Does an unspecified cosmological constant solve the problem of time in quantum gravity?

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In unimodular gravity, an unspecified cosmological constant appears as a variable canonically conjugate to a four-volume variable, the cosmological time. It was suggested that this time sets the conditions of quantum measurements and solves thereby the interpretation problems of quantum geometrodynamics. By analyzing the relationship of the cosmological time to hypertime (the collection of spacelike hypersurfaces), we highlight the difficulties of such a position. The constraint system of parametrized unimodular gravity implies that the cosmological time labels only equivalence classes formed by hypersurfaces separated by a zero four-volume, while individual spacelike hypersurfaces within an equivalence class are physically irrelevant. As a result, unless complemented by a hypertime variable, cosmological time does not uniquely set the conditions for measuring geometric variables either in the classical or in the quantum theory.

I. INTRODUCTION

In the same paper in which he set the foundations of the general theory of relativity, Einstein remarked that the law of gravitation can be simplified by a special choice of coordinates.[1] To illustrate this point, he evoked the unimodular [$\det(\gamma_{AB})=-1$] coordinate condition. Several years later, he proposed a relaxed law of gravitation which admitted an unspecified cosmological constant.[2] Such a law can be obtained by imposing the unimodular condition before rather than after the variation. The problem which the cosmological constant (or rather the experimental lack of it) presents to quantum cosmologists helps to explain why a number of authors have recently rediscovered, revived, and reviewed Einstein's procedure.[3,4] Concurrently, an alternative method of introducing an unspecified cosmological term, based on variational principles with an auxiliary three-form field, appeared in the literature.[5]

Unimodular gravity was cast into canonical form by Henneaux and Teitelboim[6] and by Unruh.[4] The cosmological constant appears in the formalism as a canonical momentum; it is accompanied by a canonically conjugate coordinate, the "cosmological time." Henneaux and Teitelboim have shown that the change of the cosmological time equals the four-volume enclosed between the initial and the final hypersurfaces.[6] (In a related development, Sorkin used the four-volume time in a path-integral approach to quantum gravity.[7]) The unimodular condition breaks the diffeomorphism invariance. Nevertheless, the canonical formulation of unimodular gravity leads to covariant spacetime action principle with auxiliary scalar and vector density fields.[6] The action principle of general relativity can be regarded as analogous to the Jacobi form of this new action principle.[8]

The appearance of the cosmological time in unimodular gravity raised hopes that this theory can solve the problem of time in quantum geometrodynamics. In the canonical formulation of general relativity, the problem of time arises because nothing in the structure of the resulting constraints helps us to distinguish the true dynamical degrees of freedom from the quantities which determine the hypersurface. Such a formalism does not offer a suitable framework for describing experiments which would measure a given dynamical variable at a given instant. In the Dirac constraint quantization, the constraints yield the Wheeler-DeWitt equation. This is a second-order functional differential equation for the states $\Psi[\mathbf{g}]$ considered as functionals of the three-geometry. It resembles the Klein-Gordon equation for a relativistic particle on a curved background. Like the Klein-Gordon equation, the Wheeler-DeWitt equation leads (at least formally) to a conserved current.[9] However, such a current gives a positive-definite inner product only on stationary backgrounds.[10] Superspace, which plays the role of the background for the Wheeler-DeWitt equation, is not stationary,[11] and the probabilistic interpretation of the state functional $\Psi[\mathbf{g}]$ becomes questionable.

Unruh and Wald suggested that unimodular gravity avoids these difficulties.[4,12] They argued that any reasonable quantum theory should contain a parameter (which they called Heraclitian time) whose role is to set the conditions for measuring quantum variables and provide the temporal ordering of such measurements. They surmised that the cosmological time of unimodular gravity is such a parameter.[13] Because the Hamiltonian constraint is linear in the cosmological constant, its imposition on the state functional gives a Schrödinger equation in the cosmological time τ. The Schrödinger equation implies that the ordinary quadratic norm is conserved in τ. It is thus tempting to interpret $\Psi(\tau;\mathbf{g}]$ as the probability amplitude for the outcomes of measurements carried on geometric variables at a given instant of τ.[14]

The problem with this suggestion is that the cosmological time is not in any obvious way related to the standard concept of time in the theory of relativity. In classical relativity, time is a foliation of the spacetime M by hypersurfaces, while space is a congruence of worldlines in M. A spacelike hypersurface is an instant of time, a timelike worldline is a point of space. Time (and space) are relative, because neither the time foliation nor the space congruence are unique. To deal with all possible instants of time at once, it is best to identify time with a collection of all spacelike hypersurfaces. We shall call such a collection hypertime, or many-fingered time. To label the elements of hypertime, one needs to use functions of three coordinates rather than a single real parameter of Newtonian mechanics.[15] The basic canonical variables $g_{ab}(x)$ and $p^{ab}(x)$ which unimodular gravity shares with geometrodynamics are always supposed to be measured *on a single spacelike hypersurface* rather than *at a single value of the cosmological time*. It thus remains obscure in what sense the cosmological time "sets the conditions of quantum measurements."[16] Our aim is to clarify its relation to hypertime and, by doing that, to show that the claim that unimodular gravity solves the problem of time is misleading.

The basic tool of our analysis is the technique which we developed for discussing quantum gravity obtained by imposing a set of coordinate conditions prior to variation. The technique was introduced in conjunction with Gaussian coordinate conditions[17] and later applied to harmonic coordinate conditions.[18] The coordinate conditions are adjoined to the action with a set of Lagrange multipliers. The additional terms in the action break the diffeomorphism invariance of the theory and introduce a source into the Einstein law of gravitation. The source can be interpreted as a material system, the reference fluid, which is coupled to gravity. Diffeomorphism invariance is then restored by parametrizing the action.[19] The privileged coordinates stipulated by the coordinate conditions are expressed as functions of arbitrary label coordinates and promoted to field variables side by side with the metric. These functions play the role of potentials whose variation yields the equations of motion of the reference fluid.

The fluid potentials mark the spacetime events. In their role of canonical coordinates, they specify an embedding in the encompassing spacetime, i.e., they represent hypertime. For suitable coordinate conditions (like the Gaussian or the harmonic coordinate conditions) the diffeomorphism-invariant parametrized action leads to the familiar super-Hamiltonian and supermomentum constraints on the phase space of the geometric variables g_{ab}, p^{ab} extended by the embedding variables X^A, P_A. These constraints can be resolved with respect to the embedding momenta P_A and then imposed on the states $\Psi[X^A,g_{ab}]$. This yields a functional Schrödinger equation which, with some qualifications, can be said to solve the problem of time.

We then apply the same procedure to the unimodular coordinate conditions and show why it does not work. We adjoin the unimodular condition to the Hilbert action with a Lagrange multiplier Λ and restore the diffeomorphism invariance by turning the unimodular coordinates X^A into field variables (Sec. II). The parametrized action $S[X^A,\Lambda,\gamma_{\alpha\beta}]$ is then cast into canonical form on the extended phase space g_{ab}, p^{ab}, X^A, P_A. The Lagrange multiplier Λ becomes thereby a dynamical variable on the embedding sector of the phase space. The embedding variables are subject to $3\infty^3$ (primary) supermomentum constraints. The variation of the lapse and the shift leads to $3\infty^3$ (secondary) gravitational supermomentum constraints and ∞^3 (secondary) super-Hamiltonian constraints. The total system of constraints is first class. It is related to the Henneaux-Teitelboim system by a canonical transformation (Appendix).

The constraints can be split into $4\infty^3-1$ constraints on the embedding variables, $4\infty^3-1$ constraints on the geometric variables, and one single constraint coupling the geometric variables to the embedding variables. This count is radically different from a single set of $4\infty^3$ constraints of the Gaussian or harmonic gravity. It explains why unimodular gravity does not solve the problem of time (Sec. III). The $4\infty^3-1$ constraints on the embedding variables generate displacements of embeddings under which a displaced embedding is separated from an old one by a zero four-volume. The $4\infty^3-1$ constraints on the geometric variables generate changes of the intrinsic geometry and the extrinsic curvature under such displacements. The embedding constraints can be cast by a canonical transformation into the form in which they assert that $4\infty^3-1$ embedding momenta vanish. The conjugate embedding coordinates label the embeddings within an equivalence class $\mathcal{E}$ of embeddings with a zero four-volume separation. Because the $4\infty^3-1$ constraints on the embedding variables are totally independent of the $4\infty^3-1$ constraints on the metric variables, the displacements of embeddings which the former constraints generate are in no way correlated to the changes of geometry. This is to be interpreted as the statement that the coordinates canonically conjugate to the $4\infty^3-1$ new embedding momenta, i.e., the individual embeddings within an equivalence class $\mathcal{E}$, are physically irrelevant.

