PHYSICAL REVIEW D VOLUME 43, NUMBER 2 15 JANUARY 1991

Vacuum for a massless scalar field outside a collapsing body in de Sitter spacetime

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(Received 2 April 1990; revised manuscript received 27 July 1990)

We present a general way to define regular vacuum states of a quantized massless scalar field in two-dimensional spacetimes with horizons. We discuss in more detail the cases of Schwarzschild–de Sitter spacetime and especially the exterior of a massive shell that collapses to form a black hole in a two-dimensional de Sitter spacetime. In the latter case a vacuum state is defined using modes of positive frequency with respect to the past cosmological horizon's affine parameter. In this vacuum static observers, long after the shell has collapsed, detect thermal fluxes coming from both the cosmological and the black-hole horizons, and characterized by the corresponding Hawking temperatures. The renormalized stress-energy tensor in this vacuum state is regular everywhere and has precisely the form one would expect from prior experience with de Sitter spacetime and with gravitational collapse spacetimes that have a vanishing cosmological constant.

I. INTRODUCTION

During the sixteen years since Hawking[1] discovered that black holes should emit thermal radiation, a clear understanding of the quantum-field-theory properties of spacetime horizons has emerged.[2–6] Among the horizons that have been studied are those of Schwarzschild spacetime, Kerr spacetime, and de Sitter spacetime, as well as gravitational collapse spacetimes that are asymptotically Schwarzschild or Kerr in the future. In each of these spacetimes, when a quantum field is in an "Unruh-type"[2] vacuum state, the field's properties are remarkably simple and aesthetic: (i) Particle detectors at rest just above the horizon, and also detectors at rest far from the horizon ("static detectors"), measure the horizon to emit perfectly thermal radiation at the "Hawking temperature;" (ii) the renormalized stress-energy tensor $T^{\mu\nu}$ is regular at the horizon; (iii) far from the horizon $T^{\mu\nu}$ has the form of outgoing, thermal radiation; (iv) near the horizon $T^{\mu\nu}$ has the form of downgoing thermal radiation, but with negative energy density rather than positive—as is required by energy-momentum conservation; (v) the near-horizon renormalized $T^{\mu\nu}$ can be regarded as the naive flat spacetime stress-energy tensor corresponding to the quanta measured by static particle detectors, minus a contribution from vacuum polarization which is precisely thermal at the horizon's temperature.[6,7]

Recently Hiscock[8] has argued that there is a breakdown in these properties in the case of a body that collapses to form a black hole in de Sitter spacetime. Such a spacetime is more complex than those studied previously because it has two horizons, the black-hole horizon and the cosmological horizon, with two different temperatures T_h and T_c. The temperature difference, Hiscock speculated, forces the renormalized stress-energy tensor in every vacuum state to be divergent at least at one of the horizons. Correspondingly, Hiscock implied, if nature chooses a vacuum state that is well behaved at the cosmological horizons (as one would expect), gravitational collapse will produce a black-hole horizon that has a divergent renormalized $T^{\mu\nu}$. If true, this would mean that quantum field theory produces an instability of the black-hole horizon. As evidence for this speculation, Hiscock enumerated several possible vacuum states for a massless scalar field and showed that each of them had a divergent $T^{\mu\nu}$ at one of the horizons.

In this paper we show that Hiscock's conjecture is incorrect: There do exist states in both the "eternal" black-hole–de Sitter spacetime (otherwise called Schwarzschild–de Sitter spacetime) and the gravitational-collapse–de Sitter spacetime which are regular on all of the horizons, cosmological and black hole. These states have the nice properties enumerated above, including renormalized stress-energy tensors $T_{\mu\nu}$, which are regular at all of the horizons.

Throughout this paper we will restrict our attention to two-dimensional model spacetimes in which the θ,ϕ dependence of the metric is suppressed. Consider, first, the two-dimensional eternal black-hole–de Sitter spacetime.[3] In this spacetime the left- and right-moving modes are uncoupled. As an aid in defining our vacuum state we choose two null geodesics which cross in the region between the black-hole horizon and the cosmological horizon. For each of the left- and right-moving modes, we define positive frequency with respect to the affine parameters along these null geodesics. Since the affine parameters are regular along each of these null geodesics as they cross the horizons, the vacuum state defined with respect to these affine parameters will also be regular on the horizons, leading to regular stress-energy tensors at each of the horizons. One can easily see that this vacuum state is not invariant with respect to the time translation isometry of Schwarzschild–de Sitter spacetime. This property is indeed inevitable according to Kay and Wald,[9] who have proved the nonexistence of stationary, nonsingular states in Schwarzschild–de Sitter spacetime.

