B}}. \quad \text{(B14)}$$

The weightless tangent space arrays $\sigma^a_{A\dot{B}}$ and $\sigma^{aA\dot{B}}$ are proportional to the identity and the three Pauli matrices for $a = 1,2,3$, respectively. Then, from $\sigma^\mu_{A\dot{B}} = e^\mu_a \sigma^a_{A\dot{B}}$ and $\sigma^{\mu A\dot{B}} = g^{\mu\nu}\sigma^{A\dot{B}}_\nu$, it follows that $\omega(\sigma^\mu_{A\dot{B}}) = \omega(e)$ and $\omega(\sigma^{\mu A\dot{B}}) = -1 - \omega(e)$. From the property $\omega(D\xi) = \omega(\xi)$ and the condition (3) one finds that

$$2\omega(\xi) + \omega(e) = -2, \quad 2\omega(\phi) - 1 - \omega(e) = -2, \quad \text{(B15)}$$

which immediatly shows that (B13) and (B14) both vanish identically. Hence there is no minimal coupling between two-spinors and the Weyl vector. The same conclusion holds for Dirac spinors

$$\psi = \begin{pmatrix} \phi_A \\ \xi^{\dot{A}} \end{pmatrix}, \quad \text{(B16)}$$

since the Dirac matrices (in the Weyl basis) are realized as

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a_{A\dot{B}} \\ \sigma^{aA\dot{B}} & 0 \end{pmatrix}. \quad \text{(B17)}$$

Finally, requiring that ψ have a well-defined Weyl transformation,

$$\psi' = e^{\omega(\psi)\Lambda(x)\psi}, \quad \text{(B18)}$$

fixes the value of *all* the spinors and tetrad weights at once: $\omega(\psi) = \omega(\xi) = \omega(\phi) = -\frac{3}{4}$ and $\omega(e) = -\frac{1}{2}$. In particular, this value for the weight of the Dirac spinor field [in units of $\omega(g)$] is just what one expects from a naive dimensional analysis, but its raison d'être can now be traced back to the particulars of the Weyl-transformation properties of two-spinors. Thus, the covariant derivative operator for Dirac spinors in a Weyl space reduces to the expression given in (11).

[1] H. Weyl, Sitzung. d. Preuss. Akad. d. Wiss. **465** (1918).
[2] H. Weyl, *Raum Zeit Materie* (Springer-Verlag, Heidelberg, New York, 1970).
[3] A. Einstein, Sitzung. d. Preuss. Akad. d. Wiss. **478** (1918).
[4] R. Utiyama, Prog. Theor. Phys. **50**, 2080 (1973); P. G. O. Freund, Ann. Phys. (N.Y.) **84**, 440 (1974); R. Utiyama, Prog. Theor. Phys. **53**, 565 (1975).
[5] P. A. M. Dirac, Proc. R. Soc. London **A333**, 403 (1973).
[6] K. Hayashi, M. Kasuya, and T. Shirafugi, Prog. Theor. Phys. **57**, 431 (1977); K. Hayashi and T. Kugo, *ibid.* **61**, 334 (1979).
[7] H. Cheng, Phys. Rev. Lett. **61**, 2182 (1988).
[8] D. Hochberg and J. T. Wheeler, Bartol Research Institute Report No. BA-89-24, 1989 (unpublished); D. Hochberg, in *Last Workshop on Grand Unification*, Chapel Hill, North Carolina, 1989, edited by P. H. Frampton (World Scientific, Singapore, 1989).
[9] J. T. Wheeler, Phys. Rev. D **41**, 431 (1990).
[10] J. Ehlers, F. A. E. Pirani, and A. Schild, in *General Relativity*, edited by L. O'Raifeartaigh (Oxford, London, 1972); M. Enosh and A. Kovetz, J. Math. Phys. **14**, 572 (1973); J. Ehlers, in *Relativity, Astrophysics and Cosmology*, edited by W. Israel (Reidel, Dordrecht, 1973).
[11] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).
[12] See, e.g., A. Zee, Phys. Rev. Lett. **42**, 417 (1979).
[13] G. A. Vilkovisky, Nucl. Phys. **B234**, 125 (1984).
[14] B. S. DeWitt, in *Architecture of Fundamental Interactions at Short Distances*, Proceedings of the Les Houches Summer School, Les Houches, France, 1985, edited by P. Ramond and R. Stora (North-Holland, Amsterdam, 1987), Vol. 44.
[15] H. T. Cho, Phys. Rev. D **40**, 3302 (1989).
[16] V. Fock and D. Iwanenko, Comptes Rendus **188**, 1470 (1929).
[17] If a gauge-invariant regularization scheme exists, the vacuum sector would be renormalizable, since $\mathcal{L}_{\text{quadratic}}$ would then be the most general counterterm Lagrangian—the number of possible counterterms is finite. The "nonstandard" contractions appearing there are due to the symmetry properties of the curvature tensor.
[18] J. Audretsch, Phys. Rev. D **27**, 2872 (1983).
[19] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[20] E. Schmutzer, *Relativistische Physik* (Teubner, Leipzig, 1968).
[21] J.-M. Levy-Leblond, Commun. Math. Phys. **6**, 286 (1967).

[22]M. Greiner, W. Scheid, and R. Herrmann, J. Phys. A **21**, 3227 (1988).

[23]R. Herrmann, G. Plunien, M. Greiner, and W. Scheid, J. Mod. Phys. A **4**, 4961 (1989).

[24]V. Bargmann, Sitzung. d. Preuss. Akad. Math. Kl. **346** (1932).

[25]T. Fulton, F. Rohrlich, and L. Witten, Rev. Mod. Phys. **34**, 442 (1962).

[26]D. Hochberg and G. Plunien, work in progress.

[27]S. Pakvasa, W. A. Simmons, and T. J. Weiler, Phys. Rev. D **39**, 1761 (1989).