The single remaining embedding momentum is the cosmological constant and its conjugate coordinate is the cosmological time. The cosmological time labels the equivalence classes $\mathcal{E}_\tau$ by giving the four-volume separation of an arbitrary embedding in $\mathcal{E}_\tau$ from a fiducial embedding in $\mathcal{E}_0$. The single remaining constraint which couples the embedding variables to the geometric variables correlates the changes of geometric variables to the change of τ, i.e., to the passage from one equivalence class $\mathcal{E}_\tau$ to another. The only new physical objects introduced by unimodular gravity are thus the equivalence classes $\mathcal{E}_\tau$ (labeled by the cosmological time τ), not the individual embeddings [labeled by the hypertime $X^A(x)$].

When imposed on the state functional $\Psi[X^A,g_{ab}]$, the $4\infty^3-1$ embedding constraints tell us that Ψ can depend on the unimodular embeddings X^A only via the cosmological time τ. The single constraint which couples the embeddings with geometry yields the Schrödinger equation for Ψ as a function of τ. If this were the only equation on the state, one could interpret Ψ as the probability amplitude that the metric $g_{ab}(x)$ has a given distribution at a time τ. However, Ψ must also satisfy the additional

$4\infty^3-1$ geometric constraints. The three-metric operator (or, rather, the three-geometry operator) does not commute with these constraints and the interpretation of Ψ as the probability amplitude becomes untenable. The $4\infty^3-1$ geometric constraints are actually a Wheeler-DeWitt equation which describes the evolution of states between embeddings of a single equivalence class $\mathcal{E}$. The cosmological time does not set the conditions of the measurement uniquely because it does not tell us on which one of the infinitely many hypersurfaces of the equivalence class the geometric variables are to be measured. The hypersurfaces with different values of τ are allowed to intersect and the cosmological time thus even does not provide the causal ordering required of the Heraclitian time. The fundamental reason why unimodular gravity does not and cannot solve the problem of time is simple: time in relativity is a collection of all spacelike hypersurfaces (hypertime) and no single parameter, like the cosmological time τ, is able to label uniquely so many instants.

II. PARAMETRIZED UNIMODULAR GRAVITY

Unimodular gravity is obtained by varying the Hilbert action

$$S^G[\gamma_{AB}]=\int_M d^4x\,|\det(\gamma_{AB})|^{1/2}R[\gamma_{AB}] \tag{2.1}$$

under the auxiliary condition

$$-|\det(\gamma_{AB})|^{1/2}+1=0\ . \tag{2.2}$$

The coordinates $X^A=(T,Z^k)$ in which Eq. (2.2) holds are called unimodular coordinates. We choose them so that they satisfy the Hilbert conditions:[20] The hypersurfaces $T=$const are spacelike and the reference lines $Z^k=$const are timelike.

One can incorporate Eq. (2.2) into the action principle by adjoining it to the Hilbert action by a Lagrange multiplier Λ:

$$S[\gamma_{AB},\Lambda]=S^G[\gamma_{AB}]+\int_M d^4X\,\Lambda(-|\det(\gamma_{AB})|^{1/2}+1)\ . \tag{2.3}$$

The additional term breaks the diffeomorphism invariance of S^G. The invariance is restored by parametrization: The privileged (unimodular) coordinates X^A are expressed in terms of arbitrary (label) coordinates x^α and turned into field variables.[19] The parameterized action $S[X^A,\gamma_{\alpha\beta},\Lambda]$ is defined by requirements that it be invariant under transformations of x^α and reduce to the old action (2.3) when the unimodular coordinates are used as labels:

$$S[X^A=\delta^A_\alpha x^\alpha,\gamma_{\alpha\beta},\Lambda]=S[\gamma_{AB},\Lambda]\ . \tag{2.4}$$

These two conditions fix the form of $S[X^A,\gamma_{\alpha\beta},\Lambda]$ to

$$S[X^A,\gamma_{\alpha\beta},\Lambda]=\int_M d^4x(|\det(\gamma_{\alpha\beta})|^{1/2}(R[\gamma_{\alpha\beta}]-\Lambda)+\Lambda\tilde{X}[X^A])\ , \tag{2.5}$$

where

$$\tilde{X}=\frac{1}{4!}\delta_{ABCD}X^A_\alpha X^B_\beta X^C_\gamma X^D_\delta\delta^{\alpha\beta\gamma\delta},\quad X^A_\alpha:=X^A{}_{,\alpha} \tag{2.6}$$

is the Jacobian of the transformation $x^\alpha\to X^A$.

The diffeomorphism invariance of the action (2.5) ensures that the field equations obtained by varying $X^A(x^\alpha)$ follow from those obtained by varying $\gamma_{\alpha\beta}$ and Λ. The variation of $\gamma_{\alpha\beta}$ yields the Einstein law

$$R^{\alpha\beta}-\tfrac{1}{2}R\gamma^{\alpha\beta}+\tfrac{1}{2}\Lambda\gamma^{\alpha\beta}=0\ . \tag{2.7}$$

The Bianchi identities imply that $\frac{1}{2}\Lambda$ is a spacetime constant. This can be identified with the cosmological constant. The variation of Λ leads to the parametrized unimodular condition

$$|\det(\gamma^{\alpha\beta})|^{1/2}=\tilde{X}[X^A]\ . \tag{2.8}$$

Let us bring the parametrized action into canonical form. We assume that $M=R\times\Sigma$, where Σ is compact. We subject the label coordinates $x^\alpha=(t,x^a)$ to the Hilbert conditions and decompose $\gamma_{\alpha\beta}$ into the lapse N, shift N^a, and the induced metric g_{ab}. We introduce the momentum p^{ab} conjugate to g_{ab} and arrive thus at the action

$$S[X^A;g_{ab},p^{ab},N,N^a;\Lambda]=\int_R dt\int_\Sigma d^3x(p^{ab}\dot{g}_{ab}-N(H^G+\Lambda g^{1/2})-N^aH^G_a+\Lambda\tilde{X}[X^A])\ . \tag{2.9}$$

The dynamical variables H^G and H^G_a are the standard gravitational super-Hamiltonian

$$H^G=g^{-1/2}(p^{ab}p_{ab}-\tfrac{1}{2}p^2)-g^{1/2}R[g_{cd}] \tag{2.10}$$

and supermomentum

$$H^G_a=-2p_a{}^b{}_{|b}\ . \tag{2.11}$$

To handle the embedding variables, we rearrange the Jacobian $\tilde{X}[X^A]$ in two different ways. The aim of the first rearrangement is to express $\tilde{X}$ as a divergence,

$$\tilde{X}=\tau^\alpha{}_{,\alpha},\quad\text{with}\quad \tau^\alpha:=\frac{1}{3!}T\delta^{\alpha\beta\gamma\delta}Z^k_\beta Z^l_\gamma Z^m_\delta\delta_{klm}\ . \tag{2.12}$$

The space-time vector density τ^α has the components

$$\tau^\alpha=(\tau^0,\tau^a)=(T\tilde{Z},\ T\tilde{Z}Z^a_k\dot{Z}^k)\ , \tag{2.13}$$

where Z^a_k is the inverse to $Z^k_a:=Z^k{}_{,a}$ and $\tilde{Z}:=\det(Z^k_a)$. Equations (2.12) and (2.13) are the starting point for passing to the Henneaux-Teitelboim action[6] (Appendix). They enable us to determine the volume $\tau^{\rm FIN}_{\rm IN}$ of the spacetime region between two embeddings $X^A_{\rm IN}(x^a)$ and $X^A_{\rm FIN}(x^a)$. To get $\tau^{\rm FIN}_{\rm IN}$, we connect the embeddings by a foliation $X^A(t,x^a)$ and integrate the parametrized unimodular condition (2.8) from $t_{\rm IN}$ to $t_{\rm FIN}$:

$$\tau_{\rm IN}^{\rm FIN}=\int_{t_{\rm IN}}^{t_{\rm FIN}}dt\int_\Sigma d^3x\, Ng^{1/2}$$
$$=\int_{t_{\rm IN}}^{t_{\rm FIN}}dt\int_\Sigma d^3x\, \tau^0{}_{,0}(t,x)=\tau_{\rm FIN}-\tau_{\rm IN}\ ,$$
$$\tau:=\int_\Sigma d^3x\, \tilde{Z}(x)T(x)\ . \tag{2.14}$$

The variable τ is the four-volume between the fiducial hypersurface $T(x)=0$ and the embedding $X^A(x)$.