In the case of the gravitational-collapse–de Sitter spacetime, to which we will devote more attention, we define the vacuum state with respect to the affine parameter along the past cosmological horizon. The positive-frequency modes come into the collapsing star, reflect off the origin $r=0$ within the collapsing body, and propagate outward again. These modes will again be regular along any null line traveling from the past cosmological horizon into the future black-hole horizon. This regularity of the outward propagating positive-frequency modes along inward propagating surfaces at the horizon again ensures that the stress energy tensor will be regular at the black-hole horizon, in addition to being regular at the past and future cosmological horizons—just as in the case of usual black-hole spacetimes.[2]

This paper is organized as follows. In Sec. II we examine the stress-energy tensor for "vacuum" states for a massless scalar field in a static spacetime with a horizon. As did Hiscock, we restrict attention to a two-dimensional version of the spacetime in which the angles (θ, ϕ) are suppressed[10] (this simplifies the calculations). We show that for the stress energy to be regular on the horizon, the null coordinate used to define the vacuum state in the manner of Davies, Fulling, and Unruh[11] (DFU) must have certain smoothness properties across the horizon. As mentioned above, the null coordinates defined as the affine parameters of a pair of crossing null geodesics indeed have these smoothness properties. We also briefly outline the application of these ideas to the eternal black-hole–de Sitter spacetime. In Sec. III we specialize to the geometry of the gravitational-collapse–de Sitter spacetime. For simplicity we take the collapsing body to be a thin, spherical, massive shell. In Sec. IV we introduce the vacuum state which we designate by $|\,\mathcal{V}\rangle$, and in Sec. V we explore its properties at early times, before the collapse begins, and at late times, after the black-hole horizon forms. Among other things, we show that in this vacuum state static particle detectors measure incoming modes to be precisely thermally populated at the temperature of the cosmological horizon (which is where these modes originate). Outgoing modes, by contrast, are measured by static detectors to be populated in different manners before the collapse and after the collapse: before, they are thermally populated at the cosmological temperature; afterward, they are thermally populated at the black-hole temperature. In Sec. VI we evaluate the renormalized stress-energy tensor in the $|\,\mathcal{V}\rangle$ vacuum, and show that it has all the nice properties that one might expect from prior experience: It is regular at all horizons; and near each future horizon it is the $T^{\mu\nu}$ of radiation flowing into the horizon—radiation that is a superposition of perfectly thermal radiation at the black-hole temperature and at the cosmological temperature, with positive energy associated with the temperature of the distant horizon and negative energy with that of the nearby one. This $T^{\mu\nu}$ has just the form that one expects from the "membrane paradigm:"[6,7] it is the naive stress-energy corresponding to the measurements made by static particle detectors, minus that of perfectly thermal radiation at the temperature of the nearby horizon.

II. REGULAR STATES ON A STATIC HORIZON

In this section we will be concerned with requirements that a state must satisfy in order that the stress-energy tensor be regular on a horizon. We will show that the requirement is that the state itself be regular on the horizon. We will work in a two-dimensional static spacetime, with the field of interest being a massless scalar field.

By a suitable choice of the spatial coordinate r, we can bring the metric into the form

$$ds^2 = f(r)\, dt^2 - \frac{dr^2}{f(r)} \,, \tag{1}$$

where $f(r)$ is assumed to be zero at $r = r_h$ (the horizon), and to be smooth (have a power-series expansion in $r - r_h$) near $r = r_h$.

Using the usual null coordinates in the region exterior to the horizon,

$$u = t - r_* \tag{2}$$

and

$$v = t + r_* \,, \tag{3}$$

where $dr_* = dr/f$, we get the two-dimensional metric

$$ds^2 = f(r(v-u))du\, dv \,. \tag{4}$$

If we define new null coordinates

$$V = \int h_v(v) dv \tag{5}$$

and

$$U = \int h_u(u) du \,, \tag{6}$$

where $h_v(v)$ and $h_u(u)$ are some as yet unspecified functions, we can rewrite the metric as

$$ds^2 = \frac{f(r(v-u))}{h_v(v)h_u(u)} dU\, dV \equiv C dU\, dV \ . \tag{7}$$

As Davies, Fulling, and Unruh[11] (DFU) we can define a vacuum state with respect to these UV coordinates by choosing the positive-frequency modes of the Klein-Gordon field to have the form $e^{-i\omega U}$ and $e^{-i\omega V}$ for $\omega > 0$. The energy-momentum tensor in this UV vacuum is then given in DFU by

$$T_{UU} = -\frac{1}{12\pi} C^{1/2} \partial_U \partial_U C^{-1/2} \ , \tag{8}$$

$$T_{VV} = -\frac{1}{12\pi} C^{1/2} \partial_V \partial_V C^{-1/2} \ , \tag{9}$$

$$T_{UV} = -\frac{1}{96\pi} C\mathcal{R} \ , \tag{10}$$

where $\mathcal{R}$ is the curvature scalar for the spacetime.

This tensor obeys the usual conservation law

$$T^{\mu\nu}{}_{;\nu} = 0 \ , \tag{11}$$

which in this case reduces to

$$T_{\tilde{U}\tilde{U},\tilde{V}} = \frac{\tilde{C}}{96\pi} \mathcal{R}_{,\tilde{U}} \ , \qquad T_{\tilde{V}\tilde{V},\tilde{U}} = \frac{\tilde{C}}{96\pi} \mathcal{R}_{,\tilde{V}} \ , \tag{12}$$

in any null coordinate system $\tilde{U}\tilde{V}$. Thus along the null ray $\tilde{U}$=const, we have

$$T_{\tilde{U}\tilde{U}} = T^0_{\tilde{U}\tilde{U}} + \frac{1}{96\pi} \int \tilde{C} \mathcal{R}_{,\tilde{U}} d\tilde{V} \ . \tag{13}$$

and similarly for $T_{\tilde{V}\tilde{V}}$ along a $\tilde{V}$=const ray.

We are interested in the behavior of the stress-energy tensor at the horizon, $r = r_h$. We define a $\tilde{U}\tilde{V}$ coordinate system in which the metric coefficients are regular along the horizon by

$$\tilde{U} = -e^{-\kappa u} \ , \quad \tilde{V} = e^{\kappa v} \ . \tag{14}$$

where $\kappa = \frac{1}{2} f'(r_h)$ and where the prime denotes a derivative with respect to r. In these coordinates the horizon $r = r_h$ is located at $\tilde{U} = 0$. Using the relation (13), the value of $T_{\tilde{V}\tilde{V}}$ on the horizon can be found from the value off the horizon in the interior region. Assuming that the curvature $\mathcal{R}$ is regular everywhere, including on the horizon, we see that $T_{\tilde{V}\tilde{V}}$ will be regular on the horizon if it is regular in the rest of the spacetime. We thus need worry only about the behavior of $T_{\tilde{U}\tilde{U}}$ on the horizon.

We have

$$T_{\tilde{U}\tilde{U}} = \left(\frac{\partial U}{\partial \tilde{U}}\right)^2 T_{UU} \ . \tag{15}$$

Writing

$$\tilde{U} = \int \tilde{h}_u(u) du \ , \tag{16}$$

this becomes

$$\begin{aligned} T_{\tilde{U}\tilde{U}} &= \frac{h_u^2(u)}{\tilde{h}_u^2(u)} T_{UU} \\ &= \frac{1}{48\pi} \frac{1}{\tilde{h}_u^2(u)} \left(2\frac{\ddot{f}}{f} - 3\frac{\dot{f}^2}{f^2} - 2\frac{\ddot{h}_u}{h_u} + 3\frac{\dot{h}_u^2}{h_u^2} \right) \\ &= \frac{1}{48\pi} \frac{e^{2\kappa u}}{\kappa^2} \left(\frac{f'' f}{2} - \frac{f'^2}{4} - 2\frac{\ddot{h}_u}{h_u} + 3\frac{\dot{h}_u^2}{h_u^2} \right) , \end{aligned} \tag{17}$$

where an overdot denotes a derivative with respect to u. The horizon occurs at $u = \infty$, and in order that the stress-energy tensor be regular on the horizon, the quantity in large parentheses must fall off at least as fast as $e^{-2\kappa u}$. The terms in f go as

$$2f''f - f'^2 \approx -f_0'^2 + O((r-r_h)^2) = -f_0'^2 + O(e^{-2\kappa u}) \ , \tag{18}$$

where $f_0' = df/dr\,|_{r=r_h}$. We must then have

$$2\frac{\ddot{h}_u}{h_u} - 3\frac{\dot{h}_u^2}{h_u^2} = -\frac{f_0'^2}{4} + O(e^{-2\kappa u}) = -\kappa^2 + O(e^{-2\kappa u}) \ , \tag{19}$$

or

$$4 h_u^{1/2} \partial_u^2 h_u^{-1/2} = \kappa^2 + O(e^{-2\kappa u}) \ . \tag{20}$$

Thus for the stress energy to be regular, we must have

$$h_u(u) = e^{\mp\kappa u}[\text{const} + O(e^{-2\kappa u})] \ , \tag{21}$$

or

$$\begin{aligned} U &= \int h_u(u) du \\ &= e^{\mp\kappa u}[\text{const} + O(e^{-2\kappa u})] \\ &= \tilde{U}^{\pm 1}[\text{const}' + O(\tilde{U}^2)] \ . \end{aligned} \tag{22}$$

U is the coordinate which is used to define the vacuum state, while $\tilde{U}$ is the coordinate regular on the horizon. Thus this relation states that the coordinates defining the state must be related to the coordinate regular on the horizon either by direct or inverse proportionality. Direct proportionality is easily obtained. Take any null geodesic which intersects the horizon. Define the U coordinate to be the affine parameter along this geodesic. This coordinate will obey the required condition.

The inverse relation is somewhat harder to arrange. The simplest way is to place reflective boundary conditions on the scalar field at some point $r = r_0$ within the spacetime. If we now choose the state such that positive frequencies are defined with respect to the affine parameter $\tilde{V}$ along the $\tilde{U} = 0$ horizon, then the boundary conditions at $r_0 = r(v-u) = r[\ln(-\tilde{V}\tilde{U})/\kappa]$ mean that positive frequencies in the $\tilde{U}$ direction are given by $e^{i\omega'\tilde{U}^{-1}}$ for $\omega' > 0$, which is the required condition. The state we shall examine below for the collapse–de Sitter spacetime is of just such a form.

This inverse relationship is a general one. If we have two coordinates U and $\bar{U}$ where $\bar{U} = 1/U$, then states defined with respect to these two coordinates lead to exactly the same stress-energy tensors.

$$\bar{T}_{UU} = \left(\frac{\partial \bar{U}}{\partial U}\right)^2 \bar{T}_{\bar{U}\bar{U}} = -\frac{1}{12\pi}U^{-4}\bar{C}^{1/2}\partial_{\bar{U}}^2\bar{C}^{-1/2} = -\frac{1}{12\pi}U^{-3}C^{1/2}(U^2\partial_U)^2C^{-1/2}/U$$
$$= -\frac{1}{12\pi}\frac{1}{U}C^{1/2}\partial_U U^2\partial_U\frac{C^{-1/2}}{U} = -\frac{1}{12\pi}C^{1/2}\partial_U^2 C^{-1/2} = T_{UU} \ , \tag{23}$$

while T_{VV} is obviously left unchanged, as is $T_{UV} \propto \mathcal{R}$.