The second rearrangement exhibits $\tilde{X}$ as a linear function of the embedding velocities $\dot{X}^A$:

$$\tilde{X}[X^A]=n_A\dot{X}^A,\quad n_A:=\frac{1}{3!}\delta_{ABCD}Z_b^BZ_c^CZ_d^D\delta^{bcd}\ . \tag{2.15}$$

The spacetime covector n_A is normal to the embedding. It behaves as a scalar density under DiffΣ. Its components are

$$n_A=\tilde{Z}(1,\ Z_k^aT_{,a})\ . \tag{2.16}$$

From Eqs. (2.9) and (2.15) we get the momentum P_A conjugate to X^A:

$$P_A=\Lambda n_A\ . \tag{2.17}$$

We see that the tangential projection of P_A must vanish:

$$P_a:=X_a^AP_A=0\ . \tag{2.18}$$

This is a primary constraint on the embedding variables. Inversely, Eq. (2.18) implies that P_A is directed along the normal, i.e., that there exists a Λ which yields Eq. (2.17). Up to the terms in P_A, Eq. (2.18) determines Λ as a dynamical variable on the embedding phase space. We choose a particular solution for Λ which does not depend on P_A:

$$\Lambda=\tilde{Z}^{-1}P_T\ . \tag{2.19}$$

By substituting this Λ into (2.9), we cast the action into the canonical form

$$S[X^A,P_A,g_{ab},p^{ab};N,N^a]=\int_R dt\int_\Sigma d^3x(P_A\dot{X}^A+p^{ab}\dot{g}_{ab}-N(H^G+g^{1/2}\Lambda)-N^aH_a^G)\ . \tag{2.20}$$

The variation of the lapse and the shift yields the constraints

$$H_a^G=0 \tag{2.21}$$

and

$$g^{-1/2}H:=g^{-1/2}H^G+\Lambda=0\ . \tag{2.22}$$

In Eq. (2.22), the scalar $g^{-1/2}H^G$ depends only on the metric variables and the scalar $\Lambda=\tilde{Z}^{-1}P_T$ only on the embedding variables.

The supermomentum constraints (2.18) and (2.21) generate the Lie derivative change of arbitrary dynamical variables on the embedding sector and the metric sector of the phase space. Each set thus represents LDiffΣ and, moreover, the constraints (2.18) have vanishing Poisson brackets with the constraints (2.21). The scalar constraints (2.22) close into the supermomentum constraints (2.21),

$$\{g^{-1/2}H(x),g^{-1/2}H(x')\}$$
$$=g^{-1/2}(x)g^{ab}(x)H_a^G(x)\delta_{,b}(x,x')g^{-1/2}(x')-(x\leftrightarrow x')\ . \tag{2.23}$$

However, because $H_a^G(x)$ and $P_a(x)$ generate LDiffΣ, and $g^{-1/2}H^G(x)$ and $\Lambda(x)$ are scalars under DiffΣ,

$$\{g^{-1/2}H(x),H_a^G(x)\}=H^G{}_{,a}(x)\delta(x,x')\ ,$$
$$\{\Lambda(x),P_a(x)\}=\Lambda_{,a}(x)\delta(x,x')\ . \tag{2.24}$$

The constraints (2.18), (2.21), and (2.22) thus imply secondary constraints

$$(g^{-1/2}H^G(x))_{,a}=0\quad\text{and}\quad\Lambda_{,a}(x)=0\ . \tag{2.25}$$

With these additional constraints, the total system of constraints is first class.

The constraints (2.25) ensure that the gravitational super-Hamiltonian $g^{-1/2}H^G$ and the dynamical variable Λ are constant on Σ. They can be written in an integral form

$$H_0^G(x):=H^G(x)+\lambda^Gg^{1/2}(x)=0 \tag{2.26}$$

and

$$P_0(x):=P_T(x)-\lambda\tilde{Z}(x)=0\ , \tag{2.27}$$

where λ^G and λ are the dynamical variables

$$\lambda^G:=-\int_\Sigma d^3x\, H^G(x)\Big/\int_\Sigma d^3x\, g^{1/2}(x)$$
$$\lambda=\int_\Sigma d^3x\, P_T(x)\Big/\int_\Sigma d^3x\, \tilde{Z}(x)\ . \tag{2.28}$$

The constraint (2.22) then reduces to a single global relation

$$\lambda-\lambda^G=0 \tag{2.29}$$

among the metric variables and the embedding variables.

To summarize, the complete constraint system consists of the supermomentum constraints (2.18) and (2.21), the integral constraints (2.26) and (2.27), and the global constraint (2.29). In the supermomentum and integral constraints, the metric variables and the embedding variables are completely separated. The only connection between these two classes of variables is provided by the global constraint (2.29).

The supermomentum constraints generate the Lie derivative change under the shift N^a within the hypersurface. We want to interpret the changes generated by the integral constraints (2.26) and (2.27).

The integral constraint (2.26) generates the normal change of a geometric dynamical variable $F[g_{ab},p^{ab}]$ within a class of hypersurfaces which have a zero four-volume separation from each other. Indeed,

$$\int_\Sigma d^3x\, N_0(x)H_0^G(x) =: H_0^G(N_0) = H^G(N) := \int_\Sigma d^3x\, N(x)H^G(x) , \tag{2.30}$$

where

$$N(x)=N_0(x)-\int_\Sigma d^3x'\, g^{1/2}(x')N_0(x') \Big/ \int_\Sigma d^3x'\, g^{1/2}(x') . \tag{2.31}$$

The integral constraint (2.26) smeared by $N_0(x)$ thus generates the same change as the gravitational super-Hamiltonian smeared by $N(x)$. However, the four-volume between the hypersurfaces separated by the lapse function (2.31) vanishes:

$$\int_\Sigma d^3x\, N(x)g^{1/2}(x)=0 . \tag{2.32}$$

Similarly, the integral constraint (2.27) generates the change of the embedding along the worldlines $Z^k=\text{const}$ of the unimodular reference frame within a class of hypersurfaces which again have a zero four-volume separation from each other. To show that, smear $P_0(x)$ by an arbitrary function $M(x)$,

$$P_0(M) := \int_\Sigma d^3x\, M(x)P_0(x) , \tag{2.33}$$

and determine the change of the embedding which it produces:

$$\begin{aligned}\dot T &= \{T(x), P_0(M)\} \\ &= M(x) - \int_\Sigma d^3x'\, \tilde Z(x')M(x') \Big/ \int_\Sigma d^3x'\, \tilde Z(x') , \\ \dot Z^k(x) &= \{Z^k(x), P_0(M)\} = 0 .\end{aligned} \tag{2.34}$$

We see that $Z^k(x)$ remains unchanged, and the four-volume between the initial and the displaced hypersurfaces determined from Eq. (2.14) vanishes:

$$d\tau = dt \int_\Sigma d^3x\, \tilde Z(x)\dot T(x) = 0 . \tag{2.35}$$

The supermomenta (2.18), (2.21) and the integral constraints (2.26), (2.27) keep us always within the equivalence class $\mathcal{E}$ of embeddings separated by the zero four-volume. To get from one such equivalence class to another, we must evolve the data by the global constraint (2.29).

The dynamical variables λ and λ^G given by Eqs. (2.28) have vanishing Poisson brackets with all the constraints (2.18), (2.21), (2.26), (2.27), and (2.29). They are therefore constants of the motion. They are not independent, because the constraint (2.29) forces them to be the same. The constant of the motion $\frac{1}{2}\lambda = \frac{1}{2}\lambda^G$ is simply the cosmological constant, once calculated as a functional of the embedding variables, the other time as a functional of the geometric variables. The canonical formalism reproduces thereby the result which followed from the Einstein law of gravitation (2.7).

III. INTERPRETATION OF CONSTRAINTS AND THE PROBLEM OF TIME

To interpret the constraint system of unimodular gravity, we shall compare it on one hand with the constraint system of general relativity, and on the other hand with the constraint system of Gaussian gravity.

In general relativity, the phase space (g_{ab},p^{ab}) does not contain any embedding variables. The geometric variables satisfy the usual super-Hamiltonian and supermomentum constraints

$$H^G(x;g_{bc},p^{bc}]=0=H_a^G(x;g_{bc},p^{bc}] . \tag{3.1}$$

Of course, we can artificially enlarge the configuration space by the embedding variables $X^A(x)$ and parametrize the Hilbert action. However, because the Hilbert action is already invariant in the metric variables, its parametrized version $S[\gamma_{\alpha\beta},X^A]$ does not actually depend on the embedding variables. As a result, the embedding momenta vanish:

$$P_A(x)=0 . \tag{3.2}$$

We thus end with two sets of constraints, (3.1) and (3.2), in the extended phase space $(X^A,P_A;g_{ab},p^{ab})$. The first set depends only on the geometric variables, the second set only on the embedding variables. Altogether, we have $2\times 4\infty^3$ constraints for $(12+8)\infty^3$ canonical variables.