The energy-momentum tensor can be simply calculated everywhere in the static spacetime if we choose U and V to be affine parameters along some curves $u = u_0$ and $v = v_0$. In the uv coordinates, the Ricci curvature $\mathcal{R}$ depends only on $v - u$. Thus the conservation equation reads

$$T_{uu},_v = \frac{f}{96\pi}\mathcal{R}(v-u),_u = -\frac{f}{96\pi}\mathcal{R}(v-u),_v \tag{24}$$

so that

$$T_{uu} + \frac{f}{96\pi}\mathcal{R} = \int \frac{f,_v}{96\pi}\mathcal{R}dv \tag{25}$$

along a u =const null surface. Now, along the surface $v = v_0$, we choose U to be the affine parameter. Thus the metric $C = 2g_{UV}$ is independent of U along $v = v_0$, and thus T_{UU} (and T_{uu}) is zero along this surface. Furthermore, f depends only on r, so that $f,_v = \frac{1}{2}f'f$. We thus have

$$T_{uu} = -\frac{1}{96\pi}\left(f(v-u)\mathcal{R}(v-u) - f(v_0-u)\mathcal{R}(v_0-u) - \int_{v_0}^{v}\frac{1}{2}f'f\mathcal{R}dv\right) , \tag{26}$$

and similarly for T_{vv}.

Finally, we calculate the change in the energy-momentum tensor component $T_{\tilde{U}\tilde{U}}$ along the $\tilde{U} = 0$ horizon. We have

$$T_{\tilde{U}\tilde{U}} = T^0_{\tilde{U}\tilde{U}} + \frac{1}{96\pi}\int \tilde{C}\mathcal{R},_{\tilde{U}}\, d\tilde{V} \ . \tag{27}$$

For our spacetime, $\mathcal{R}$ is a function only of r, and is independent of where one is on the horizon. Thus we can write

$$\begin{aligned} \tilde{C}\mathcal{R},_{\tilde{U}} &= \mathcal{R},_r \tilde{C}f(r)\left(-\frac{1}{2\kappa}\right)e^{\kappa u} \\ &\approx -\tilde{C}\mathcal{R},_r (r - r_h)e^{\kappa u} \\ &\approx -\frac{1}{\kappa}\tilde{C}\mathcal{R},_r e^{\kappa v} \\ &\approx -\frac{1}{\kappa}\tilde{C}\mathcal{R},_r \tilde{V} \ , \end{aligned} \tag{28}$$

and, since $\tilde{C}$ is constant along the horizon, we finally get

$$T_{\tilde{U}\tilde{U}} = T^0_{\tilde{U}\tilde{U}} - \frac{1}{96\pi\kappa}\tilde{C}\mathcal{R},_r \frac{\tilde{V}^2}{2} \ . \tag{29}$$

Thus in this affine parameter coordinate system, this component of the stress-energy tensor diverges as one travels along the horizon, if $\mathcal{R},_r$ is not equal to zero at the horizon. However, if we transform to the proper reference frames of freely falling observers whose four-velocities are

$$\mathbf{u} = \frac{\sqrt{f_0}}{f}\frac{\partial}{\partial t} - \sqrt{f_0 - f}\frac{\partial}{\partial r} \ , \tag{30}$$

where f_0 is f at the points from which the observers are dropped, we find the components of the stress-energy tensor to be finite as $\tilde{V} \to 0$. Specifically, at the horizon we have the relation

$$\partial_{\hat{u}} \propto \frac{1}{\tilde{V}}\partial_{\tilde{U}} \ , \tag{31}$$

where $\partial_{\hat{u}}$ is the $\tilde{U}$-directed proper null vector of a freely falling observer. This ensures the finiteness of $T_{\hat{u}\hat{u}}$ as $\tilde{V} \to \infty$ along the horizon.

In order to examine the behavior of $T_{\hat{v}\hat{v}}$, where $\partial_{\hat{v}}$ is the $\tilde{V}$-directed proper null vector of a freely falling observer, we need to see what happens along the null ray $u = u_0$ as $v \to \infty$. There we might approach another horizon, located at $r = r_h' > r_0$, whose surface gravity is κ'. This is indeed the case in the Schwarzschild–de Sitter spacetime.[3] Here r_h is the position of the black-hole horizon while r_h' corresponds to r_c, the position of the cosmological horizon (see Sec. III). Alternatively, there could be a future null infinity ($\kappa' = 0$). Again using the relations among $\tilde{V}$-directed vectors at the $r = r_h$ horizon

$$\partial_{\hat{v}} \propto \tilde{V}\partial_{\tilde{V}} \propto \partial_v \ , \tag{32}$$

and the DFU formula we have

$$\begin{aligned} T_{\hat{v}\hat{v}} &= \left(\frac{\partial V}{\partial \hat{v}}\right)^2 T_{VV} \\ &\propto h_v^2 C^{1/2}\partial_V^2 C^{-1/2} \\ &\propto \left(e^{\mp\kappa' v}\right)^2 \left(\frac{e^{\kappa(v-u)}}{e^{\mp\kappa u}e^{\mp\kappa' v}}\right)^{1/2} \\ &\quad \times \left(\frac{1}{e^{\mp\kappa' v}}\partial_v\right)^2 \left(\frac{e^{\kappa(v-u)}}{e^{\mp\kappa u}e^{\mp\kappa' v}}\right)^{-1/2} \propto e^{\pm 2\kappa' v}e^{\mp 2\kappa' v} , \end{aligned} \tag{33}$$

as $v \to \infty$. Here the upper signs correspond to the choice of affine parameters along $u = u_0$ or $v = v_0$ rays. For instance, near the $v = \infty$ horizon we have either $V \propto \tilde{V}'$ or $V \propto 1/\tilde{V}'$ as discussed above in this section. Hence $h_v(v) \propto \exp(\mp\kappa' v)$. From Eq. (33) we see that $T_{\hat{v}\hat{v}}$ is finite at the horizon located at $r = r_h$ even as $\tilde{V} \to \infty$. Together with the finiteness of $T_{\hat{u}\hat{u}}$ and $T_{\hat{u}\hat{v}}$ this guaranties the regularity of the stress-energy tensor everywhere on the horizon. Similar arguments can be applied to any horizon that might exist at $u = \pm\infty$ or $v = \pm\infty$.

The eternal black-hole–de Sitter spacetime can be regarded as a special case of the above analysis. By choosing U and V coordinates to be the affine parameters of null lines that cross at the point t_0, r_0 ($r_h < r_0 < r_c$), we define a vacuum state whose renormalized stress-energy tensor is, according to the above discussion, regular at both the black-hole and cosmological horizons. We shall return to this vacuum state at the end of Sec. IV, after first discussing vacua for the gravitational-collapse–de Sitter spacetime.

III. THE GRAVITATIONAL-COLLAPSE–DE SITTER SPACETIME

According to the generalization[12] of Birkhoff's theorem to nonzero values of the cosmological constant Λ, the exterior region of a spherically symmetric body of mass M has the Schwarzschild–de Sitter line element

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2) , \tag{34}$$

where $f(r)$ is

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 . \tag{35}$$

For the sake of simplicity, we will assume that the massive body is a shell of radius R whose interior is described by the de Sitter line element

$$ds^2 = g(r)d\tilde{t}^2 - \frac{dr^2}{g(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2) , \tag{36}$$

where $g(r)$ is

$$g(r) = 1 - \frac{\Lambda}{3}r^2 . \tag{37}$$

Requiring that the proper time as measured by a clock on the shell be independent of whether it is calculated using the exterior or interior metric, we obtain

$$\frac{dt}{d\tilde{t}} = \left\{ \frac{1}{f}\left[g + \left(\frac{1}{f} - \frac{1}{g}\right)\left(\frac{dR}{d\tilde{t}}\right)^2 \right] \right\}^{1/2}_{r=R(\tilde{t})} . \tag{38}$$

We suppose that the radius of the shell is fixed ($R = R_0$) until the moment $\tilde{t}$=0 when the shell starts imploding. At the moment $\tilde{t} = \tilde{t}_h$ it crosses its horizon radius r_h and forms a black hole.

Of the three zeros of the equation $f(r)$=0 one, r_-, is negative. The two others, r_h and r_c, are the locations of

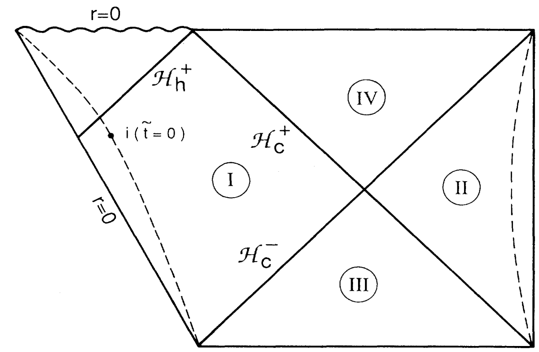

FIG. 1. Penrose diagram of a gravitational-collapse–de Sitter spacetime. The shell in region I starts its collapse at point $\tilde{t} = 0$, while the shell in region II remains static forever.

the black-hole and cosmological horizons, respectively.[13] The corresponding surface gravities are

$$\kappa_h = \frac{1}{2}\left(\frac{df}{dr}\right)_{r=r_h} = \frac{\Lambda(r_c - r_h)(r_h - r_-)}{6r_h} ,$$
$$\kappa_c = -\frac{1}{2}\left(\frac{df}{dr}\right)_{r=r_c} = \frac{\Lambda(r_c - r_h)(r_c - r_-)}{6r_c} . \tag{39}$$

The Penrose diagram describing the causal structure of the spacetime is shown in Fig. 1. In this diagram one sees the past ($\mathcal{H}_c^-$) and future ($\mathcal{H}_c^+$) cosmological horizons as well as the future black-hole horizon ($\mathcal{H}_h^+$) which is created by the collapse. Note that in this spacetime, by contrast with pure de Sitter, $r = 0$ is uniquely determined: it is the center of the spherical shell.

IV. THE VACUUM STATE $|\mathcal{V}\rangle$

As in Sec. II, we will ignore the spherical coordinates ϕ and θ, reducing the spacetime to two dimensions.

In addition to u and v, Eqs. (2)–(4), we shall need a second set of coordinates U and V defined as follows; see Fig. 2.