In unimodular gravity, the parametrized unimodular coordinate condition is adjoined to the Hilbert action with a Lagrange multiplier. The canonical analysis of the last section leads to $4\infty^3-1$ constraints on the geometric variables,

$$\begin{aligned}&H_a^G(x)=0, \quad H^G(x)+\lambda^G g^{1/2}(x)=0 , \\ &\lambda^G := \int_\Sigma d^3x\, H^G(x) \Big/ \int_\Sigma d^3x\, g^{1/2}(x) ,\end{aligned} \tag{3.3}$$

$4\infty^3-1$ constraints on the embedding variables,

$$\begin{aligned}&P_a(x)=0, \quad P_T(x)-\lambda\tilde Z(x)=0 , \\ &\lambda := \int_\Sigma d^3x\, P_T(x) \Big/ \int_\Sigma d^3x\, \tilde Z(x) ,\end{aligned} \tag{3.4}$$

and one constraint coupling the geometric variables with the embedding variables,

$$\lambda[X^A,P_A]-\lambda^G[g_{ab},p^{ab}]=0 . \tag{3.5}$$

Altogether, we have $2\times 4\infty^3-1$ constraints for $(12+8)\infty^3$ canonical variables. To reduce unimodular gravity to general relativity we must impose a single additional constraint

$$\lambda[X^A,P_A]=0 . \tag{3.6}$$

Gaussian gravity follows the pattern of unimodular gravity, the parametrized Gaussian coordinate conditions replacing the unimodular condition.[17] They are again adjoined to the action with Lagrange multipliers. The canonical analysis of the action leads to a single set of $4\infty^3$ constraints which couple the geometric variables to the embedding variables,

$$P_A(x) - n_A(x;X^B,g_{bc}]H^G(x;g_{bc},p^{bc}] + X^a_A(x;X^B,g_{bc}]H^G_a(x;g_{bc},p^{bc}] = 0 \ . \quad (3.7)$$

The coefficients n_A and X^a_A have the meaning of the normal and the tangent covectors to an arbitrary embedding. By virtue of the Gaussian coordinate conditions, these are quite definite functionals of the configuration variables $X^A(x)$ and $g_{ab}(x)$. Altogether, we have $4\infty^3$ constraints (3.7) for $(12+8)\infty^3$ canonical variables. To reduce Gaussian gravity to general relativity, we should impose $4\infty^3$ additional constraints (3.2).

The problem of time in general relativity arises when one tries to implement the Dirac constraint quantization in terms of the $4\infty^3$ geometric constraints (3.1). The constraints generate the change of the $12\infty^3$ geometric variables $g_{ab}(x),p^{ab}(x)$ under an arbitrary displacement of the embedding, but the embedding does not enter into the canonical description of the system. One would like to say that $4\infty^3$ combinations $\phi^A(x;g_{ab},p^{ab}]$, $A=0,1,2,3$, of the geometric variables specify the embedding and $2\infty^3$ combinations $g_r(x;g_{ab},p^{ab}]$, $r=1,2$, describe the true degrees of freedom of the gravitational field on that embedding.[21] The embedding variables ϕ^A ought to be four independent spacetime scalars.[22] They play the role of an internal many-fingered time.

The transition

$$g_{ab}(x);p^{ab}(x) \to \phi^A(x),g_r(x);\pi_A(x),p^r(x) \quad (3.8)$$

should be accomplished by a canonical transformation. The constraints (3.1) should then be resolved with respect to the embedding momenta π_A, i.e., written in the form

$$\pi_A(x) + h_A(x;\phi^B,g_r,p^r] = 0 \ . \quad (3.9)$$

The functionals $h_A(x)$ represent a true many-fingered Hamiltonian (the energy density and the energy flux through the embedding).

To quantize the system, we should turn the constraints into operators and impose them as restrictions on the physical states. When we apply this procedure to the geometric constraints (3.1),

$$\hat{H}^G_a(x)\Psi[g_{ab}] = 0, \quad \hat{H}^G(x)\Psi[\mathbf{g}] = 0 \ , \quad (3.10)$$

we recover the familiar scheme of quantum geometrodynamics. The first equation (3.10) ensures that the physical states depend only on the three-geometry $\mathbf{g}$.[23] The second equation (3.10) is the Wheeler-DeWitt equation.[24] As a second-order functional differential equation in $\mathbf{g}$, this equation (at least formally) yields a conserved current,[9] but not a positive-definite inner product.[10,11,22] The probabilistic interpretation of the solutions $\Psi[\mathbf{g}]$ thus remains problematic.

These difficulties motivated the quest for an internal time (3.8). When the constraints are imposed on the state functional in their resolved form (3.9), they yield a many-fingered time Schrödinger equation

$$\frac{i\delta\Psi[\phi^B,g_r]}{\delta\phi^A(x)} = h_A(x;\phi^B,\hat{g}_r,\hat{p}^r]\Psi[\phi^B,g_r] \ . \quad (3.11)$$

This equation, at least formally, keeps the norm of the state Ψ,

$$\langle\Psi|\Psi\rangle = \int Dg_r \Psi^*[\phi^A,g_r]\Psi[\phi^A,g_r] \ , \quad (3.12)$$

independent of the embedding. The integrand of the functional integral (3.12) can then be interpreted as the probability density for the gravitational degrees of freedom $g_r(x)$ to have a given distribution on the embedding $\phi^A(x)$.

Unfortunately, one does not know a canonical transformation (3.8) for which the quantization program (3.9), (3.11), and (3.12) would be technically feasible. Moreover, different splits (3.8) of the geometric variables into dynamical degrees of freedom and a many-fingered time are expected to lead to inequivalent quantum theories.[22] This constitutes the problem of time in quantum geometrodynamics.

The parametrization of the Hilbert action by the external embedding variables $X^A(x)$ does not by itself resolve the problem of time. The extension of the phase space into $(X^A,P_A;g_{ab},p^{ab})$ is counterbalanced by the doubling of the constraints, Eq. (3.2). The constraints (3.2) on the external embedding variables are totally separated from the constraints (3.1) on the metric variables. The change of an external embedding $X^A(x)$, generated by the constraints (3.2), is thus in no way correlated to the change of the geometry generated by the constraints (3.1). In quantum theory, the state functional $\Psi[X^A,g_{ab}]$ (or, with internal embeddings identified, $\Psi[X^A,\phi^A,g_r]$), is subject to a double set of constraints, (3.10) [or (3.11)], and

$$\hat{P}_A(x)\Psi = 0 \ . \quad (3.13)$$

The constraints (3.13) tell us that Ψ does not depend on external embeddings $X^A(x)$. The expression

$$\Psi^*[X^A,g_{ab}]\Psi[X^A,g_{ab}] = \Psi^*[g_{ab}]\Psi[g_{ab}] \quad (3.14)$$

is the same on every embedding, and thus automatically conserved. However, it cannot be interpreted as the probability density for the metric $g_{ab}(x)$ to have a given distribution on the embedding $X^A(x)$ because $\Psi[g_{ab}]$ is subject to the old constraints (3.10) and the metric $g_{ab}(x)$ is thus not a measurable quantity. At best, one can again separate the internal embedding variables $\phi^A(x)$ from the true degrees of freedom $g_r(x)$, and interpret $\Psi^*[\phi^A,g_r]\Psi[\phi^A,g_r]$ as the probability density for the true degrees of freedom $g_r(x)$ to have a given distribution on an internal embedding $\phi^A(x)$. The internal embedding $\phi^A(x)$ is in no way related to the external embedding $X^A(x)$; when we know $X^A(x)$, we have no idea about what $\phi^A(x)$ may be. The mere parametrization of the Hilbert action does not help us in resolving the problem of time.

The situation is entirely changed when we first break the invariance of the Hilbert action by suitably chosen coordinate conditions, like the Gaussian conditions, and only then parametrize it by $X^A(x)$.[16] The constraints in the extended phase space $(X^A(x),P_A(x);g_{ab}(x),p^{ab}(x))$ are then not doubled, but form a single set of $4\infty^3$ constraints (3.7) which couple the geometric variables to the embedding variables. The embeddings $X^A(x)$ written in terms of the Gaussian coordinates X^A describe the physical state of a material system, the Gaussian reference fluid, which interacts with gravity. The Gaussian reference fluid has the structure of a heat-conducting dust. The embeddings $X^A(x)$ are anchored in this material medium. The embedding variables $X^A(x)$ and the metric are independent canonical coordinates and one thus does not need to perform a canonical transformation (3.8) to distinguish time from dynamical degrees of freedom. The constraints (3.7) are already resolved with respect to the embedding momenta, as in Eq. (3.9). The Dirac constraint quantization thus leads to a functional Schrödinger equation

$$i\delta\Psi/\delta X^A(x)=h_A(x;X^B,\hat{g}_{ab},\hat{p}^{ab})\Psi \tag{3.15}$$

in the many-fingered Gaussian time $X^A(x)$. The expression

$$\Psi^*[X^A,g_{ab}]\Psi[X^A,g_{ab}] \tag{3.16}$$

can be interpreted as the probability density that, on the embedding $X^A(x)=(T(x),Z^k(x))$, the metric $g_{ab}(x)$ [which is measured in the system of coordinates x^a connected to the Gaussian system of coordinates Z^k by the transformation $Z^k=Z^k(x^a)$] is found in the cell $Dg_{ab}(x)$. The only difficulties arise from the energy conditions which must be satisfied in order that the Gaussian reference fluid be realizable as a material system.[25] These are not relevant for our present discussion of unimodular gravity.