We first define the affine parameter of the past cosmological horizon

$$\mathcal{V} \equiv -e^{-\kappa_c v} . \tag{40}$$

Then we choose an arbitrary point $\mathcal{P}$ at which the values of U and V are to be defined. The value of $\mathcal{V}$ at which the past-directed null ray propagating rightwards from $\mathcal{P}$ hits $\mathcal{H}_c^-$ is the coordinate $V(\mathcal{P})$. This construction gives

$$V = -e^{-\kappa_c v} \tag{41}$$

everywhere outside the shell. Similarly, we extend the past-directed, leftward-propagating null ray from $\mathcal{P}$ through the shell to r=0, and there we reflect it into a past-directed, rightward-propagating null ray. The value $\mathcal{V}$ at which this ray hits $\mathcal{H}_c^-$ is the coordinate $U(\mathcal{P})$.

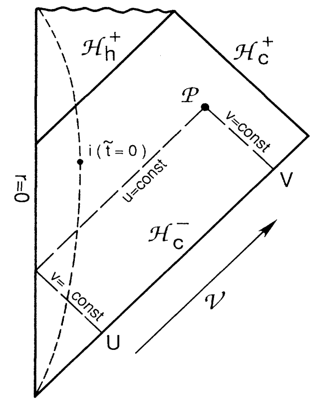

FIG. 2. The definition of U and V coordinates. For simplicity, only the region I from Fig. 1 is depicted.

Note that the coordinate V can be extended beyond the future cosmological horizon $\mathcal{H}_c^+$.

For $u < u_i$, where $U_i(u_i)$ and $V_i(v_i)$ are the coordinates of the beginning of the implosion, we have the simple relation

$$U = -e^{-\kappa_c u} . \tag{42}$$

However for $u > u_i$ this relation is changed. As the point $\mathcal{P}$ gets close to the future black-hole horizon, the past-directed, leftward-propagating ray used to define the coordinate U is strongly affected by the collapsing shell. In this region $(r_* \to -\infty, u \to \infty)$ the metric function f has the following asymptotic behavior:

$$f \propto e^{2\kappa_h r_*} = e^{\kappa_h (v-u)} . \tag{43}$$

From this relation and Eq. (38) we obtain

$$\frac{dt}{d\tilde{t}} \approx -\frac{\dot{R}}{f} \propto e^{-\kappa_h (v-u)} , \tag{44}$$

near the point where the shell crosses its black-hole horizon. Using Eqs. (43) and (44) we can, following the past-directed ray through and out of the shell,[14] derive the expression

$$\frac{d\ln(-U)}{du} \propto -e^{-\kappa_h u} , \tag{45}$$

which implies

$$-e^{-\kappa_h u} \propto -\ln\left(\frac{U}{U_h}\right) \approx \left(1 - \frac{U}{U_h}\right) \tag{46}$$

for values of U very close to U_h, the position of the black-hole horizon.

We now quantize the real scalar field with respect to the modes proportional to $\exp(-i\omega V)$ or $\exp(-i\omega U)$. Both types of modes originate at the past cosmological horizon $\mathcal{H}_c^-$, where they are positive frequency with respect to the past horizon affine parameter $\mathcal{V}$. Thus, they are a natural extension to our gravitational-collapse spacetime of de Sitter–invariant modes in a pure de Sitter spacetime. The quantum state which contains no particles in these modes we will call the $|\, \mathcal{V}\rangle$ vacuum. It is a natural generalization of a de Sitter–invariant vacuum of de Sitter spacetime. Namely, in the case where the shell has zero mass (pure de Sitter), $|\, \mathcal{V}\rangle$ reduces to the conformal de Sitter vacuum.[15] On the other hand, in the case of a collapsing body in an asymptotically flat spacetime $(\Lambda = 0)$, the past horizon affine parameter would be replaced by the advanced time at the past null infinity and our state $|\, \mathcal{V}\rangle$ would become the vacuum state originally discussed by Hawking[1] and subsequently by Unruh.[2]

The vacuum $|\, \mathcal{V}\rangle$ is closely related to the vacuum state for the eternal black-hole–de Sitter spacetime discussed in Sec. II. The vacuum defined there using the affine parameters of any pair of crossing null rays can be shown to exibit late time behavior identical to that of the vacuum $|\, \mathcal{V}\rangle$. By contrast with Schwarzschild–de Sitter, however, in the gravitational-collapse–de Sitter spacetime we need only one null ray propagating from the past to the future cosmological horizon to specify the vacuum state $|\, \mathcal{V}\rangle$. This null ray is chosen to coincide with the past cosmological horizon, thereby defining the state's initial conditions in the past relative to all static observers outside the massive body.

V. PROPERTIES OF $|\, \mathcal{V}\rangle$ AS MEASURED BY STATIC OBSERVERS

Because the proper time τ of a static observer $(r = r_{\rm so})$ is proportional to the time coordinate t,

$$d\tau = \sqrt{f(r_{\rm so})}dt , \tag{47}$$

and hence is also proportional to $u = t - r_*$ and to $v = t + r_*$, a particle detector carried by such an observer detects particles that are of positive frequency with respect to u and v. The corresponding vacuum state $|\, \mathcal{S}\rangle$ ($\mathcal{S}$ for "static observers") is one in which the particle detector sees no quanta in the modes $e^{-i\omega u}$ and $e^{-i\omega v}$.