The imposition of the Gaussian coordinate conditions before parametrization thus solves (apart from the difficulty posed by the energy conditions) the problem of time in quantum gravity. Other coordinate conditions, like the harmonic ones, serve the same purpose.[18] The crucial question which we are facing in this paper is whether the unimodular condition works in the same way as the Gaussian conditions or the harmonic conditions. The answer to this question is no.

In parametrized general relativity, all $4\infty^3$ external embedding variables $X^A(x)$ are physically irrelevant because the conjugate momenta are subject to $4\infty^3$ constraints. Their change is not correlated with the change of geometry. In Gaussian gravity, all $4\infty^3$ embedding variables are physically significant. The embedding momenta are not limited by any separate constraints, and the constraints (3.7) perfectly correlate the changes of $X^A(x)$ to the change of geometry. In unimodular gravity, the embedding momenta are subject to $4\infty^3-1$ constraints (3.4). This counting is much closer to parametrized general relativity (which does not solve the problem of time) than to Gaussian gravity (which solves it). As a result, in unimodular gravity almost all embedding variables, namely, $4\infty^3-1$ of them, are physically irrelevant. Only one single variable is correlated with an observable change of geometry.

To carry this argument to the bitter end, we must separate the physically irrelevant part of $X^A(x)$ from the physical part of $X^A(x)$ by a point transformation. At the same time, we shall cast the embedding constraints (3.4) into new momenta.

The point transformation is performed in two stages. The first one is a transition

$$\begin{bmatrix} T(x), & Z^k(x) \\ P_T(x), & P_k(x) \end{bmatrix} \leftrightarrow \begin{bmatrix} T(\bar{Z}), & \bar{Z}^k(x) \\ P(\bar{Z}), & \bar{P}_k(x) \end{bmatrix} . \tag{3.17}$$

The new momenta $\bar{P}_k(x)$ are obtained by multiplying the supermomenta $P_a(x)$ by the inverse Z^a_k to the matrix $Z^k_a:=Z^k{}_{,a}$:

$$\bar{P}_k(x)=Z^a_k(x)P_a(x)=P_k(x)+Z^a_k(x)T_{,a}(x)P_T(x) . \tag{3.18}$$

Unlike $P_a(x)$, the dynamical variables $\bar{P}_k(x)$ have vanishing Poisson brackets among themselves,

$$\{\bar{P}_k(x),\bar{P}_l(x')\}=0 . \tag{3.19}$$

This can either be checked directly, or can be shown to hold by a general argument: Because the supermomentum constraints close, and $\bar{P}_k(x)=0$ are equivalent to $P_a(x)=0$, the Poisson bracket (3.19) must weakly vanish, modulo the constraints $\bar{P}_k(x)=0$. However, because $P_k(x)$ in Eq. (3.18) are separated from the remaining variables, the Poisson bracket $\{\bar{P}_k(x),\bar{P}_l(x')\}$ does not depend on $P_k(x)$. The constraint $\bar{P}_k(x)=0$ thus cannot help the bracket to vanish, and Eq. (3.19) must hold strongly rather than weakly.

It is also easy to check that the embedding variables

$$\bar{Z}^k(x)=Z^k(x) \tag{3.20}$$

are canonically conjugate to the new momenta (3.18):

$$\{\bar{Z}^k(x),\bar{P}_l(x')\}=\delta^k_l\delta(x,x') . \tag{3.21}$$

To complete Eqs. (3.18) and (3.20) into a canonical transformation (3.17), we construct a number of dynamical variables which commute both with the new coordinates $\bar{Z}^k(x)$ and the new momenta $\bar{P}_k(x)$. Let $F(Z^k)$ and $G(Z^k)$ be arbitrary functions of Z^k. On an embedding $Z^k(x)$, we turn them into smearing functions and define

$$T_F:=\int_\Sigma d^3x\, \tilde{Z}(x)F(Z^k(x))T(x) \tag{3.22}$$

and

$$P_G:=\int_\Sigma d^3x\, G(Z^k(x))P_T(x) . \tag{3.23}$$

These variables are invariant under DiffΣ and hence their Poisson brackets with the generators $P_a(x)$ of LDiffΣ vanish. Because they do not depend on the $P_k(x)$'s,

$$\{T_F,\bar{Z}^k(x)\}=0=\{P_G,\bar{Z}^k(x)\} , \tag{3.24}$$

their Poisson brackets with the new momenta (3.18) also vanish:

$$\{T_F,\bar{P}_k(x)\}=0=\{P_G,\bar{P}_k(x)\} . \tag{3.25}$$

The Poisson brackets among the different pairs of the variables (3.22) and (3.23) yield

$$\{T_F,T_G\}=0=\{P_F,P_G\} \tag{3.26}$$

and

$$\{T_F,P_G\}=\int_\Sigma d^3x\,\tilde{Z}(x)F(Z^k(x))G(Z^k(x)) \ . \qquad (3.27)$$

We now pass to the limit in which we choose for $F(Z^k)$ a family of δ functions labeled by three parameters $\bar{Z}^k$:

$$F_{\bar{Z}^l}(Z^k)=\delta(Z^k-\bar{Z}^k) \ . \qquad (3.28)$$

The corresponding dynamical variables (3.22) and (3.23) are labeled by the same parameters:

$$T(\bar{Z}^k)=\int_\Sigma d^3x\,\tilde{Z}(x)\delta(Z^k(x)-\bar{Z}^k)T(x) \qquad (3.29)$$

and

$$P(\bar{Z}^k)=\int_\Sigma d^3x\,\delta(Z^k(x)-\bar{Z}^k)P_T(x) \ . \qquad (3.30)$$

Equation (3.29) tells us that $T(\bar{Z}^k)$ is a functional of $Z^k(x)$ and $T(x)$ obtained by solving the equation $\bar{Z}^k=Z^k(x^a)$ for x^a and substituting this solution back into $T(x^a)$:

$$T(\bar{Z}^k)=T(x^a(\bar{Z}^k)) \ . \qquad (3.31)$$

Equations (3.24) and (3.25) reduce to the form

$$\{T(\bar{Z}),\bar{Z}^k(x)\}=0=\{T(\bar{Z}),\bar{P}_k(x)\} \ ,$$
$$\{P(\bar{Z}),\bar{Z}^k(x)\}=0=\{P(\bar{Z}),\bar{P}_k(x)\} \ , \qquad (3.32)$$

and similarly, Eqs. (3.26) and (3.27) yield

$$\{T(\bar{Z}),T(\bar{Z}')\}=0=\{P(\bar{Z}),P(\bar{Z}')\} \ ,$$
$$\{T(\bar{Z}),P(\bar{Z}')\}=\delta(\bar{Z}-\bar{Z}') \ . \qquad (3.33)$$

They tell us that the transformation (3.17) given by Eqs. (3.18), (3.20), (3.29), and (3.30) is a canonical transformation.

The variables $T(\bar{Z})$ label the hypersurfaces by giving the unimodular time as a function of spatial unimodular coordinates. The variables $\bar{Z}^k(x)$ are the mappings from Σ into the unimodular coordinates; as such, they assign the coordinates x^a to the points of a hypersurface.

In the second stage of the canonical transformation, we decompose $T(\bar{Z})$ into its mean value, $\tau/\int d^3\bar{Z}$, and the deviation $T_0(\bar{Z})$ from this mean value:

$$T(\bar{Z})=T_0(\bar{Z})+\tau\Big/\int d^3\bar{Z}, \quad \int d^3\bar{Z}\;T_0(\bar{Z})=0 \ ; \qquad (3.34)$$

$$\tau=\int d^3\bar{Z}\;T(\bar{Z}), \quad T_0(\bar{Z})=T(\bar{Z})-\frac{\int d^3\bar{Z}\;T(\bar{Z})}{\int d^3\bar{Z}} \ . \qquad (3.35)$$

This is similar to decomposing the configuration of a system of particles into the position of their center of mass and the relative positions of the particles with respect to this center.