The relation (41) between the null coordinates V of the $|\, \mathcal{V}\rangle$ vacuum and v of the $|\, \mathcal{S}\rangle$ vacuum implies that when the field is in the $|\, \mathcal{V}\rangle$ state, static observers will see the modes $e^{-i\omega v}$ thermally populated at the cosmological temperature

$$T_c = \frac{\kappa_c}{2\pi} . \tag{48}$$

Stated more precisely, when studying observables confined to region I of the spacetime, one can regard the pure state $|\, \mathcal{V}\rangle$ as being equal to a mixed state that is obtained from $|\, \mathcal{S}\rangle\langle\mathcal{S}\,|$ by populating all the incoming modes $e^{-i\omega v}$ thermally at temperature T_c.[16]

Similarly, the asymptotic relations between U and u at early and late times imply that, for outgoing modes $e^{-i\omega u}$, the $|\, \mathcal{V}\rangle$ vacuum is seen by static observers as obtained by populating $|\, \mathcal{S}\rangle\langle\mathcal{S}\,|$ thermally at the cosmological temperature T_c at early times (before the collapse), and at the black-hole temperature

$$T_h = \frac{\kappa_h}{2\pi} \tag{49}$$

at late times (after the collapse).

These thermal population properties of $| \mathcal{V}\rangle$ show up not only in the mathematical expressions for $| \mathcal{V}\rangle$ in terms of $| \mathcal{S}\rangle$, but also in the behavior of static particle detectors. Consider, for concreteness, a model particle detector that is adiabatically switched on at late times (long after the collapse). When the quantum field $\phi(x)$ is in the state $| \mathcal{V}\rangle$, its influence on the detector is described by the Wightman function[17]

$$D^+_{\mathcal{V}}(x,x') \equiv \langle \mathcal{V} \mid \phi(x)\phi(x') \mid \mathcal{V}\rangle = -\frac{1}{4\pi}\ln[(V - V' - i\epsilon)(U - U' - i\epsilon)] \ . \quad (50)$$

Using this Wightman function and the asymptotic expression (46) for large u, we find that the transition rate from the ground state of the detector to an excited state of energy E is proportional to the response function

$$\mathcal{F}(E) = \frac{1}{2E}\left(\frac{1}{e^{E\alpha/T_h}-1} + \frac{1}{e^{E\alpha/T_c}-1}\right) \ . \quad (51)$$

Here $\alpha(r_{so}) \equiv \sqrt{f(r_{so})}$ is the "lapse function," which blue shifts the temperature. Expression (51) confirms that the detector behaves as though it were bathed by cosmological and black-hole thermal fluxes coming from opposite directions and having the temperatures T_c and T_h, respectively. That this conclusion does not depend on the position of the detector relative to the horizons is due to the fact that in two spacetime dimensions the quanta propagate freely, without encountering any centrifugal barrier and without scattering off spacetime curvature.[2,5]

VI. THE VACUUM STRESS-ENERGY TENSOR

The renormalized stress-energy tensor of the conformally coupled scalar field in the $| \mathcal{V}\rangle$ vacuum of our two-dimensional spacetime is given by the DFU formulas, Eqs. (8)–(10).

Before the collapse starts ($U < U_i$) this stress-energy tensor, transformed to the proper reference frame of a static observer, has the following time-independent form:

$$T_{\hat\mu\hat\nu} = \frac{1}{24\pi}\frac{1}{f}\left(\kappa_c^2 - \frac{\Lambda}{3} + \frac{2M\Lambda}{r} - \frac{2M}{r^3} + \frac{3M^2}{r^4}\right) I_{\hat\mu\hat\nu} - \frac{\mathcal{R}}{48\pi} g_{\hat\mu\hat\nu} \ , \quad (52)$$

where $I_{\hat\mu\hat\nu}$ is the unit 2×2 matrix.

As we approach the cosmological horizon ($r \to r_c$) for $U < U_i$ the first term in Eq.(52) vanishes and we are left with the simple expression

$$T_{\mu\nu} = -\frac{\mathcal{R}}{48\pi} g_{\mu\nu} \ . \quad (53)$$

This $T_{\mu\nu}$ is obviously regular at the horizon.

Long after the collapse starts, at $u \gg u_i$, we have

$$T_{tt} = \frac{1}{48\pi}[\kappa_c^2 + \kappa_h^2 - 2F(r)] - \frac{1}{48\pi}\mathcal{R}f + O(e^{-2\kappa_h u}) \ ,$$

$$T_{rr} = \frac{1}{f^2}T_{tt} + \frac{1}{48\pi}\frac{2\mathcal{R}}{f} \ , \quad (54)$$

$$T_{tr} = \frac{1}{48\pi}\frac{1}{f}(\kappa_c^2 - \kappa_h^2) + O(e^{-2\kappa_h u}) \ ,$$

where

$$F(r) \equiv \frac{1}{4}f'^{\,2} - \frac{1}{2}ff'' \ . \quad (55)$$

At $u \gg u_i$ static observers observe the thermal radiation coming from the black hole, in addition to the already existing cosmological Hawking radiation. Very close to the black-hole horizon, $r \to r_h$ ($v - u \to -\infty$) we have

$$F(r) = \kappa_h^2 + O(e^{2\kappa_h(v-u)}) \ . \quad (56)$$

In that region the renormalized stress-energy tensor (54) in a static observer's proper frame is

$$T_{\hat t\hat t} = \frac{\pi}{12}\left[\left(\frac{T_c}{\alpha}\right)^2 - \left(\frac{T_h}{\alpha}\right)^2\right] - \frac{\mathcal{R}}{48\pi} + \text{h.o.} \ ,$$

$$T_{\hat r\hat r} = \frac{\pi}{12}\left[\left(\frac{T_c}{\alpha}\right)^2 - \left(\frac{T_h}{\alpha}\right)^2\right] + \frac{\mathcal{R}}{48\pi} + \text{h.o.} \ , \quad (57)$$

$$T_{\hat t\hat r} = T_{\hat r\hat t} = \frac{\pi}{12}\left[\left(\frac{T_c}{\alpha}\right)^2 - \left(\frac{T_h}{\alpha}\right)^2\right] + \text{h.o.} \ ,$$

where h.o. denotes terms of second order or higher in $\alpha \equiv \sqrt{f}$.