The variable τ is the four-volume (2.14) separating the hypersurface $T(\bar{Z})$ from the fiducial hypersurface $T(\bar{Z})=0$. It labels the equivalence classes $\mathcal{E}_\tau$ of hypersurfaces which have a zero four-volume separation from each other, while $T_0(\bar{Z})$ labels the individual hypersurfaces within each equivalence class τ=const. As a new canonical coordinate, $T_0(\bar{Z})$ has a disadvantage that it cannot be freely prescribed because its integral must vanish. To avoid this shortcoming, we overlabel the members of each equivalence class by redundant coordinates $\bar{T}(\bar{Z})$ which generate a $T_0(\bar{Z})$ that automatically satisfies the integral constraint. This we do by putting

$$T_0(\bar{Z})=\bar{T}(\bar{Z})-\int d^3\bar{Z}\;\bar{T}(\bar{Z})\Big/\int d^3\bar{Z} \ . \qquad (3.36)$$

We see that two functions, $\bar{T}_1(\bar{Z})$ and $\bar{T}_2(\bar{Z})$, which differ by a constant, label the same hypersurface. The constant itself is unobservable. By comparing (3.36) with (3.35), we see that the unimodular time $T(\bar{Z})$ differs from the redundant coordinate $\bar{T}(\bar{Z})$ by such an unobservable constant.

Because of the redundancy, the mapping from the ∞^3+1 variables $\bar{T}(\bar{Z}),\tau$ to the ∞^3 variables $T(\bar{Z})$ cannot be one to one. To have a one-to-one mapping, we must extend the original configuration space $T(\bar{Z})$ by an unphysical degree of freedom σ. We state that σ is physically irrelevant by requiring that the momentum π conjugate to σ vanish:

$$\pi=0 \ . \qquad (3.37)$$

Through this formal device, we are able to write a point transformation which meets our needs:

$$\begin{bmatrix} T(\bar{Z}), & \sigma \\ P(\bar{Z}), & \pi \end{bmatrix} \leftrightarrow \begin{bmatrix} \bar{T}(\bar{Z}), & \tau \\ \bar{P}(\bar{Z}), & \lambda \end{bmatrix} . \qquad (3.38)$$

We put

$$T(\bar{Z})=\bar{T}(\bar{Z})-\int d^3\bar{Z}\;\bar{T}(\bar{Z})\Big/\int d^3\bar{Z}+\tau\Big/\int d^3\bar{Z} \ ,$$
$$\sigma=\int d^3\bar{Z}\;\bar{T}(\bar{Z}) \ , \qquad (3.39)$$

and, inversely,

$$\bar{T}(\bar{Z})=T(\bar{Z})-\int d^3\bar{Z}\;T(\bar{Z})\Big/\int d^3\bar{Z}+\sigma\Big/\int d^3\bar{Z} \ ,$$
$$\tau=\int d^3\bar{Z}\;T(\bar{Z}) \ . \qquad (3.40)$$

This induces the transformation of the momenta

$$\bar{P}(\bar{Z})=P(\bar{Z})-\int d^3\bar{Z}\;P(\bar{Z})\Big/\int d^3\bar{Z}+\pi \ ,$$
$$\lambda=\int d^3\bar{Z}\;P(\bar{Z})\Big/\int d^3\bar{Z} \ . \qquad (3.41)$$

We see that σ and π on the left-hand side of Eq. (3.38) are unphysical, corresponding to one unphysical degree of freedom in the canonical pair $\bar{T}(\bar{Z}),\bar{P}(\bar{Z})$ on the right-hand side of Eq. (3.38). As a compensation, the pair τ,λ on the right-hand side of Eq. (3.38) is physical, τ being the four-volume variable and λ the cosmological constant (2.28).

The canonical transformation (3.38) casts the old constraints (3.4) and (3.37) into a new constraint

$$\bar{P}(\bar{Z})=0 \ . \qquad (3.42)$$

We have thereby achieved our aim of transforming all the embedding constraints into the statement that the new momenta, $\bar{P}_k(x)$ and $\bar{P}(\bar{Z})$, must vanish.

As in parametrized general relativity, this is to be interpreted as the statement that the coordinates $\bar{Z}^k(x)$ and $\bar{T}(\bar{Z})$ are physically irrelevant. The individual hypersurfaces within an equivalence class $\mathcal{E}_\tau$ of hypersurfaces with a zero four-volume separation, as well as spatial unimodular coordinates, are not physical elements of unimodular gravity. The only new physical objects introduced by unimodular gravity are the equivalence classes themselves. These are labeled by the four-volume parameter τ, the "cosmological time."

This is naturally reflected in quantum theory. In the Schrödinger representation, the states are taken as functionals $\Psi[X^A, g_{ab}]$ of the unimodular embeddings and of the metric g_{ab} (which, for the time being, we suppress in our notation). When we extend the embeddings by an unphysical variable σ, the state $\Psi[X^A]$ is extended into $\Psi(\sigma; X^A]$. The extended states, however, must be subject to the constraint (3.37).

$$\hat{\pi}\Psi(\sigma;X^A] = -i\frac{\partial}{\partial\sigma}\Psi(\sigma;X^A] = 0 \ . \tag{3.43}$$

As a result, $\Psi(\sigma;X^A]$ does not depend on σ and reduces thus to an old state $\Psi[X^A]$.

We now express the extended state in terms of the new coordinates:

$$\Psi(\sigma;X^A(x)] = \Phi(\tau;\bar{T}(\bar{Z}),\bar{Z}^k(x)] \ . \tag{3.44}$$

The functional Φ is obtained from Ψ by substituting for the old arguments σ; $X^A(x)$ the expressions obtained from the canonical transformation (3.17) and (3.38). The functional Φ is subject to the new constraints

$$\hat{\bar{P}}(\bar{Z})\Phi = 0 = \hat{\bar{P}}_k(x)\Phi \tag{3.45}$$

which tell us that Φ does not depend on the function variables $\bar{T}(\bar{Z})$ and $\bar{Z}^k(x)$. This means that Φ is actually a function of a single real variable τ. In terms of the original variables $X^A(x)$,

$$\tau = \int_\Sigma d^3x\, \tilde{Z}(x)T(x) \ . \tag{3.46}$$

By putting Eqs. (3.43)–(3.46) together, we conclude that $\Psi[X^A(x)]$ can depend on the unimodular coordinates only through the combination (3.46), i.e., that

$$\Psi[X^A] = \Phi\left[\int_\Sigma d^3x\, \tilde{Z}(x)T(x)\right] \ . \tag{3.47}$$

This takes care of all of the embedding constraints.

We now reintroduce the metric argument and impose on $\Phi(\tau;g_{ab}]$ the remaining constraints (3.3) and (3.5):

$$\hat{H}^G_a(x)\Phi(\tau;g_{bc}] = 0 \ , \tag{3.48}$$

$$\left[\hat{H}^G(x) + \int_\Sigma d^3x'\, \hat{H}^G(x') \Big/ \int_\Sigma d^3x'\, \hat{g}^{1/2}(x')\right]\Phi(\tau;\mathbf{g}] = 0 \ , \tag{3.49}$$

and

$$i\frac{\partial\Phi(\tau;\mathbf{g}]}{\partial\tau} = \frac{\int d^3x\, \hat{H}^G(x)}{\int d^3x\, \hat{g}^{1/2}(x)}\Phi(\tau;\mathbf{g}] \ . \tag{3.50}$$

Equation (3.48) implies that Φ depends only on the equivalence classes of metrics $g_{ab}(x)$ modulo spatial diffeomorphisms DiffΣ, i.e., only on the three-geometry $\mathbf{g}$. We are thus entitled to use the notation $\Phi(\tau;\mathbf{g}]$ in the remaining two equations.

Equation (3.50) has the form of a Schrödinger equation in the cosmological time τ. The proponents of the view that unimodular gravity solves the problem of time in quantum geometrodynamics base their claim on this fact. They propose that $\Phi(\tau;\mathbf{g}]$ describes the statistical distribution of measurements performed at a given instant τ.

To see whether such a claim can be justified, let us grant that one can factor order the Hamiltonian on the right-hand side of Eq. (3.50) as a self-adjoint operator on the space of square-integrable functionals of $\mathbf{g}$. It follows that the norm

$$\langle\Phi|\Phi\rangle = \int D\mathbf{g}\, \Phi^*(\tau;\mathbf{g}]\Phi(\tau;\mathbf{g}] \tag{3.51}$$

does not depend on τ. Then, if Eq. (3.50) were the only equation for the state Φ, one could interpret the integrand

$$\Phi^*(\tau;\mathbf{g}]\Phi(\tau;\mathbf{g}] \tag{3.52}$$

as the probability density that, when observed at an instant τ, the three-geometry $\mathbf{g}$ has a given distribution.

However, besides the Schrödinger equation (3.50), the states must also satisfy the constraints (3.49). Only those states which solve Eq. (3.49) belong to the physical space $\mathcal{F}_0$. Unfortunately, the geometry operator $\hat{\mathbf{g}} = \mathbf{g}\times$ throws the state out of $\mathcal{F}_0$. To show that, one must first define $\hat{\mathbf{g}}$ as a multiplication operator. This is done indirectly by turning an arbitrary invariant functional $G[g_{ab}]$ of the metric into a multiplication operator. Because $G[g_{ab}]$ depends only on the equivalence classes of metrics modulo DiffΣ,

$$\hat{H}^G_a(x)G[g_{ab}] = 0 \ , \tag{3.53}$$

it can be interpreted as a functional $G[\mathbf{g}]$ of geometry. An implementation of $\hat{G}$ as a multiplication operator

$$\hat{G} := G[\mathbf{g}]\times \tag{3.54}$$

then amounts to defining $\hat{\mathbf{g}}$ as a multiplication operator.

After this is done, it becomes clear that a generic $\hat{G}$ does not weakly commute with the constraint $\hat{H}^G_0(x)$ of Eq. (3.49):

$$\Phi \in \mathcal{F}_0 \not\Rightarrow [\hat{G},\hat{H}^G_0(x)]\Phi = 0 \ . \tag{3.55}$$

Therefore, $\hat{G}\Phi$ does not necessarily lie in the physical space $\mathcal{F}_0$, and (3.52) can no longer be interpreted as the probability density for the geometry $\mathbf{g}$.

One can easily see the reason why the three-geometry $\mathbf{g}$ is not observable. The constraint (3.49) is actually a Wheeler-DeWitt equation describing the evolution of state between hypersurfaces separated from each other by a zero four-volume. Equation (3.50) describes then the evolution of state from one equivalence class of such hypersurfaces to another. It is only this equation which has the Schrödinger form; the remaining constraints (3.49) are still of the Klein-Gordon type. The unimodular coordinate condition (unlike the Gaussian coordinate conditions) is much too weak to solve the problem of time. The essential feature of canonical gravity engrained into its formalism is that the fundamental variables $g_{ab}(x)$, $p^{ab}(x)$ must be given on a single spacelike embedding. The embeddings $X^A(x)$ expressed in unimodular coordinates fail to correlate with the geometric variables. Most of the unimodular coordinates are just arbitrary labels without physical significance; only one particular combination of them, namely, the variable τ, is physically relevant. However, no single variable can uniquely label ∞^3 many spacelike hypersurfaces. Time is a functional variable in general relativity, not a single variable as in Newtonian physics. As a result, the cosmological time τ does not label individual hypersurfaces, but only (∞^3-1)-dimensional equivalence classes $\mathcal{E}_\tau$ of such hypersurfaces. To say that the geometric variables are measured at an instant τ does not tell us on which one of the infinitely many hypersurfaces of the equivalence class $\mathcal{E}_\tau$ they are going to be measured. The cosmological time τ thus does not set the conditions of the measurement uniquely. This is why the question about the distribution of the geometric variables (like **g** or **p**) at a given "time" τ does not make sense. Unruh and Wald saw τ as a primary example of their Heraclitian time variable which was supposed to provide a causal ordering between the measurements.[12] This is exactly what the cosmological time fails to do: The individual hypersurfaces of the equivalence class τ=const must necessarily intersect, and parts of them lie to the future and parts of them to the past of each other. The hypersurfaces with different values of τ are also allowed to intersect: $\tau_{\rm FIN}>\tau_{\rm IN}$ is thus no guarantee that a hypersurface from the equivalence class $\tau_{\rm FIN}$=const be entirely lying to the future of a hypersurface from the equivalence class $\tau_{\rm IN}$=const. The cosmological time τ does not provide a causal ordering of spacelike hypersurfaces in the embedding space time.

The cosmological time τ can set the conditions of the measurement properly only in conjunction with some other ∞^3-1 time variables whose assignment uniquely selects a single spacelike hypersurface from each equivalence class $\mathcal{E}_\tau$. There are two alternative ways of providing such a supplementary time variable:

(1) One could devise supplementary coordinate conditions which, in conjunction with the unimodular coordinate condition (2.2), would uniquely fix the foliation of the spacetime once the initial embedding is chosen. One would then adjoin these supplementary coordinate conditions together with the unimodular condition to the Hilbert action with Lagrange multipliers. Next, one would parametrize the action by introducing the privileged coordinates X^A expressed as functions of arbitrary label coordinates x^α as additional field variables. The canonical version of the action should then yield a single set of constraints of the form (3.7) encountered in Gaussian gravity. The supplementary coordinate conditions would make all of the embeddings $X^A(x)$, not only their particular combination τ, physically relevant. One would then interpret the ensuing quantum constraints in full analogy with those of Gaussian gravity.

Of course, the supplementary coordinate conditions can be expected to introduce other source terms on the right-hand side of the Einstein law of gravitation (2.7). Such terms would lead us outside the simple geometric picture of pure unimodular gravity. In effect, the supplementary conditions introduce a material system, a reference fluid, and couple it to unimodular gravity. This is the price to be paid for making the external embedding variables $X^A(x)$ physical.

(2) One can try to identify the supplementary time variables from among the geometric canonical variables, i.e., as internal time variables. The general counting goes as follows: Altogether, the classical constraints (3.3) impose $4\infty^3-1$ restrictions on $2\times3\infty^3$ geometric variables g_{ab}, $p^{ab}(x)$. One can surmise that there exists a canonical transformation

$$\begin{bmatrix} g_{ab}(x) \\ p^{ab}(x) \end{bmatrix} \leftrightarrow \begin{bmatrix} \phi^0(x), & \phi^k(x); & g_r(x), & \lambda^G \\ \pi_0(x), & \pi_k(x); & p^r(x), & \tau^G \end{bmatrix} \tag{3.56}$$

which, like the transformation (3.8), separates the internal embedding variables $\phi^A(x)$ from the true dynamical degrees of freedom. The internal coordinates $\phi^k(x)$ assign the coordinates x^a to the points of space, the internal time $\phi^0(x)$ [which, like the external time variable $T_0(\overline{Z}^k)$, should really have only ∞^3-1 independent components] labels the hypersurfaces separated from each other by a zero four-volume, and the $2\infty^3+1$ variables $g_r(x),\lambda^G$ represent the true degrees of freedom of unimodular gravity. Here, we have explicitly identified λ^G of Eq. (3.3) as that single degree of freedom which unimodular gravity allows and Einsteinian gravity suppresses. Indeed, in unimodular gravity, $\lambda^G[g_{ab},p^{ab}]$ can be freely prescribed, while in general relativity it is constrained to vanish.

Following the canonical transformation (3.56), the geometric constraints (3.3) should be resolved with respect to the internal embedding momenta π_A and in this form imposed as restrictions on the state functional $\Phi(\tau,\lambda^G;\phi^A,g_r]$. Such a process would replace the constraints (3.48) and (3.49) by a $(4\infty^3-1)$-fingered time Schrödinger equation

$$i\frac{\delta\Phi(\tau,\lambda^G;\phi^B,g_r]}{\delta\phi^A(x)}=h_A(x,\hat{\lambda}^G;\phi^B,\hat{g}_r,\hat{p}^r]\Phi(\tau,\lambda^G;\phi^B,g_r]\ . \tag{3.57}$$

This equation would yield the evolution of state between hypersurfaces separated by a zero four-volume, while the Schrödinger equation (3.50),

$$i\frac{\partial}{\partial\tau}\Phi(\tau,\lambda^G;\phi^A,g_r]=-\lambda^G\Phi(\tau,\lambda^G;\phi^A,g_r]\ , \tag{3.58}$$

would yield the evolution of state between equivalence classes of such hypersurfaces, from one value of the four-volume time τ to another. In the λ^G representation, the solution of Eq. (3.58) is a stationary state in τ,

$$\Phi(\tau,\lambda^G;\phi^A,g_r]=\Theta(\lambda^G;\phi^B,g_r]e^{i\lambda^G\tau} . \tag{3.59}$$

The amplitudes $\Theta(\lambda^G;\phi^B,g_r]$ then satisfy the τ-independent Schrödinger equation (3.57).

With the usual proviso that there exists a factor ordering which makes the functional Schrödinger equation integrable and the Hamiltonian $\hat{h}_A$ self-adjoint under the norm

$$\langle\Phi|\Phi\rangle=\int d\lambda^G\int Dg_r\,\Phi^*(\tau,\lambda^G;\phi^A,g_r]\Phi(\tau,\lambda^G;\phi^a,g_r] , \tag{3.60}$$

the norm (3.60) does not depend on the hypertime $(\tau,\phi^A(x))$, and its integrand can be interpreted as the probability density that the embedding $(\tau,\phi^A(x))$ carries the cosmological constant λ^G and the gravitational degrees of freedom g_r.

The problem with such a proposal is that the construction of a supplementary time is as complicated, if not more so, as finding an internal time (3.8) in general relativity. This reinforces our conclusion that unimodular gravity, whatever other purposes it may serve, does not help the least in resolving the problem of time.