The leading $O(\alpha^{-2})$ terms in (57) have precisely the form of ingoing thermal radiation in two-dimensional spacetime, except that the sign of the component at temperature T_h is negative rather than positive. This result, obtained directly from the Davies-Fulling-Unruh formulas, Eqs. (8)–(10), has a simple interpretation in terms of measurements made by static observers—an interpretation embodied in the "membrane paradigm" for black holes:[6,7] The static observers, near the horizon, measure outgoing modes to be precisely thermally populated at temperature T_h, and incoming modes precisely thermal at temperature T_c. The rule for renormalization, in terms of these static observers' measurements, is to subtract off, in all modes, a thermal contribution with temperature T_h. Doing so leaves zero net renormalized stress energy in the outgoing modes, and leaves in the incoming modes the difference between a thermal flux at temperature T_c and that at T_h—which is precisely the $O(\alpha^{-2})$ contribution to expression (57).

Turn attention now from the vicinity of the black-hole horizon to the vicinity of the future cosmological horizon, long after the collapse. Near the cosmological horizon, $r \to r_c$ ($v - u \to \infty$) we have

$$F(r) = \kappa_c^2 + O(e^{-2\kappa_c(v-u)}) \ . \quad (58)$$

There the renormalized stress-energy tensor (54) in the proper reference frame of a static observer takes the form

$$T_{\hat{t}\hat{t}} = \frac{\pi}{12}\left[\left(\frac{T_h}{\alpha}\right)^2 - \left(\frac{T_c}{\alpha}\right)^2\right] - \frac{\mathcal{R}}{48\pi} + \text{h.o.}\ ,$$
$$T_{\hat{r}\hat{r}} = \frac{\pi}{12}\left[\left(\frac{T_h}{\alpha}\right)^2 - \left(\frac{T_c}{\alpha}\right)^2\right] + \frac{\mathcal{R}}{48\pi} + \text{h.o.}\ , \qquad (59)$$
$$T_{\hat{t}\hat{r}} = T_{\hat{r}\hat{t}} = -\frac{\pi}{12}\left[\left(\frac{T_h}{\alpha}\right)^2 - \left(\frac{T_c}{\alpha}\right)^2\right] + \text{h.o.}\ .$$

Like the $T_{\mu\nu}$ near the black-hole horizon, this has the simple, standard membrane-paradigm interpretation of being, at $O(\alpha^{-2})$, the stress-energy tensor measured by static observers, minus the contribution of perfectly thermal radiation at temperature T_c in all modes.

Expressions (57) and (59) exhibit the usual blueshift of temperature (factors α^{-2}), which causes the stress energy as measured by static observers to become infinite as either of the horizons is approached. This divergence, however, is an artifact of the pathological behavior of the static observers' reference frames at the horizons. To verify that the stress-energy tensor is, in fact, regular at both horizons, we can transform to the proper reference frame of a freely falling observer whose four-velocity is

$$\mathbf{u} = \frac{\sqrt{f_0}}{f}\frac{\partial}{\partial t} \pm \sqrt{f_0 - f}\,\frac{\partial}{\partial r}\ . \qquad (60)$$

Here f_0 is f at the starting point of the free fall, the plus and minus signs pertain to the observer falling towards the cosmological or black-hole horizons, respectively. We can either repeat the argument presented following Eq. (29), or use the late time expressions Eqs. (54), as well as Eq. (56) (at the black-hole horizon) or Eq. (58) (at the cosmological horizon), to verify that in a freely falling observer's reference frame the stress-energy tensor remains finite as the observer crosses any of the horizons, regardless of where the crossing point is located.

VII. CONCLUSION

In this paper we have seen that quantum-field theory does not induce an instability of the black-hole horizon formed by gravitational collapse in de Sitter spacetime. Rather, when a massless scalar field is in the natural generalization of a de Sitter vacuum (also a natural generalization of the Unruh vacuum), it remains everywhere well behaved—and, indeed, behaves in just the manner one would expect from the study of quantum-field theory in other horizon-endowed spacetimes.

After the original version of this paper was submitted for publication, we received a paper by Shin-ichi Tadaki and Shin Takagi which reaches the same principal conclusions as we derive in Secs. IV and VI—but describes them in somewhat different language.[18]

ACKNOWLEDGMENTS

D.M. wishes to thank Kip S. Thorne for advice about this research. D.M. was supported in part by the National Science Foundation, Grant No. AST-8817792. W.G.U. was supported in part by Natural Sciences and Engineering Research Grant No. 580441, by the Canadian Institute for Advanced Research and by the LAC Minerals Corporation.

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