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APPENDIX: UNIMODULAR GRAVITY AND THE HENNEAUX-TEITELBOIM FORMALISM

The parametrized unimodular action (2.5) depends on the unimodular coordinates $X^A(x^\beta)$ which are considered as four scalar fields on M. Let us replace these fields by a vector density field $\tau^\alpha(x^\beta)$ connected with $X^A(x^\beta)$ by the transformation (2.12),

$$\tau^\alpha=\frac{1}{3!}T\delta^{\alpha\beta\gamma\delta}Z^k_\beta Z^l_\gamma Z^m_\delta\delta_{klm} . \tag{A1}$$

This casts the action (2.5) into the Henneaux-Teitelboim form[6,8]

$$S[\tau^\alpha,\gamma_{\alpha\beta},\Lambda]=\int_M d^4x(|\det(\gamma_{\alpha\beta})|^{1/2}(R[\gamma_{\alpha\beta}]-\Lambda)+\Lambda\tau^\alpha{}_{,\alpha}) . \tag{A2}$$

The variation of (A2) with respect to $\tau^\alpha(x^\beta)$ yields equations which are equivalent to those obtained by varying $X^A(x^\beta)$. This can be shown by comparing the two ways of varying $\tilde{X}$:

$$(\tilde{X}X^\alpha_A\delta X^A)_{,\alpha}=\delta\tilde{X}=(\delta\tau^\alpha)_{,\alpha} , \tag{A3}$$

where X^α_A is the inverse of $X^A_\alpha:=X^A{}_{,\alpha}$. Now, if S is *any* action which depends on X^A only through $\tilde{X}$, it holds that

$$\frac{\delta S}{\delta\tau^\alpha(x^\beta)}=\frac{\delta S}{\delta X^A(x^\beta)}\tilde{X}(x^\beta)X^A_\alpha(x^\beta) . \tag{A4}$$

Because X^A_α is a regular matrix,

$$\frac{\delta S}{\delta\tau^\alpha(x^\beta)}=0 \iff \frac{\delta S}{\delta X^A(x^\beta)}=0 . \tag{A5}$$

This establishes the equivalence of the parametrized unimodular gravity with the Henneaux-Teitelboim formalism at the spacetime level.

The ADM (Arnowitt, Deser, and Misner) decomposition of the Henneaux-Teitelboim action (A2) leads to an alternative canonical description of gravity with an unspecified cosmological constant:

$$S[\mu,\tau^0,g_{ab},p^{ab};\tau^a]=\int_R dt\int_\Sigma d^3x(\tau^0\dot{\mu}+p^{ab}\dot{g}_{ab}-N(H^G-\mu g^{1/2})-N^aH^G_a+\tau^a\mu_{,a}) . \tag{A6}$$

The cosmological field

$$\mu_0:=\mu:=-\Lambda \tag{A7}$$

(which we denote here as μ) appears as the coordinate canonically conjugate to τ^0, while τ^a plays the role of a Lagrange multiplier. The variation of N, N^a and τ^a yields the constraints

$$H^G-\mu g^{1/2}=0 , \tag{A8}$$

$$H^G_a=0 , \tag{A9}$$

and

$$\mu_{,a}=0 \tag{A10}$$

on the physical space $(\mu,\tau^0;g_{ab},p^{ab})$. This system of constraints is first class.

We would like to understand the transition from (X^A,P_A) to (μ,τ^0) as a canonical transformation. However, this cannot work, because the variables do not match. At the very least, we should extend the new phase space by a canonical pair (μ_a,τ^a), and adjoin to the constraints (A8)–(A10) the primary constraint

$$\mu_a=0 \tag{A11}$$

expressing the fact that $\dot{\tau}^a$ does not appear in the action (A6). Even then, (X^A,P_A) and (μ_α,τ^α) cannot be connected by a canonical transformation because the transition (A1) involves the embedding velocities $\dot{Z}^k$. These must be determined from the equations of motion. The only constraint containing P_k is $P_a=0$ and hence

$$\dot{Z}^k = M^a Z^k_a \ , \tag{A12}$$

where M^a is the Lagrange multiplier with which we adjoin the constraint (2.18) to the old action. We are thus obliged to extend the old phase space by the canonical pair (M^a, Π_a) and enlarge the old system of constraints by the primary constraint

$$\Pi_a = 0 \ . \tag{A13}$$

To match the variables, we extend the new phase space by the embedding variables

$$z^k = Z^k \tag{A14}$$

and their conjugate momenta p_k. The transformation (A1) then reads

$$\tau^0 = T\tilde{Z} \ , \qquad \tau^a = T\tilde{Z}M^a \ . \tag{A15}$$

Our task is to complete the transformation (A14) and (A15) into a canonical transformation

$$\begin{bmatrix} T, & Z^k, & M^a \\ P_T, & P_k, & \Pi_a \end{bmatrix} \leftrightarrow \begin{bmatrix} \mu, & z^k, & \mu_a \\ \tau^0, & p_k, & \tau^a \end{bmatrix} \ . \tag{A16}$$

This is best done by writing the generating function

$$F[X^A, M^a; \mu_\alpha, p_k] = \int_\Sigma d^3x \, (-T\tilde{Z}\mu - T\tilde{Z}M^a\mu_a + Z^k p_k) \tag{A17}$$

in terms of the old coordinates X^A, M^a, the new coordinates μ_α, and the new momenta p_k. The canonical transformation generated by F is given by

$$\tau^\alpha(x) = -\delta F/\delta\mu_\alpha(x), \quad z^k(x) = \delta F/\delta p_k(x) \tag{A18}$$

and

$$P_A(x) = \delta F/\delta X^A(x), \quad \Pi_a(x) = \delta F/\delta M^a(x) \ . \tag{A19}$$

Equations (A18) dutifully reproduce the transformation equations (A14) and (A15). Equations (A19) complete them into a canonical transformation:

$$P_T = -\tilde{Z}\mu \ , \tag{A20}$$

$$P_k = p_k + (T(\mu + M^b\mu_b)\tilde{Z}Z^a_k)_{,a} \ , \tag{A21}$$

$$\Pi_a = -T\tilde{Z}\mu_a \ . \tag{A22}$$

The transformation (A14) and (A15) and (A20)–(A22) connects the parametrized unimodular gravity with the Henneaux-Teitelboim formalism at the canonical level. It does so by casting the old constraints into the new constraints and vice versa. Equation (A22) maintains the equivalence of the old constraint (A13) with the new constraint (A11). Equation (A20) identifies $-\mu$ with the old cosmological field (2.19). It transfers the old constraint (2.25) into the new constraint (A10) and the old constraint (2.22) into the new constraint (A8). Finally, modulo the constraint $\mu_a=0$, Eq. (A21) shows that the old constraint $P_a=0$ [Eq. (2.18)] is equivalent to the new constraint

$$p_k = 0 \ . \tag{A23}$$

The only purpose of the constraint (A23) is to propagate the new coordinate z^k by an arbitrarily chosen multiplier.

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[13]To separate the ordering role of the cosmological time from its metric significance, Unruh and Wald identified the Heraclitian time with an arbitrary monotonically increasing function of the cosmological time. This distinction is not important in our further discussion.

[14]The arguments of functions are enclosed by parentheses, the function arguments of functionals by square brackets. We often meet quantities which are simultaneously functions of one set of arguments and functionals of another set of arguments. In such a case, we separate the two sets of arguments by a semicolon and close the parenthesis against the square bracket, as in $\Psi(\tau;\mathbf{g}]$.

[15]Hypersurfaces are best represented by embeddings $\Sigma \rightarrow M$ of the three-space Σ in M. A hypersurface is an equivalence class of embeddings modulo DiffΣ. If x^a are the coordinates in Σ and X^A the coordinates in M, an embedding is described by four functions $X^A(x^a)$ of three coordinates x^a.

[16]Unruh and Wald imply that the Heraclitian time labels events along the worldline of a single observer, not the leaves of a spacelike foliation. If the observer has to measure data outside his worldline, he is expected to send out probes, retrieve them later, and complete the measurement at a fixed value of his private Heraclitian time. There is a tacit assumption this can be done in unimodular gravity in such a way that (I) the observer is able to measure dynamical variables constructed from the standard data $g_{ab}(x)$, $p^{ab}(x)$ which refer to a single spacelike hypersurface, (II) to perform and order all such measurements at definite instants of his private Heraclitian time which coincides with (a given monotonic function of) the cosmological time, i.e, with the four-volume between spacelike hypersurfaces, and (III) recover in the end the (classical and quantum) predictions which follow from the standard canonical formalism of unimodular gravity. We feel that our analysis of the relation between the cosmological time and hypertime indicates that these requirements are not likely to be realizable.

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